ANALYSIS 1

(ENGLISH)

LECTURE 6

TODAY:

- LIMIT OF A SEQUENCE
  (CONVERGENCE)

- ALGEBRA WITH LIMITS
- SQUEEZE THEOREM
- EXAMPLES & COMPUTATIONS

PROP $q \in \mathbb{R}$, $q > 1$. THEN

BERNOULLI INEQ. $q^n \geq 1 + n(q-1) \quad \forall n \in \mathbb{N}$

Q. HOW DO WE PROVE IT?

$$q^n = (1 + (q-1))^n$$

INDUCTION

$P(K)$ = PROPERTY [STATEMENT WE WANT TO PROVE IS TRUE FOR $K \in \mathbb{N}$]

$\cap$
$\mathbb{N}$

EXAMPLE

- $P(K)$: "$(K+1)^2 \geq 2K$"
- $P(K)$: "$1+2+3+\dots+K = \dfrac{(K+1)\cdot K}{2}$"

$P(1)$: "$1=1$"; $P(2)$: "$1+2 = \dfrac{2\cdot 3}{2}$"; $P(50)$: $1+2+3+\dots+50$
$\parallel$
$25\cdot 51 =$

INDUCTION

$P(K)$ = PROPERTY [STATEMENT WE WANT TO PROVE IS TRUE FOR $K \in \mathbb{N}$]

$K \in \mathbb{N}$

HOW TO PROVE $P(K)$ IS TRUE $\forall K \in \mathbb{N}$?

2-STEP RECIPE

1. STARTING POINT : $P(0)$ IS TRUE
2. INDUCTION STEP :

K=27 START. POINT.

$P(n)$ TRUE $\Longrightarrow$ $P(n+1)$ TRUE

2 - STEP RECIPE

1. STARTING POINT : $P(0)$ IS TRUE
2. INDUCTION STEP :

$$P(n) \text{ TRUE} \implies P(n+1) \text{ TRUE}$$

<u>EXAMPLE</u> $P(K): (K+1)^2 \geq K+1$

2-STEP RECIPE

① STARTING POINT : $P(0)$ IS TRUE

② INDUCTION STEP :

$P(n)$ TRUE $\Longrightarrow P(n+1)$ TRUE

EXAMPLE $\quad P(K): (K+1)^2 \geq K+1$

① $P(0): (0+1)^2 \geq 0+1 \Longleftrightarrow 1 \geq 1$

$K=0$

2 - STEP RECIPE

① STARTING POINT : $P(0)$ IS TRUE

② INDUCTION STEP :

$$P(n) \text{ TRUE} \Longrightarrow P(n+1) \text{ TRUE}$$

<u>EXAMPLE</u> $P(K): (K+1)^2 \geq K+1$

① $P(0): (0+1)^2 \geq 0+1$ TRUE

② ASSUME $(n+1)^2 \geq (n+1)$ TRUE

$K = n$

2-STEP RECIPE

① STARTING POINT : $P(0)$ IS TRUE

② INDUCTION STEP :

$P(n)$ TRUE $\Longrightarrow$ $P(n+1)$ TRUE

EXAMPLE $P(K): (K+1)^2 \geq K+1$

① $P(0): (0+1)^2 \geq 0+1$ TRUE

② ASSUME $(n+1)^2 \geq (n+1)$ TRUE

NEED TO PROVE :

2-STEP RECIPE

① STARTING POINT: $P(0)$ IS TRUE

② INDUCTION STEP:

$$P(n) \text{ TRUE} \implies P(n+1) \text{ TRUE}$$

EXAMPLE $P(K): (K+1)^2 \geq K+1 \quad \forall n \in \mathbb{N}$

① $P(0): (0+1)^2 \geq 0+1$ TRUE

② ASSUME $*$ $\underline{\underline{(n+1)^2 \geq (n+1)}}$ TRUE

NEED TO PROVE $*$ $((n+1)+1)^2 \overset{?}{\geq} n+1+1$

$$(n+1)^2 \cancel{+1} + 2(\underset{\geq 0}{n+1}) \overset{?}{\geq} (n+1) \cancel{+1} \quad \text{TRUE}$$

