ANALYSIS 1

(ENGLISH)

LECTURE 5

LAST TIME:

- TRIANGLE INEQUALITY
- EXTENDED REAL LINE
- COMPLEX NUMBERS
  - DEFINITION: $a+ib$, $i^2=-1$
  - OPERATIONS: $+, -, \times, {}^{-1}, |\cdot|$
  - PROPERTIES

TODAY:

- SOLVING EQUATIONS W/ $\mathbb{C}$
- SEQUENCES:
  - DEFINITION
  - EXAMPLES
  - BOUNDEDNESS
  - INDUCTION

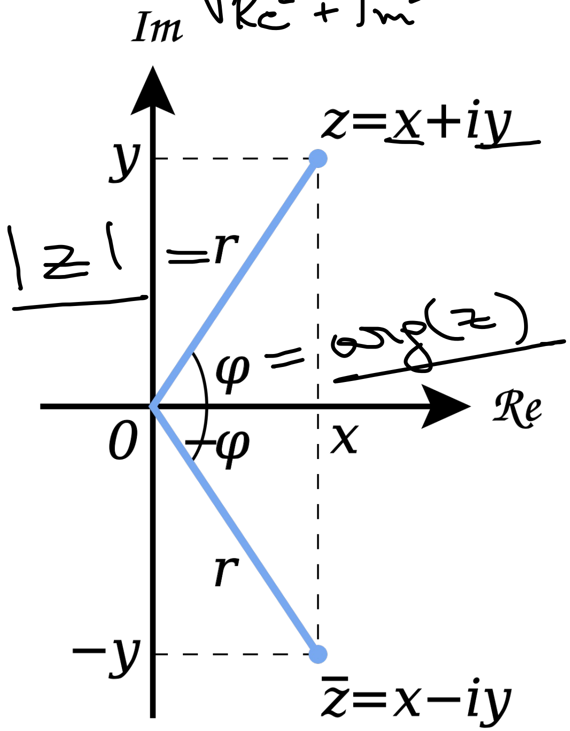
POLAR FORM: $z \in \mathbb{C}$

$$z = \underset{\sqrt{Re^2 + Im^2}}{\underset{\shortparallel}{|z|}} \cdot \left(\cos(\arg z) + \sin(\arg z)\, i\right)$$

$(\cos \theta, \sin \theta)$

$\uparrow$

all pts on the unit circle



$$\left| \frac{z}{|z|} \right| = 1$$

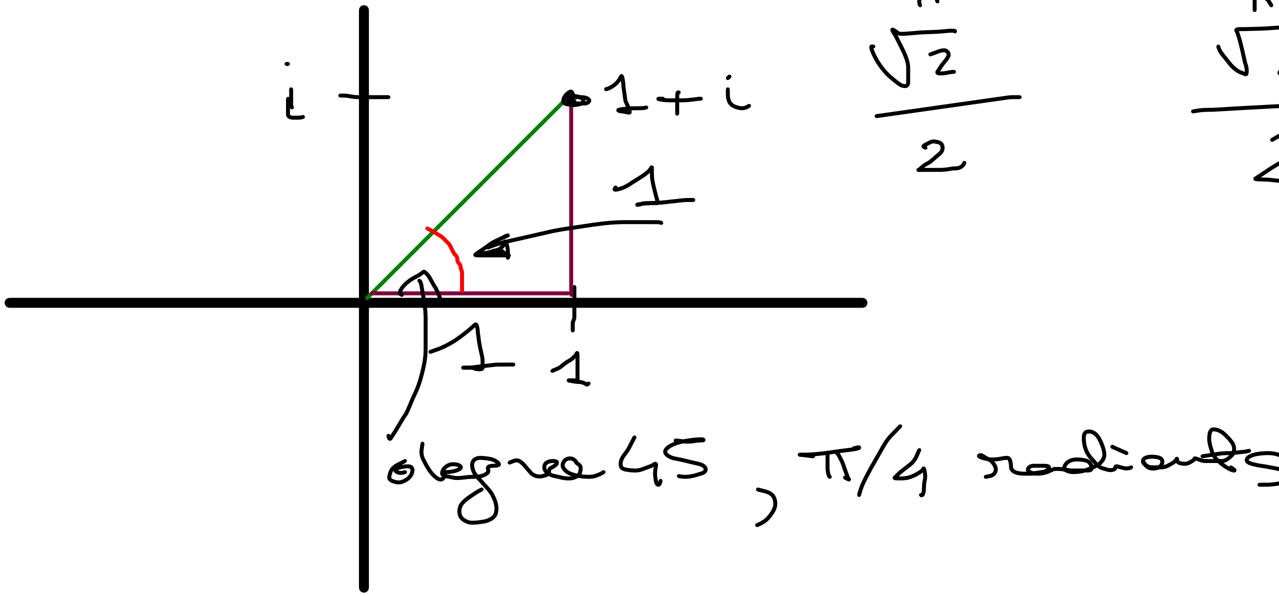
$$\Uparrow$$

$$|\lambda \cdot z| = |\lambda| \cdot |z|, \quad \lambda \in \mathbb{R}$$

$$\frac{z}{|z|} \in \text{unit circle} \subseteq \mathbb{C}$$

EXAMPLE

$$1+i=\sqrt{2}\left(\underbrace{\cos\left(\frac{\pi}{4}\right)}_{=\frac{\sqrt{2}}{2}}+\underbrace{\sin\left(\frac{\pi}{4}\right)}_{=\frac{\sqrt{2}}{2}}i\right)$$