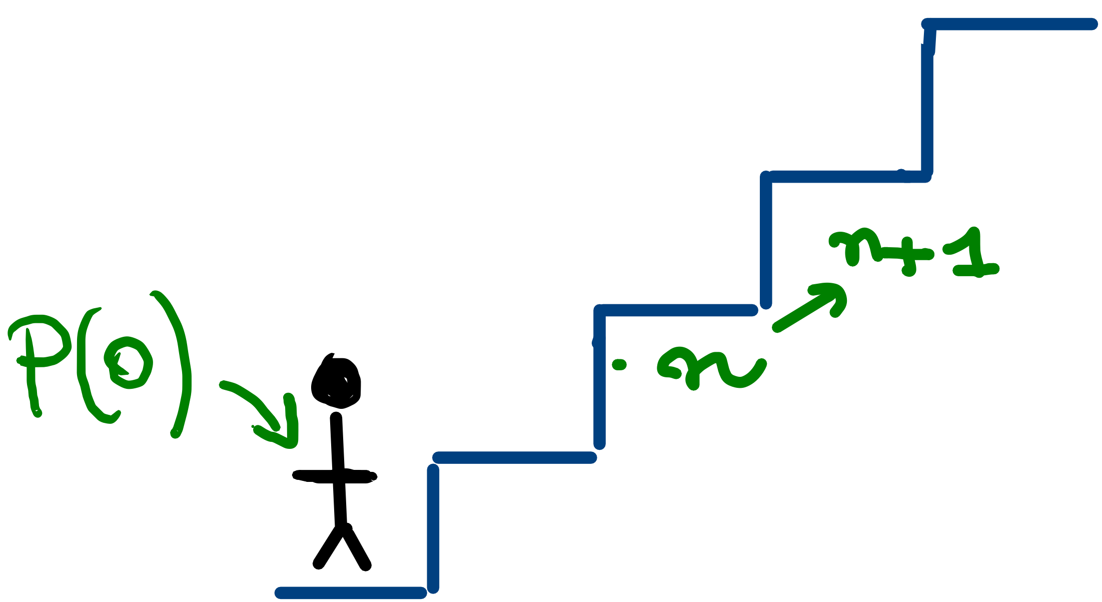
2-STEP RECIPE

① STARTING POINT : $P(0)$ IS TRUE

② INDUCTION STEP :

$P(n)$ TRUE $\Longrightarrow$ $P(n+1)$ TRUE

"LIKE A STAIR"



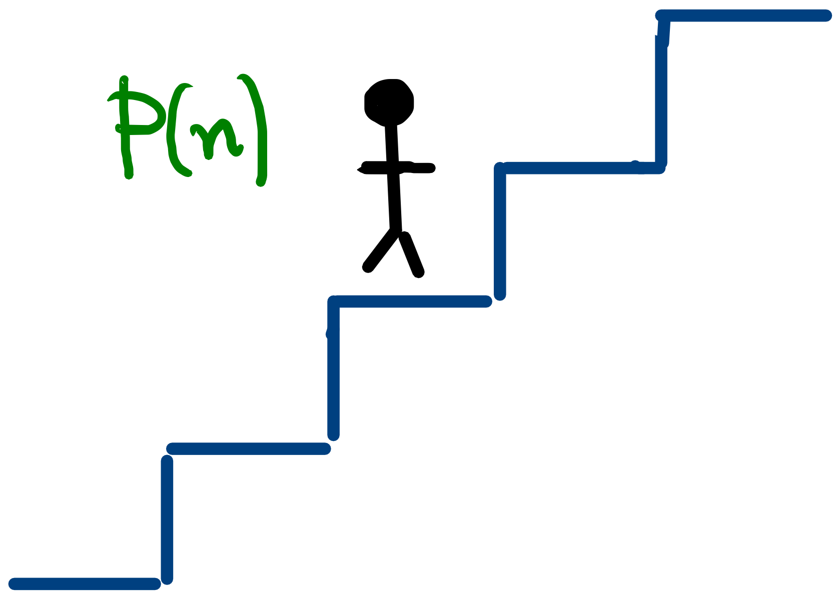
2-STEP RECIPE

① STARTING POINT : $P(0)$ IS TRUE

② INDUCTION STEP :

$P(n)$ TRUE $\Longrightarrow$ $P(n+1)$ TRUE

" LIKE A STAIR "