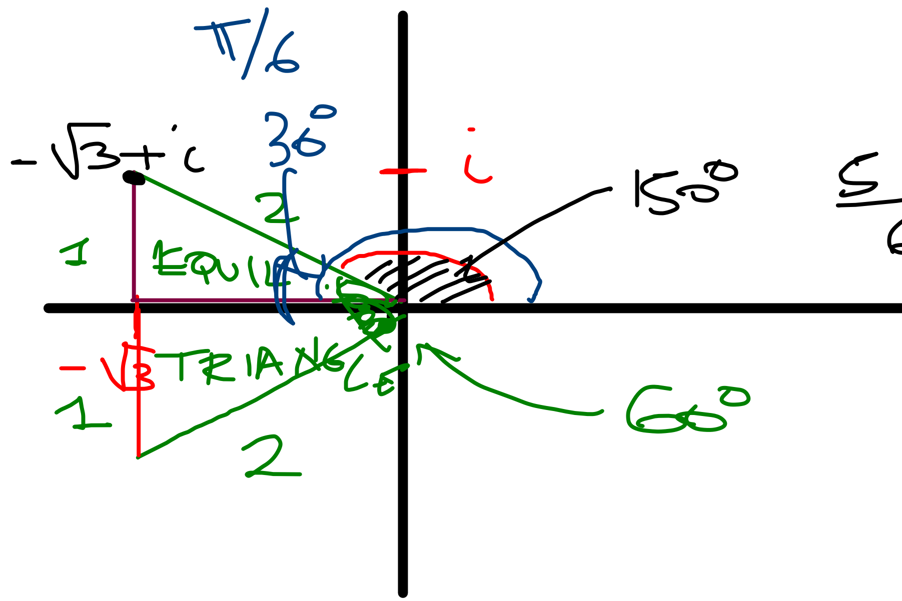


$$|1+i|=\sqrt{1^2+1^2}=\sqrt{2}$$

EXAMPLE

$$-\sqrt{3} + i = 2\left(\cos\frac{5\pi}{6} + \sin\frac{5\pi}{6}\, i\right)$$

$$\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} \qquad \sin\frac{5\pi}{6} = \frac{1}{2}$$



$$\frac{5\pi}{6}$$

$$|\sqrt{3} + i| = \sqrt{3+1} = 2$$

MULTIPLICATION IN POLAR FORM

$$z_1 = |z_1| \left(\cos(\varphi_1) + \sin(\varphi_1) \cdot i \right)$$

$$z_2 = |z_2| \left(\cos(\varphi_2) + \sin(\varphi_2) \cdot i \right)$$

$$z_1 \cdot z_2 = ?$$

MULTIPLICATION IN POLAR FORM

$$z_1 = |z_1|\left(\cos(\varphi_1) + \sin(\varphi_1)\cdot i\right) \qquad \varphi_i = \arg(z_i)$$

$$z_2 = |z_2|\left(\cos(\varphi_2) + \sin(\varphi_2)\cdot i\right)$$

$$z_1 \cdot z_2 =$$

$$|z_1|\,|z_2| \cdot \left(\cos(\varphi_1) + \sin(\varphi_1)\cdot i\right)\left(\cos(\varphi_2) + \sin(\varphi_2)\cdot i\right)$$

MULTIPLICATION IN POLAR FORM

$$z_1 = |z_1| \left(\cos(\varphi_1) + \sin(\varphi_1) \cdot i\right)$$

$$z_2 = |z_2| \left(\cos(\varphi_2) + \sin(\varphi_2) \cdot i\right)$$

$$z_1 \cdot z_2 =$$

$$\underline{\underline{|z_1|\,|z_2|}} \cdot \underbrace{\left(\cos(\varphi_1) + \sin(\varphi_1) \cdot i\right)\left(\cos(\varphi_2) + \sin(\varphi_2) \cdot i\right)}$$

$$\|$$

$$\left(\cos(\varphi_1)\cos(\varphi_2) - \sin(\varphi_1)\sin(\varphi_2)\right) +$$

$$\left(\cos(\varphi_1)\sin(\varphi_2) + \cos(\varphi_2)\sin(\varphi_1)\right) \cdot i$$

$$z_1 \cdot z_2 =$$

$$|z_1|\,|z_2| \cdot \underbrace{\left(\cos(\varphi_1) + \sin(\varphi_1)\cdot i\right)\left(\cos(\varphi_2) + \sin(\varphi_2)\cdot i\right)}$$

$$\|$$

$$\left(\cos(\varphi_1)\cos(\varphi_2) - \sin(\varphi_1)\sin(\varphi_2)\right) +$$

$$\left(\cos(\varphi_1)\sin(\varphi_2) + \cos(\varphi_2)\sin(\varphi_1)\right)\cdot i$$

$\|$ ADDITION FORM.

$$\cos(\varphi_1 + \varphi_2) = \cos(\varphi_1)\cos(\varphi_2) - \sin\varphi_1 \sin\varphi_2$$

$$\sin(\varphi_1 + \varphi_2) = \cos\varphi_1 \sin\varphi_2 + \cos\varphi_2 \sin\varphi_2$$

$$z_1 \cdot z_2 =$$

$$|z_1|\,|z_2| \cdot \underbrace{\left(\cos(\varphi_1) + \sin(\varphi_1)\cdot i\right)\left(\cos(\varphi_2) + \sin(\varphi_2)\cdot i\right)}$$

$$\|$$

$$\left(\cos(\varphi_1)\cos(\varphi_2) - \sin(\varphi_1)\sin(\varphi_2)\right) +$$

$$\left(\cos(\varphi_1)\sin(\varphi_2) + \cos(\varphi_2)\sin(\varphi_1)\right)\cdot i$$

$$\|$$

$$|z_1|\,|z_2| \cdot \underline{\left(\cos(\varphi_1+\varphi_2) + \sin(\varphi_1+\varphi_2)\cdot i\right)}$$

$$|z_1 \cdot z_2| = |z_1| \cdot |z_2|$$

$$z_1 \cdot z_2 =$$

$$|z_1|\,|z_2| \cdot \underbrace{\left(\cos(\varphi_1) + \sin(\varphi_1)\cdot i\right)\left(\cos(\varphi_2) + \sin(\varphi_2)\cdot i\right)}$$

$$\|$$

$$\left(\cos(\varphi_1)\cos(\varphi_2) - \sin(\varphi_1)\sin(\varphi_2)\right) +$$

$$\left(\cos(\varphi_1)\sin(\varphi_2) + \cos(\varphi_2)\sin(\varphi_1)\right)\cdot i$$

$$\|$$

$$|z_1|\,|z_2| \cdot \left(\cos(\varphi_1 + \varphi_2) + \sin(\varphi_1 + \varphi_2)\cdot i\right)$$

IN PARTICULAR: $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$

EULER'S FORMULA

$$\varphi \in \mathbb{R}, \quad e^{i\varphi} \doteq \cos(\varphi) + \sin(\varphi)\, i$$

NEPERO'S #

EULER'S FORMULA

$$\varphi \in \mathbb{R}, \quad e^{i\varphi} \doteq \cos(\varphi) + \sin(\varphi)\, i$$