2-STEP RECIPE

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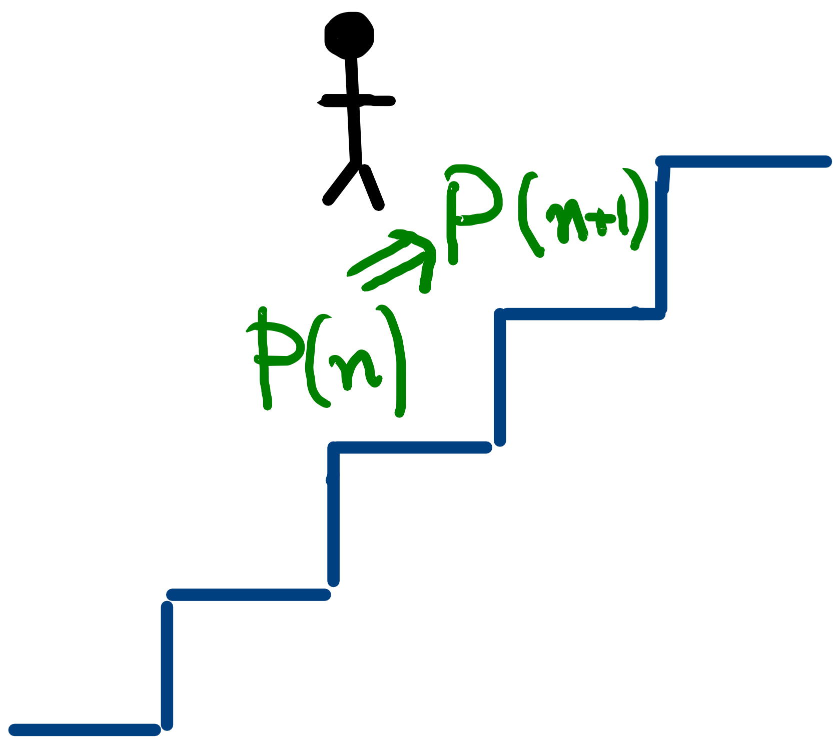

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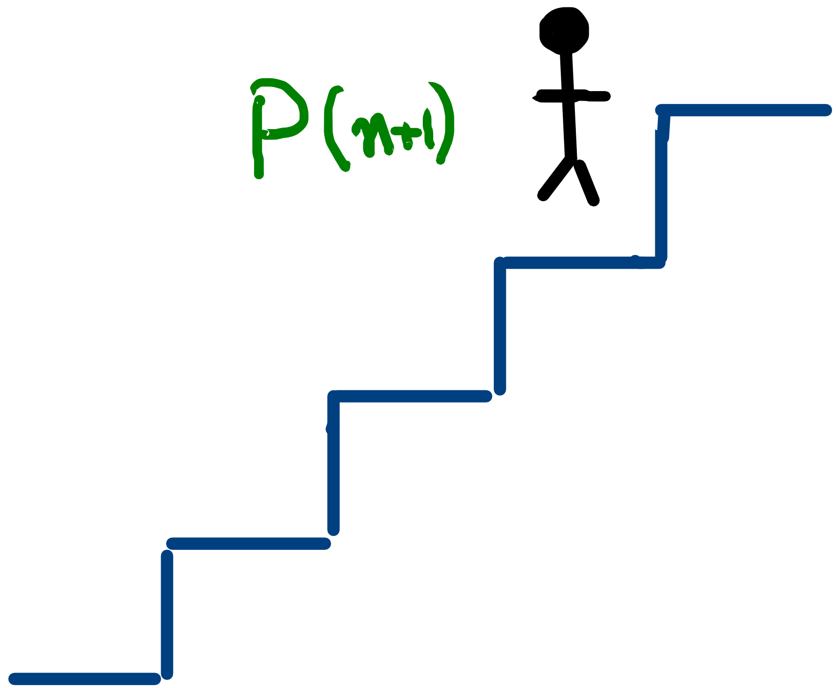


PROP $q \in \mathbb{R}$, $q > 1$. THEN

$$q^n \geq 1 + n(q-1)$$

Q. HOW DO WE PROVE IT?

$$q^n = (1 + (q-1))^n \qquad n \in \mathbb{N}$$

PROP $q \in \mathbb{R}$, $q > 1$. THEN

$$q^n \geq 1 + n(q-1)$$

Q. HOW DO WE PROVE IT?

$$q^n = (1 + (q-1))^n \qquad n \in \mathbb{N}$$

FORMULA: $(x+y)^2 = x^2 + y^2 + 2xy$

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

PROP $q \in \mathbb{R}, q > 1$. THEN

$$q^n \geqslant 1 + n(q-1)$$

Q. HOW DO WE PROVE IT?

$$q^n = (1 + (q-1))^n$$

FORMULA: $(x+y)^2 = x^2 + y^2 + 2xy$

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

GENERAL: $(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$

BINOMIAL "n choose i"

GENERAL: $(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^j y^{n-j}$

$$\binom{n}{j} \doteq \frac{n!}{j!\,(n-j)!}$$

"n choose j"

$k!$ for $k \in \mathbb{N}^*$

$\cdot\!\!\shortparallel\!\!\cdot$

$1 \cdot 2 \cdot 3 \cdot 4 \cdot \,\ldots\, \cdot (k-1) \cdot k$

GENERAL: $(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^j y^{n-j}$

$$\binom{n}{j} \doteq \frac{n!}{j!(n-j)!}$$

$$= \frac{n(n-1)(n-2)\ldots 3\cdot 2\cdot 1}{(j(j-1)(j-2)\ldots 1)((n-j)(n-j-1)\ldots 2\cdot 1)}$$

$0! = 1$

PROPERTY:

$(n+1)! = (n+1)\cdot n!$

EX $\binom{4}{2} = \frac{4!}{2!\,2!} = \frac{\cancel{4}\cdot 3\cdot 2\cdot 1}{(\cancel{2}\cdot 1)(\cancel{2}\cdot 1)} = 6$

$\binom{5}{4} = \frac{5!}{4!\,1!} = \frac{5\cdot \cancel{4!}}{\cancel{4!}\cdot 1!} = 5$

GENERAL: $(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^j y^{n-j}$

$\mathbb{N} \ni \binom{n}{j} \doteq \frac{n!}{j!(n-j)!} = \binom{n}{n-j}$

$$= \frac{n(n-1)(n-2)\ldots 3\cdot 2\cdot 1}{\left(j(j-1)(j-2)\ldots 1\right)\left((n-j)(n-j-1)\ldots 2\cdot 1\right)}$$

EX GENERAL $\binom{n}{n-1} = \frac{n!}{(n-1)!} = n$

$* \binom{n}{1} = \binom{n}{n-1} = n$

PROP $q \in \mathbb{R},\ q > 1$. THEN

$$q^n \geq \underline{\underline{1 + n(q-1)}}$$

$$q^n = (1 + (q-1))^n =$$

$$= \sum_{j=0}^{n} \binom{n}{j} 1^{n-j} (q-1)^j =$$

$$\binom{n}{0} = \frac{\cancel{n!}}{0!\,\cancel{n!}}$$

$$\binom{n}{0} = 1$$

$$\underbrace{\cancel{\binom{n}{0}}\ \cancel{1^n}\ \cancel{(q-1)^0}}_{1} + \underbrace{\binom{n}{1}\ \cancel{1^{n-1}}\ (q-1)^1}_{n(q-1)} +$$

$$\binom{n}{2} \cdot \cancel{1^{n-2}} (q-1)^2 + \dots$$

$$q^n = 1 + n(q-1) + \text{POSITIVE TERMS} \to 1 + n(q-1)$$

NEED TO PROVE:

$$\forall\; 2 \le j \le n \qquad \binom{n}{j} \underbrace{1^j}_{\substack{= \\ 1 \\ > \\ 0}} \underbrace{(q-1)^{n-j}}_{\substack{> \\ 0}} > 0$$

ASSUMPTION: $q > 1$

$$\Updownarrow$$

$$q - 1 > 0$$

$$\binom{n}{j} = \frac{\overbrace{n!}^{>0}}{\underbrace{j!}_{>0}\,\underbrace{(n-j)!}_{>0}}$$

$\blacksquare$

PROP $q \in \mathbb{R}, q > 1$. THEN

$$q^n \geq 1 + n(q-1)$$

$$q^n = (1 + (q-1))^n =$$

$$= \sum_{j=0}^{n} \binom{n}{j} 1^{n-j} (q-1)^j =$$

$$1^n + n \cdot (q-1)^1 + \binom{n}{2} (q-1)^2 +$$

$$\binom{n}{3}(q-1)^3 + \binom{n}{4}(q-1)^4 + \ldots$$

EXAMPLE $x_n \doteq 4^n$ IS UNBOUNDED SEQUENCE.

STRICTLY MONOTONE! (INCREASING) $x_{n+1} = 4 \cdot x_n > x_n$ $\left(x_n > 0 \ \forall n \in \mathbb{N}^* \right)$

"IT IS UNBOUNDED SINCE IT ALWAYS GROWS!!"

$$\forall c \in \mathbb{R}, \ \exists n \in \mathbb{N} \text{ s.t. } x_n \geq c$$

BERNOULLI INEQ: $4^n \geq 1 + n \cdot 3 \geq c$

$$n \doteq \left[\frac{c-1}{3} \right]$$

EXAMPLE $x_n = 4^n$ IS UNBOUNDED.

STRICTLY MONOTONE! $x_{n+1} = 4 \cdot x_n$

"IT IS UNBOUNDED SINCE IT ALWAYS GROWS!!"

FALSE: $x_n = 10 - \frac{1}{n}$ $n \geq 1$

(x_n) BOUNDED :

$\Updownarrow$

$S \doteq \{ 10 - \frac{1}{n} \mid n \in \mathbb{N}^* \}$ is BOUNDED

0 is A L.B FOR S, 10 IS AN U.B.