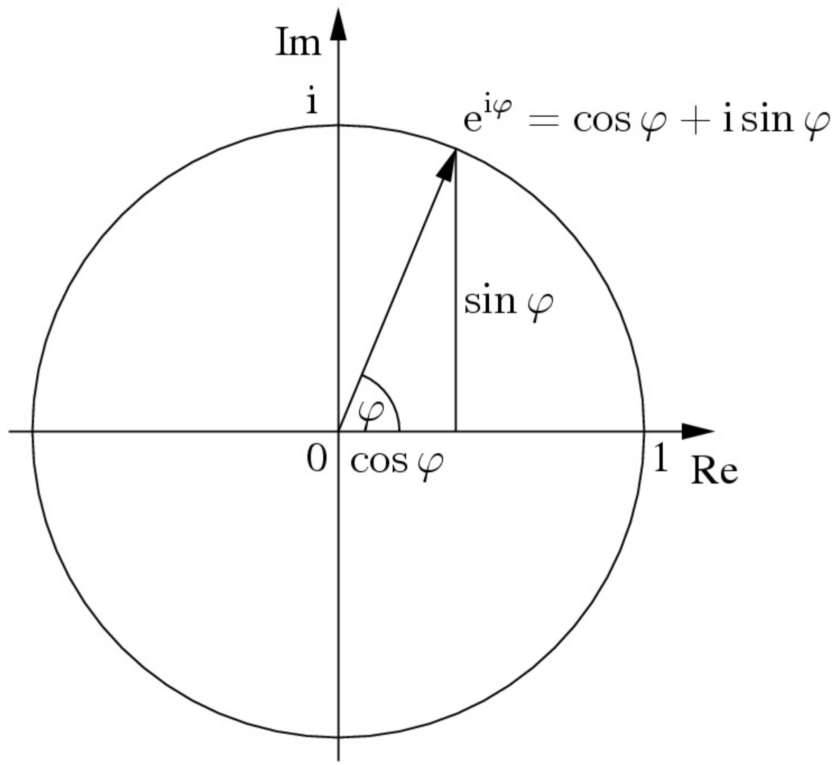
$$\left| e^{i\varphi} \right| = \sqrt{\cos^2\varphi + \sin^2\varphi} = 1$$

$e^{i\varphi}$ WHEN φ varies over $\mathbb{R}$ ($[0, 2\pi[$) COVERS ALL PTS $\in$ UNIT CIRCLE

EULER'S FORMULA

For $\alpha \in \mathbb{R}$, $e^{i\varphi} \doteq \cos(\varphi) + \sin(\varphi)\, i$

$$|e^{i\varphi}| = 1$$



PROPERTIES:

① $$e^{i\alpha} \cdot e^{i\beta} = e^{i(\alpha+\beta)} \quad \forall \alpha, \beta \in \mathbb{R}$$

$$(\cos(\alpha) + \sin(\alpha) i)(\cos\beta + \sin(\beta) i) = \cos(\alpha+\beta) + \sin(\alpha+\beta) i$$

② $$e^{i(\alpha+2\pi)} = e^{i\alpha} \quad \forall \alpha \in \mathbb{R}$$

$$\cos(\alpha+2\pi) = \cos(\alpha) \qquad \sin(\alpha+2\pi) = \sin(\alpha)$$

③ POLAR FORM:

$$z = |z| \left(\cos(\arg z) + \sin(\arg z) i\right)$$

$$\parallel$$

$$|z| \, e^{i \arg z}$$

EQUATIONS OVER $\mathbb{C}$

$$z^2 + 2z + 3 = 0$$

$$(z^2 + 2z + 1) + 2 = 0$$

$$(z+1)^2 = -2$$

$z + 1 = T$

$$T^2 = -2$$

No sol's in $\mathbb{R}$

EQUATIONS OVER $\mathbb{C}$

$$z^2 + 2z + 3 = 0$$

$$\downarrow$$

$$(z^2 + 2z + 1) = -2$$

$$\downarrow$$

$$(z+1)^2 = -2$$

$$\downarrow \quad T = z+1 \longrightarrow z = T-1$$

$$T^2 = -2 \implies T = \begin{cases} \sqrt{2}\, i \\ -\sqrt{2}\, i \end{cases}$$

EQUATIONS OVER $\mathbb{C}$

$$z^2 + 2z + 3 = 0$$

$$\downarrow$$

$$(z^2 + 2z + 1) = -2$$

$$\downarrow$$

$$(z+1)^2 = -2$$

$$\downarrow \quad T = z+1 \qquad z = T - 1$$

$$T^2 = -2 \Longrightarrow z = \begin{cases} \sqrt{2}\,i - 1 \\ -\sqrt{2}\,i - 1 \end{cases}$$

EQUATIONS OVER $\mathbb{C}$

FUNDAMENTAL THM OF ALGEBRA

EVERY POLYNOMIAL WITH COEFF'S IN $\mathbb{C}$ HAS AT LEAST 1 SOLUTION IN $\mathbb{C}$

$$a x^n + b x^{n-1} + c x^{n-2} + \dots + l x + m = 0$$

$\in \mathbb{C}$

EQUATIONS OVER $\mathbb{C}$

FUNDAMENTAL THM OF ALGEBRA

EVERY POLYNOMIAL WITH COEFF'S IN $\mathbb{C}$ HAS AT LEAST 1 SOLUTION IN $\mathbb{C}$

FALSE FOR $\mathbb{R}$!

$x^2 = -1$

IMPORTANT: A degree d POLYN. ALWAYS HAS AT MOST d SOL'S

$x^3 = 0$

EQUATIONS OVER $\mathbb{C}$

EXAMPLE

FIND SOLUTIONS FOR

INDET. $\longrightarrow$ $z^4 = 3e^{i\pi/5}$

EQUATIONS OVER $\mathbb{C}$

EXAMPLE

FIND SOLUTIONS FOR

$$z^4 = 3e^{i\pi/5}$$

$$z = r.e^{i\theta} \quad , \quad r \in \mathbb{R}^*_+ \quad , \quad \theta \in \cancel{\mathbb{R}}\ [0, 2\pi[$$

$$\theta' = \theta + 2\pi$$

EQUATIONS OVER $\mathbb{C}$

EXAMPLE

FIND SOLUTIONS FOR

$$z^4 = 3e^{i\pi/5}$$

$$z = re^{i\theta}, \quad r \in \mathbb{R}_+^*, \quad \theta \in \mathbb{R}$$

$$\Downarrow$$

$$r^4 \cdot e^{4\theta i} = 3e^{i\pi/5}$$

EQUATIONS OVER $\mathbb{C}$

EXAMPLE

FIND SOLUTIONS FOR

$$z^4 = 3 \, e^{i\pi/5}$$

$$z = r e^{i\theta}$$

$$\Downarrow$$

$$r^4 \cdot e^{4\theta i} = 3 \, e^{i\pi/5}$$

$$| \cdot | = r^4 \qquad | \cdot | = 3$$

$$r^4 = 3$$

$$r = +\sqrt[4]{3}$$

EQUATIONS OVER $\mathbb{C}$

EXAMPLE

FIND SOLUTIONS FOR

$$z^4 = 3 e^{i\pi/5}$$

$$z = r e^{i\theta}$$

$$\Downarrow$$

$$\cancel{\left(\sqrt[4]{3}\right)^4} \cdot e^{4\theta i} = \cancel{3}\, e^{i\pi/5}$$

$$1 \cdot 1 = r^4 \qquad 1 \cdot 1 = 3$$

$$r = +\sqrt[4]{3}$$

$$r^4 = 3$$

$$e^{4\theta i} = e^{i\pi/5}$$

$$e^{4\theta i} = e^{\pi/5\, i}$$

$$4\theta = \pi/5 + 2k\pi, \quad k \in \mathbb{Z}$$

$$\theta = \frac{\pi}{20} + \frac{\pi}{2}\cdot k, \quad k \in \mathbb{Z}$$

$\frac{\pi}{20}$ $\quad k=0$ $\qquad \frac{\pi}{20} + \pi$ $\quad k=2$

X $\frac{\pi}{20} + \frac{\pi}{2}$ $\quad k=1$ $\qquad \frac{\pi}{20} + \frac{3\pi}{2}$ $\quad k=3$

$\cancel{\frac{\pi}{20} + \overset{2}{4}\cdot\frac{\pi}{2} \quad k=4}$ $\qquad \pi/20 + \frac{1}{2}\pi + 2\pi$ $\quad k=5$

$$\sqrt[4]{3}\quad e^{\pi/20\, i}$$

$$\sqrt[4]{3}\quad e^{\left(\frac{\pi}{20}+\frac{\pi}{2}\right) i}$$

$$e^{\pi/20\, i}$$

$$\sqrt[4]{3}\quad e^{\left(\frac{\pi}{20}+\pi\right) i}$$

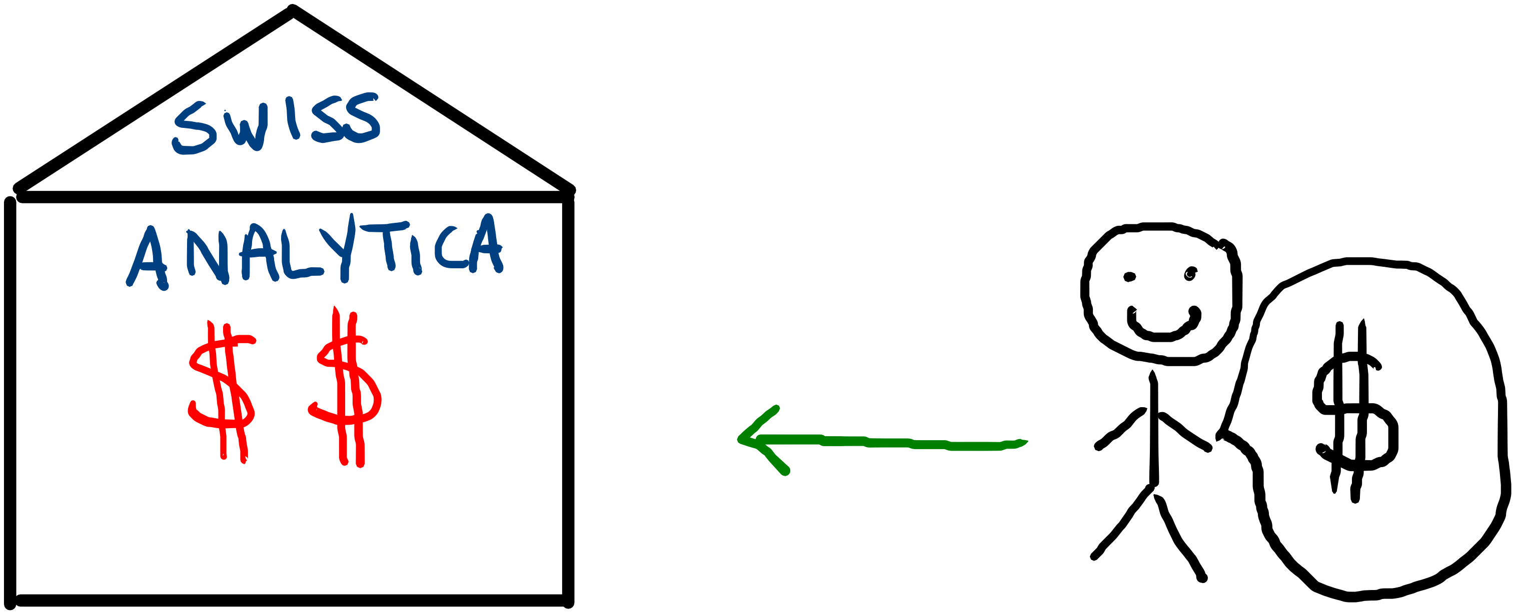
$$\shortparallel$$

$$e^{\left(\pi/20+2\pi\right) i}$$

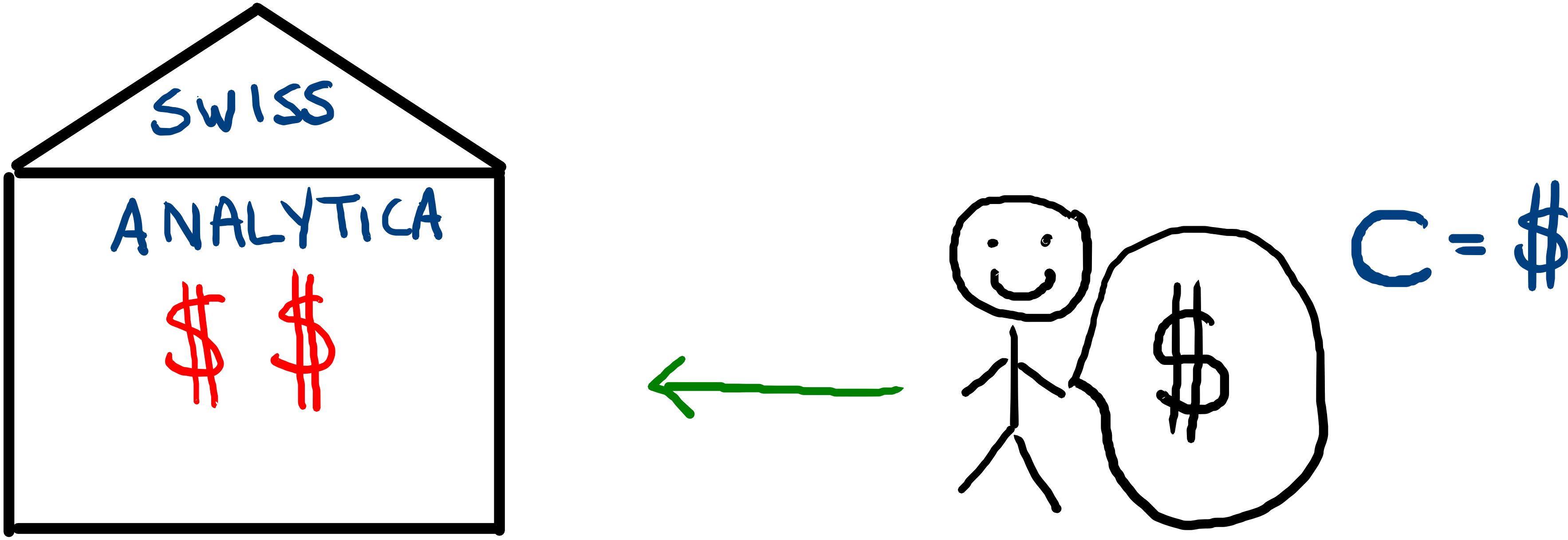
$$\sqrt[4]{3}\quad e^{\left(\pi/20+\frac{3}{2}\pi\right) i}$$