EXAMPLE

$$x_n = 10 - \frac{1}{n}$$

EXAMPLE

$$x_n = 10 - \frac{1}{n}$$

STRICTLY INCREASING:

BOUNDED:

EVEN BETTER:

EXAMPLE $x_n = 10 - \frac{1}{n}$

STRICTLY INCREASING: $\frac{1}{n+1} < \frac{1}{n} \Rightarrow x_n < x_{n+1}$

BOUNDED: $0 \leqslant x_n \leqslant 10 \quad \forall n \in \mathbb{N}$

EVEN BETTER: FOR ANY $K \in \mathbb{N}$

$$n > K \Rightarrow |x_n - 10| < \frac{1}{K}$$

$$\left|\cancel{10} - \frac{1}{n} - \cancel{10}\right|$$

$$\|$$

$$\frac{1}{n} < \frac{1}{K}$$

(x_n) GETTING CLOSER AND CLOSER TO 10

LIMIT OF A SEQUENCE

DEF A SEQUENCE (x_n) CONVERGES TO $y \in \mathbb{R}$ IF FOR EVERY $\varepsilon > 0$ $(\varepsilon \in \mathbb{R})$ THERE EXISTS $n_\varepsilon \in \mathbb{N}$ SUCH THAT

$$\forall \underset{\mathbb{N}}{\underset{\cap}{n}} \geqslant n_\varepsilon \text{ THEN } |x_n - y| \leqslant \varepsilon$$

LIMIT OF A SEQUENCE

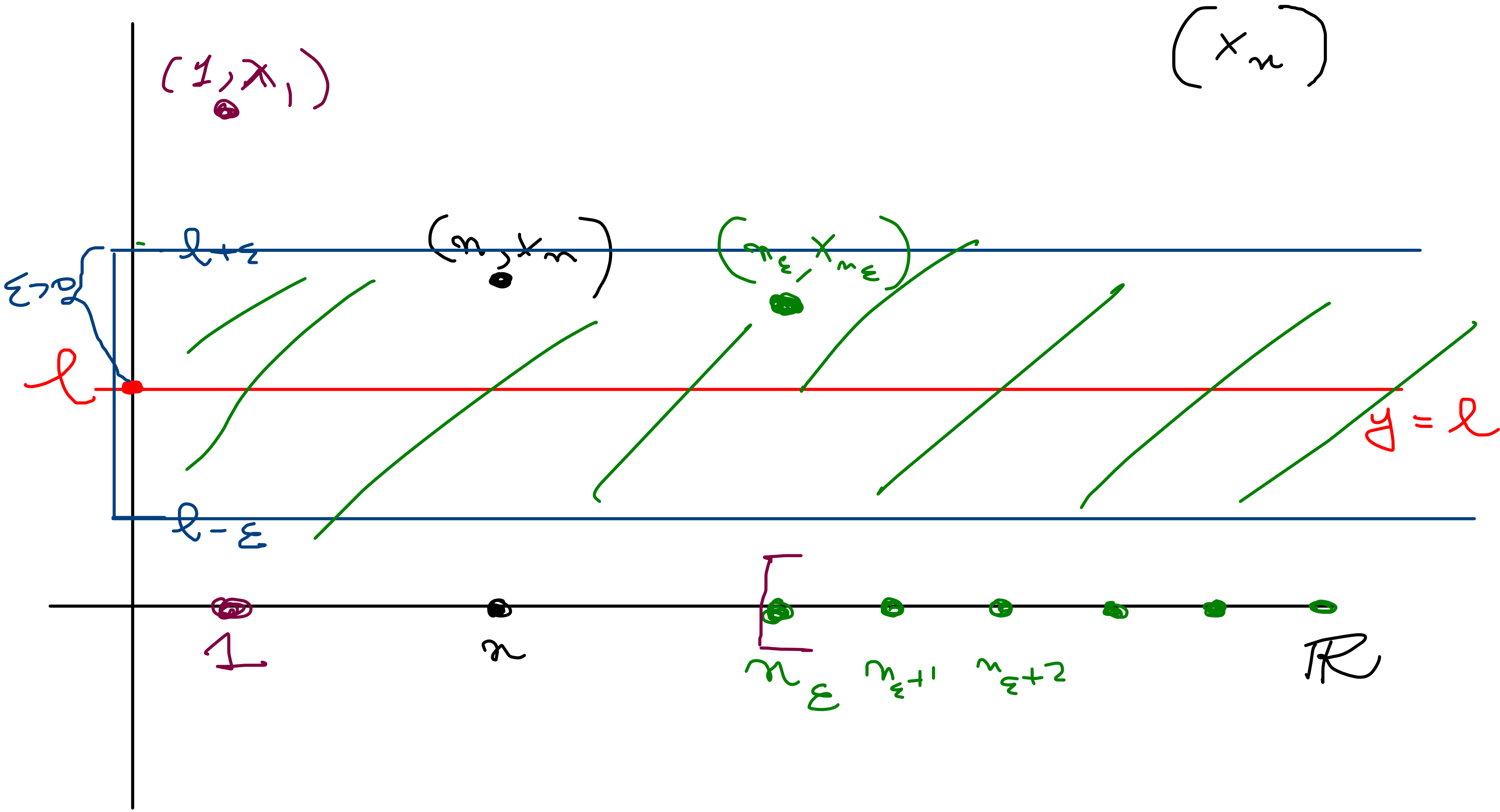
DEF A SEQUENCE (x_n) CONVERGES TO $y \in \mathbb{R}$ IF FOR EVERY $\varepsilon > 0$ THERE EXISTS $n_\varepsilon \in \mathbb{N}$ SUCH THAT $\forall n \geq n_\varepsilon$, $|x_n - y| < \varepsilon \iff x_n \in [y-\varepsilon, y+\varepsilon]$