$$\sqrt[4]{3}\quad e^{\left(\pi/20+k\frac{\pi}{2}\right) i}, \quad k\in\mathbb{Z}$$

SEQUENCES



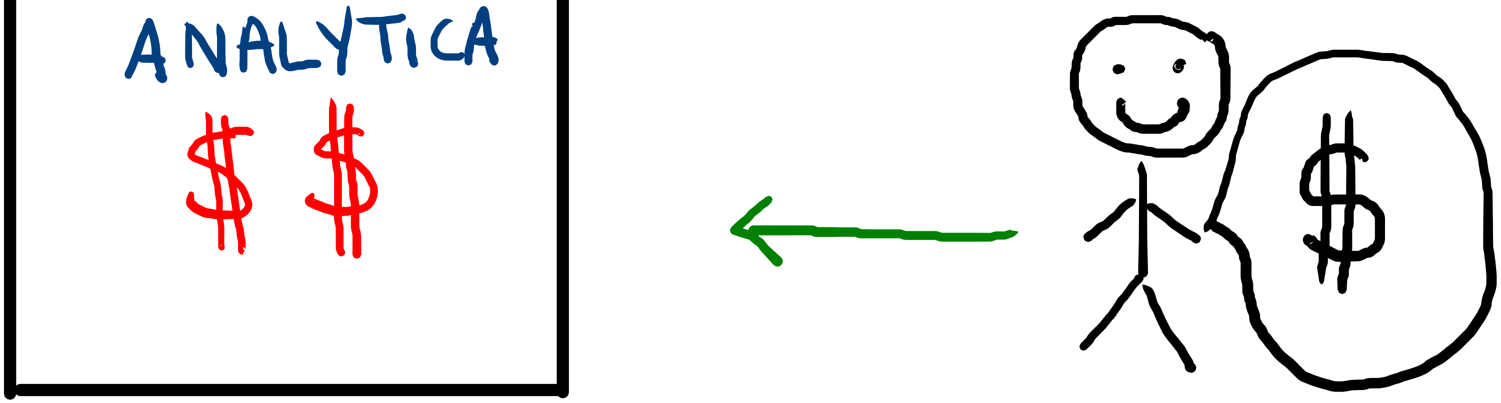
SEQUENCES



WEEK 0 = C

SEQUENCES

R = INTEREST RATE



$C = \$$

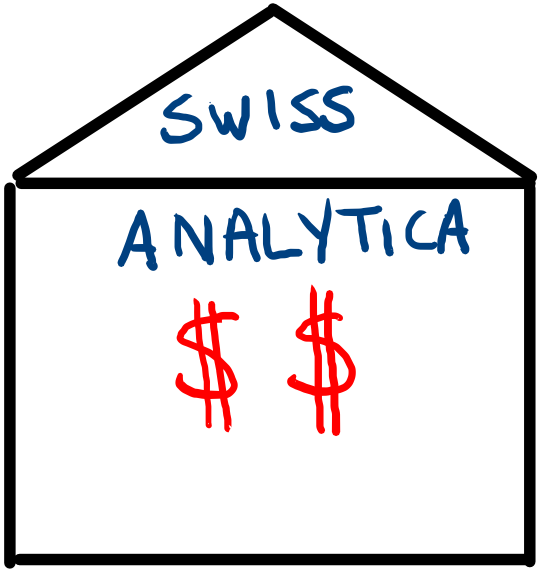
$$\text{WEEK } 0 = C$$

$$\text{WEEK } 1 = C(1+R)$$

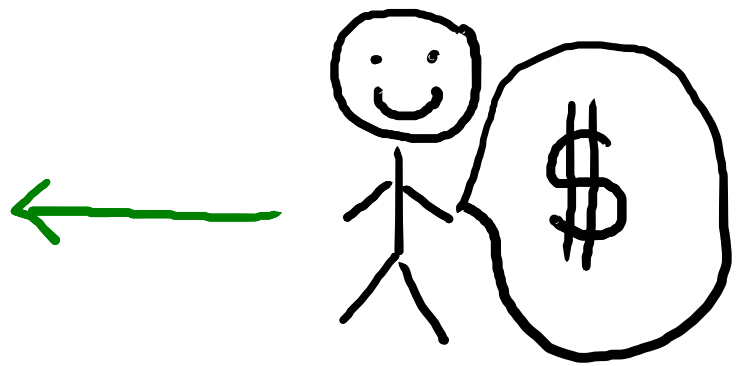
$$\text{WEEK } 2 = C(1+R)^2 - 1$$

$$\text{WEEK } 3 = C(1+R)^3 - 2$$

SEQUENCES



R = INTEREST RATE



C = $

WEEK 0 = C

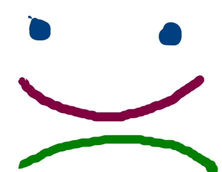
WEEK 1 = $C(1+R)$

WEEK 2 = $C(1+R)^2 - 1$

WEEK 3 = $C(1+R)^3 - 2$

WEEK N = $C(1+R)^N - (N-1)$

$N > 0$

?  ¿

SEQUENCE

FUNCTION $\mathbb{N} \longrightarrow \mathbb{R}$

ATTENTION!!!

NOT $\mathbb{R}$

BUT $\mathbb{N}$

SEQUENCE

FUNCTION $\mathbb{N} \longrightarrow \mathbb{R}$

ATTENTION!!! NOT $\mathbb{R}$ BUT $\mathbb{N}$

$0 \longmapsto x_0 \in \mathbb{R}$

$1 \longmapsto x_1 \in \mathbb{R}$

$2 \longmapsto x_2 \in \mathbb{R}$

$\vdots$

$n \longmapsto x_n \in \mathbb{R}$

SEQUENCE

FUNCTION $\mathbb{N} \longrightarrow \mathbb{R}$

$$1 \longmapsto x_1 \in \mathbb{R}$$

$$2 \longmapsto x_2 \in$$

$$\vdots$$

$$n \longmapsto x_n \in \mathbb{R}$$

EXAMPLE

$1 \mapsto 0,\ 2 \mapsto 0,\ 3 \mapsto 0, \dots$

$n \longmapsto 0, \dots$ COSTANT SEQUENCE

SEQUENCE

FUNCTION $\mathbb{N} \longrightarrow \mathbb{R}$

$$1 \longmapsto x_1 \in \mathbb{R}$$

$$2 \longmapsto x_2 \in$$

$$\vdots$$

$$n \longmapsto x_n \in \mathbb{R}$$

EXAMPLE

$1 \mapsto c,\ 2 \mapsto c,\ 3 \mapsto c, \dots$

$n \longmapsto c, \dots$

COSTANT SEQUENCE

NOTATION

x_n = VALUE AT $n \in \mathbb{N}$

NOTATION

x_n = VALUE AT $n \in \mathbb{N}$

EXAMPLE

$$x_n \doteq (-1)^n$$

$$n \longmapsto (-1)^n$$

$\Longrightarrow$

NOTATION

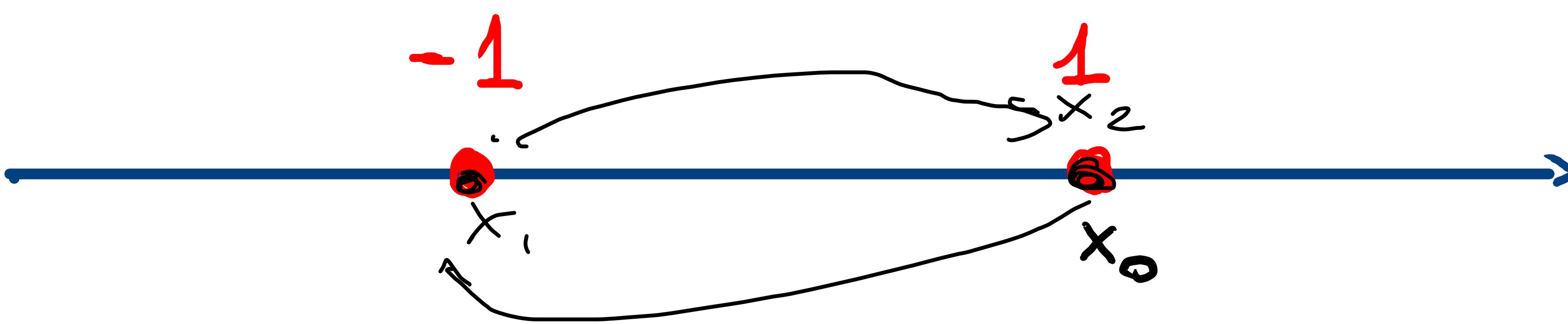
x_n = VALUE AT $n \in \mathbb{N}$

EXAMPLE

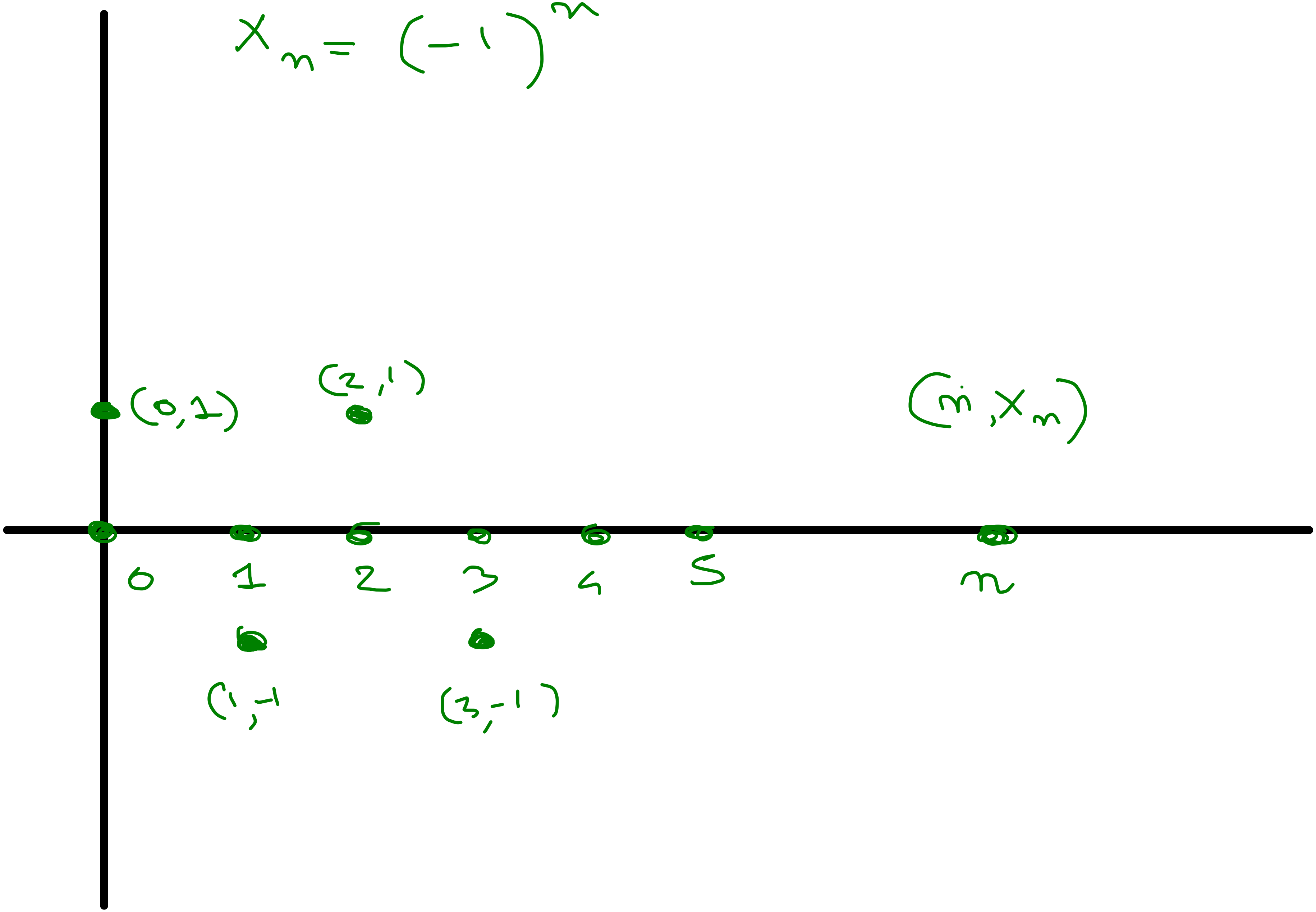
$$x_n \doteqdot (-1)^n$$

$\Longrightarrow \quad x_0 = 1 \,, \; x_1 = -1 \,, \; x_2 = +1$

$x_{odd} = -1 \qquad x_{even} = +1$



$$x_n = (-1)^n$$



NOTATION

x_n = VALUE AT $n \in \mathbb{N}$

EXAMPLE

ARITHMETIC PROGRESSION

$$x_n \doteq an + b$$

NOTATION

x_n = VALUE AT $n \in \mathbb{N}$

EXAMPLE

ARITHMETIC PROGRESSION

$$x_n \doteqdot a n + b$$

FIXED
REAL #'s

NOTATION $\quad x_n$ = VALUE AT $n \in \mathbb{N}$

EXAMPLE $\quad$ ARITHMETIC PROGRESSION

$$x_n \doteqdot a\,n + b \qquad n \in \mathbb{N}$$

FIXED
REAL #'s

THESE
ARE THE SAME!!

NOTATION x_n = VALUE AT $n \in \mathbb{N}$

EXAMPLE ARITHMETIC PROGRESSION

$$x_n \doteq an + b$$

$$\boxed{x_n = x_{n-1} + a}$$

$$\underset{a(n-1)+b}{\overset{\shortparallel}{x_{n-1}}}$$

EX. $x_n = 3n + 4$

$\Rightarrow$ $x_0 = 4$, $x_1 = 3 + 4 = 7$

$x_2 = 3 \cdot 2 + 4 = 10$

NOTATION $\quad x_n$ = VALUE AT $n \in \mathbb{N}$

EXAMPLE $\quad$ GEOMETRIC PROGRESSION

$$x_n \doteq c \cdot q^n$$

NOTATION

x_n = VALUE AT $n \in \mathbb{N}$