NOTATION

$$\lim_{n \to \infty} x_n = y$$

LIMIT OF (x_n)

"limit for n that goes to ∞"



(x_n)
$(1, x_1)$
(n, x_n)
$(n_\varepsilon, x_{n_\varepsilon})$
$l+\varepsilon$
$\varepsilon>0$
l
$y=l$
$l-\varepsilon$
1
n
n_ε
$n_\varepsilon+1$
$n_\varepsilon+2$
$\mathbb{R}$

EXAMPLE

$x_n = \frac{1}{n} \qquad n \in \mathbb{N}^*$

INTUITION: x_n GETTING SMALLER WHEN n GROWS

$$\lim_{n \to \infty} x_n \overset{?}{=} 0$$

FIX $\varepsilon > 0$: WE NEED TO FIND $n_\varepsilon \in \mathbb{N}$ S.T. $\forall n \geq n_\varepsilon \quad \underbrace{|x_n - 0|}_{= |x_n|} \leq \varepsilon$

$\varepsilon \geq 1 \Longrightarrow$ ☺

$\varepsilon \geq 1$ TAKE $n_\varepsilon = 1 \Longrightarrow \forall n \geq 1$

$$\frac{1}{n} = |x_n| \leq 1 \leq \varepsilon$$

FIX $\varepsilon < 1$: NEED TO FIND

$\frac{1}{\varepsilon} > 1 > 0$ $n_\varepsilon \in \mathbb{N}$ S.T. $\forall n \geq n_\varepsilon$

$\frac{1}{n} \leq \varepsilon \iff n \geq \frac{1}{\varepsilon}$

ARCH. PROP. : $\exists\, n_\varepsilon \in \mathbb{N}$. S.T.

$n_\varepsilon > \frac{1}{\varepsilon} \iff \frac{1}{n_\varepsilon} < \varepsilon$

THEN ALSO $\boxed{\forall n \geq n_\varepsilon}$

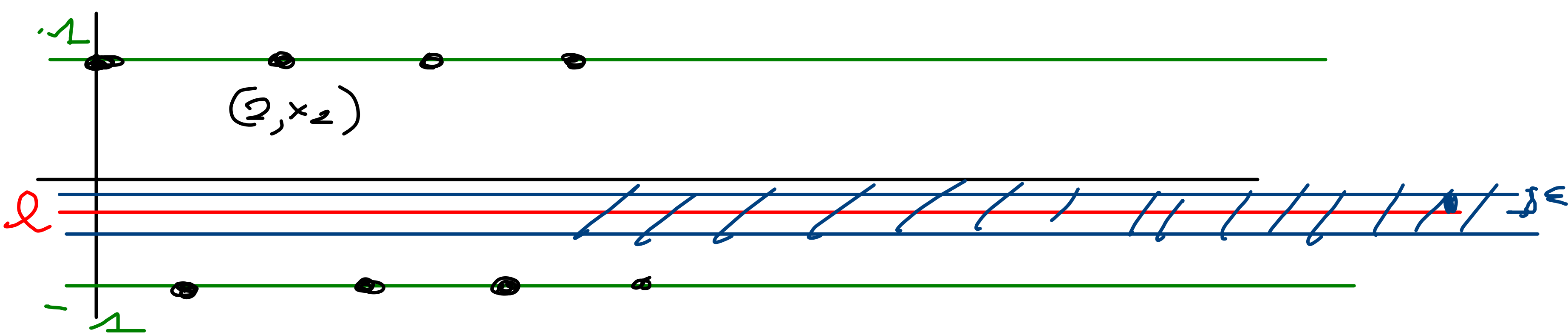
$n > \frac{1}{\varepsilon} \iff \boxed{\frac{1}{n} < \varepsilon}$