EXAMPLE

GEOMETRIC PROGRESSION

$$x_n \doteq c\, q^n$$

$n \in \mathbb{N}$

FIXED REAL #'S

NOTATION

x_n = VALUE AT $n \in \mathbb{N}$

EXAMPLE

GEOMETRIC PROGRESSION

$$x_n \doteq c\, q^n$$

$$x_n = x_{n-1} \cdot q$$

$$c \cdot q^{n-1}$$

EXAMPLE

$$x_n = 2 \cdot \left(\frac{4}{5}\right)^n$$

$$x_0 = 2, \quad x_1 = \frac{8}{5}, \quad x_2 = \frac{32}{25}, \quad \dots$$

$$2 \cdot \frac{4}{5} \qquad 2 \cdot \frac{16}{25}$$

QUESTION / GOAL : TO UNDERSTAND VALUES & BEHAVIOURS OF A SEQUENCE x_n

QUESTION / GOAL : TO UNDERSTAND VALUES & BEHAVIOURS OF A SEQUENCE x_n $(x_n)_{n \in \mathbb{N}}$

DEF THE SEQUENCE $\{x_n\}$ IS

- BOUNDED FROM ABOVE
- BOUNDED FROM BELOW
- BOUNDED

NOTATION FOR SEQUENCE

QUESTION / GOAL : TO UNDERSTAND VALUES & BEHAVIOURS OF A SEQUENCE x_n

DEF THE SEQUENCE $\{x_n\}$ IS

$\left\{\begin{array}{l}\text{BOUNDED FROM ABOVE} \\ \text{BOUNDED FROM BELOW} \\ \text{BOUNDED}\end{array}\right.$ IF

$\{\underset{\mathbb{R}}{\overset{\cap}{x_n}} \mid n \in \mathbb{N}\}$ IS $\left\{\begin{array}{l}\text{BOUNDED FROM ABOVE} \\ \text{BOUNDED FROM BELOW} \\ \text{BOUNDED}\end{array}\right.$

$$x_n = (-1)^n$$

$$y_n = \begin{cases} 1 & n = 0 \\ -1 & n > 0 \end{cases}$$

SET OF VALUES ARE THE SAME

EXAMPLE

- $x_n \doteq (-1)^n$

$\Rightarrow \{x_n \mid n \in \mathbb{N}\} = \{-1, 1\}$ BOUNDED

EXAMPLE

- $x_n \doteq (-1)^n$

$\Longrightarrow \{x_n \mid n \in \mathbb{N}\} = \{-1, 1\}$ BOUNDED

- $x_n = an + b$ IS

① BOUNDED

② BOUNDED FROM ABOVE

③ BOUNDED FROM BELOW

④ DEPENDS ON a, b

- $x_n = an + b$
- $a = 0$ $\quad x_n = b \ \forall n$. BOUNDED

- $x_n = an + b$

- $a = 0$ $\qquad x_n = b \quad \forall n$ $\quad$ BOUNDED

- $a > 0$ $\qquad x_n = an + b \geq b$ $\quad$ BOUNDED FROM BELOW

- $x_n = an + b$

- $a = 0$ $\qquad x_n = b \quad \forall n$ BOUNDED

- $a > 0$ $\qquad x_n = an + b \geqslant b$ BOUNDED FROM BELOW

NOT BOUNDED FROM ABOVE

- $x_n = an + b$

- $a = 0$ $\quad x_n = b \quad \forall n \quad$ BOUNDED

- $a > 0$ $\quad x_n = an + b \geq b \quad$ BOUNDED FROM BELOW

<u>NOT</u> BOUNDED FROM ABOVE

ARCH. PROPERTY $\Longrightarrow$ $\forall c \in \mathbb{R}, \exists n \in \mathbb{N}$ S.T.