$\lim_{n \to \infty} \frac{1}{n} = 0$

EXAMPLE

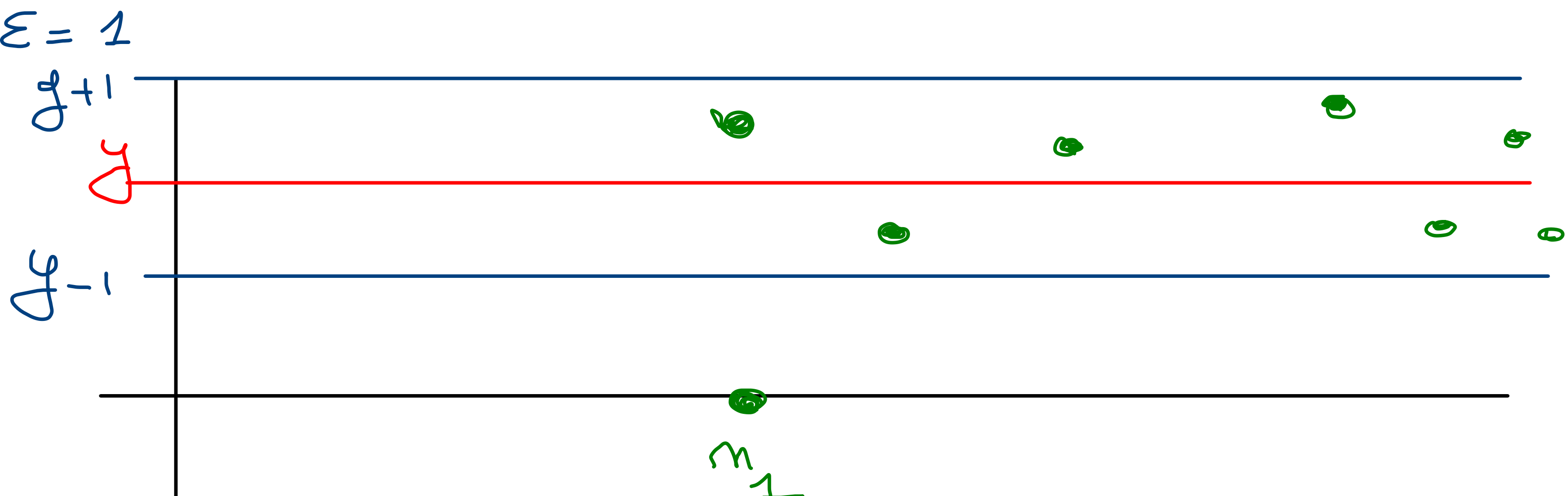
$x_n = (-1)^n$ IS [POLL]

1. CONVERGENT TO -1
2. CONVERGENT TO 1
3. NON-CONVERGENT
4. NONE OF THE ABOVE



PROP IF (x_n) IS CONVERGENT (ADMITS A LIMIT $y \in \mathbb{R}$)

$\Longrightarrow (x_n)$ IS BOUNDED



$\forall n \geq n_1$, BOUNDED SET $x_n \in [y-1, y+1]$

$\{x_n \mid n \in \mathbb{N}\} \subseteq [y-1, y+1] \cup \{x_0, x_1, \ldots, x_{n_1-1}\}$

PROP IF (x_n) IS CONVERGENT $\Longrightarrow (x_n)$ IS BOUNDED

CAVEAT: VICEVERSA IS NOT TRUE!!

(x_n) BOUNDED $\cancel{\Longrightarrow}$ (x_n) CONV.

$x_n = (-1)^n$ (x_n) NOT CONV.

EXAMPLE $x_n = n^{3/2}$ IS [POLL]

1. CONVERGENT TO 1
2. CONVERGENT TO -1
3. CONVERGENT TO 2
4. NON-CONVERGENT

x_n NOT BOUNDED

$x_n \geq n \quad \forall n \in \mathbb{N}$

$n \cdot \underset{\geq 1}{\sqrt{n}}$

LIMITS & ALGEBRA

PROP (x_n), (z_n)

$$\lim_{n \to \infty} x_n = \overline{x}, \quad \lim_{n \to \infty} z_n = \overline{z}.$$

LIMITS & ALGEBRA

PROP (x_n), (z_n)

$\lim_{n\to\infty} x_n = \bar{x}$, $\lim_{n\to\infty} z_n = \bar{z}$.

THEN: SEQ $y_n \doteq x_n + z_n$

① $\lim_{n\to\infty} (x_n + z_n) = \bar{x} + \bar{z}$

② $\lim_{n\to\infty} (x_n \cdot z_n) = \bar{x} \cdot \bar{z}$

③ $\lim_{n\to\infty} \frac{x_n}{z_n} = \frac{\bar{x}}{\bar{z}}$ [ASSUMING $\bar{z} \neq 0$]

$z_n \neq 0 \ \forall n \in \mathbb{N} \ (\forall n \geq n_0)$

limit commutes w/ the standard operations.

EXAMPLE

$$y_n = 10 - \frac{1}{n} \qquad n > 0$$

$$y_n = x_n + z_n$$

$x_n \doteq 10$ CONV. seq. $\lim x_n = 10$

$z_n \doteq -\frac{1}{n}$ CONV. SEQ. $\lim z_n = 0$

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n + z_n =$$

$$= \lim_{n\to\infty} x_n + \lim_{n\to\infty} z_n = 10$$

EXAMPLE

$$y_n = \frac{1}{n(\sqrt{n}+1)}$$

$$y_n = x_n \cdot z_n$$

$$x_n = \frac{1}{n} \qquad \lim x_n = 0$$

PROD $\Longrightarrow \lim y_n = 0$

$$z_n = (\sqrt{n}+1)^{-1} \qquad \lim z_n = 0$$

Fix $\varepsilon > 0$, $\exists\, n_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq n_\varepsilon$

$$1 \qquad |z_n| = |z_n - 0| \leq \varepsilon$$

$$\vee$$

$$\frac{1}{\sqrt{n}+1} \leq \varepsilon \iff \sqrt{n} \geq \frac{1}{\varepsilon} - 1$$

$$\frac{1}{\sqrt{n}+1} \leq \varepsilon \iff \underline{\sqrt{n} \geq \frac{1}{\varepsilon} - 1} \quad (*)$$

INTER. CASE : $0 < \varepsilon < 1 \implies \frac{1}{\varepsilon} - 1 > 0$

$$(*) \iff n \geq \left(\frac{1}{\varepsilon} - 1\right)^2$$

EXAMPLE

$$y_n = \frac{6n^2 + 3n + 2}{5n^2 + 1}$$

EXAMPLE

$$y_n = \frac{6n^2+3n+2}{5n^2+1} =$$

$$= \frac{\cancel{n^2}\left(6+3/n+2/n^2\right)}{\cancel{n^2}\left(5+1/n^2\right)} \quad \begin{matrix} = x_n \\ = z_n \end{matrix}$$

$$\lim x_n = \lim \left(\underbrace{6}_{\to 6} + \underbrace{3/n}_{\to 0} + \underbrace{2/n^2}_{= \frac{1}{n}\cdot\frac{2}{n} \to 0} \right) = 6$$

$$\lim z_n = \lim \left(\underbrace{5}_{\to 5} + \underbrace{\frac{1}{n^2}}_{= \frac{1}{n}\cdot\frac{1}{n} \to 0} \right) = 5 \neq 0$$

$$\lim y_n = \frac{6}{5}$$

EXAMPLE

$$\frac{n+6}{3n^3+n^2+1} = y_n$$

$$= \frac{\cancel{n}\left(1+6/n\right)}{n^{2\cancel{3}}\left(3+\frac{1}{n}+\frac{1}{n^3}\right)} = \underbrace{\frac{1}{n^2}}_{= x_n} \cdot \underbrace{\frac{1+6/n}{3+\frac{1}{n}+\frac{1}{n^3}}}_{= z_n}$$

$$\frac{1}{n^2} = \frac{1}{n}\cdot\frac{1}{n} \to 0 \qquad z_n \to 1/3$$

$$\lim y_n = \lim x_n \cdot z_n = 0 \cdot \frac{1}{3} = 0$$

GENERAL:

$$y_n = \frac{p(n)}{q(n)}$$

degree p POLINOMIAL IN n

degree q POLINOMIAL IN n

$$p(n) = a_p n^p + a_{p-1} n^{p-1} + a_{p-2} n^{p-2} + \dots + a_1 \cdot p + a_0 \qquad a_p \neq 0$$

$$q(n) = b_q n^q + b_{q-1} n^{q-1} + \dots + b_1 n + b_0 \qquad b_q \neq 0$$

GENERAL:

$$\frac{p(n)}{q(n)}$$

degree p POLINOMIAL IN n

degree q POLINOMIAL IN n

$$\lim_{n\to\infty} \frac{p(n)}{q(n)} = \begin{cases} \xrightarrow{p<q} 0 \\ \xrightarrow{p=q} \dfrac{a_p}{b_q} \end{cases}$$

GENERAL: $$y_n = \frac{P(n)}{q(n)}$$

$P(n)$ ⟵ degree p POLINOMIAL IN n

$q(n)$ ⟵ degree q POLINOMIAL IN n

$$\lim_{n\to\infty} \frac{p(n)}{q(n)} = \begin{cases} \xrightarrow{\;p > q\;} 0 \\ \xrightarrow{\;p = q\;} \dfrac{a_p}{b_q} \end{cases}$$

$p > q \Longrightarrow (y_n)$ IS NOT CONVERGENT.

EXAMPLE

$$\frac{6n^2 + 3n + 2}{5n^2 + 1}$$