$an + b \geq c + b$

- $x_n = an + b$

- $a = 0$ $\qquad x_n = b \quad \forall n$ BOUNDED

- $a > 0$ $\qquad x_n = an + b \geq b$ BOUNDED FROM BELOW

NOT BOUNDED FROM ABOVE

- $a < 0$ $\qquad$ BOUNDED FROM ABOVE

$an + b \leq b$ $\qquad$ NOT FROM BELOW

EXAMPLE

- $x_n \doteq (-1)^n$

$\Longrightarrow \{x_n \mid n \in \mathbb{N}\} = \{-1, 1\}$ BOUNDED

- $x_n = an + b$ IS

1. BOUNDED
2. BOUNDED FROM ABOVE
3. BOUNDED FROM BELOW
4. DEPENDS ON a, b

EXAMPLE

- $x_n \doteq a q^n$

- $a = 0$ $\qquad x_n = 0 \quad \forall n$ BOUNDED

- $|q| \leq 1$ $\qquad |x_n| = |a q^n| \leq |a| \Rightarrow$

  $\Downarrow$

  $|q^n| = |q|^n \leq 1^n$ $\qquad$ BOUNDED $\forall n \; x_n \in [-a, a]$

- $|q| > 1$ $\qquad$ UNBOUNDED

  [WE'LL SEE IT LATER]

$q > 1 \;, a > 0$

$x_2 = q \cdot x_1 > x_1 = q \cdot x_0 > x_0$

$q = 2 \quad a = 1$

IF $\quad |x| \leq c \Longrightarrow -c \leq x \leq c \qquad x_n = 2^n$

$c \in \mathbb{R}_+$

DEF $\{x_n\}$ IS

INCREASING $\qquad$ IF $\quad x_{n+1} \geq x_n \quad \forall n$

$x_0 \leq x_1 \leq x_2 \leq x_3$

STRICTLY INCREASING $\quad x_{n+1} > x_n \quad \forall n$

DECREASING $\quad x_{n+1} \leq x_n \quad \forall n$

STRICTLY DECREASING $\quad x_{n+1} < x_n \quad \forall n$

EXAMPLE • $x_n = c$

INCREAS + DECREAS.

BUT NOT STRICTLY

$c \geq c$

$\leq$

$c \ngtr c$

$c \nless c$

EXAMPLE :

$\cancel{x_n = 5}$

$$x_n = 5 - \frac{1}{n} \quad n > 0$$

STRICTLY INCREASING

$$x_n = 5 - \frac{1}{n} < 5 - \frac{1}{n+1} = x_{n+1} \quad \forall n > 0$$

$$\frac{1}{n+1} < \frac{1}{n} \quad \forall n > 0$$

$$\Updownarrow$$

$$-\frac{1}{n+1} > -\frac{1}{n}$$

EXAMPLE

- $x_n = 4n + 6$

$x_{n+1} = x_n + 4 > x_n$ STR. INCR.

$x_n = an + b$, $a < 0$ STR. DECR.

- $x_n = 13 \cdot (-1)^n$

$x_0 = 13$

$x_1 = 13 \cdot (-1) < 0$

$x_2 = 13 \cdot (1) > 0$

$x_{even} > 0$

$x_{odd} < 0$

DEF $\{x_n\}$ IS

MONOTONE IF INCREASING

DECREASING

STRICTLY MONOTONE IF STRICTLY INCREASING

STRICTLY DECREASING

EXAMPLE

- $x_n \doteq a q^n$

- $a = 0$ $\quad x_n = 0 \quad \forall n$ BOUNDED

- $|q| \leq 1$ $\quad |x_n| = |a q^n| \leq |a|$

  BOUNDED

- $|q| > 1$ UNBOUNDED

  [WE'LL SEE IT LATER]

BERNOULLI'S INEQUALITY

PROP $q \in \mathbb{R}$, $q > 1$. THEN $\forall n \in \mathbb{N}$

$$q^n \geq 1 + n(q-1)$$

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PROOF $|a q^n| = |a| |q^n| = |a| \cdot |q|^n$

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$$\cancel{\Longleftarrow} \; |a||q|^n \geq |a|\underbrace{\left(1 + n(|q| - 1)\right)}_{\text{UNBOUNDED}}$$

SET OF REALS OF THIS FORM.

- $x_n = an + b$

- $a > 0$ $\qquad x_n = an + b \geq b$ BOUNDED FROM BELOW

NOT BOUNDED FROM ABOVE

PROP $q \in \mathbb{R}$, $q > 1$.

THEN $q^n \geq 1 + n(q-1)$

$\Downarrow$

FOR $|q|>1$, $x_n = a q^n$ NOT BOUNDED

PROOF $|aq^n| = |a||q^n| = |a||q|^n$

$$|q|^n \geq \underbrace{1 + n(|q|-1)}_{\text{UNBOUNDED}} \geq C \quad \text{FOR } n \gg 0 \quad [\forall C > 0]$$

OKAY EVEN WHEN $q<0$ AS $q^{2n} > 0$.

PROP $q \in \mathbb{R}$, $q > 1$. THEN

$$q^n \geq 1 + n(q-1)$$

Q. HOW DO WE PROVE IT?

PROP $q \in \mathbb{R}$, $q > 1$. THEN

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Q. HOW DO WE PROVE IT?

NEED NEW TOOL :

INDUCTION

METHOD FOR PROOFS [USING STRUCTURE OF $\mathbb{N}$]

INDUCTION

INDUCTION

$P(K)$ = PROPERTY [STATEMENT WE WANT TO PROVE IS TRUE FOR $K \in \mathbb{N}$]

$\underset{\mathbb{N}}{\overset{\cap}{K}}$

INDUCTION

$P(K)$ = PROPERTY [STATEMENT WE WANT TO PROVE IS TRUE FOR $K \in \mathbb{N}$]

$\mathbb{N}$

EXAMPLE

- $P(K)$: "$(K+1)^2 \geq 2K$"

$\mathbb{N}$ $\qquad$ $K \in \mathbb{N}$

INDUCTION

$P(K) =$ PROPERTY [STATEMENT WE WANT TO PROVE IS TRUE FOR $K \in \mathbb{N}$]

$\mathbb{N}$

EXAMPLE

- $P(K)$: "$(K+1)^2 \geq 2K$"
- $P(K)$: "$1+2+3+\dots+K = \dfrac{(K+1)\cdot K}{2}$"

INDUCTION

$P(K) =$ PROPERTY [STATEMENT WE WANT TO PROVE IS TRUE FOR $K \in \mathbb{N}$]

$K \in \mathbb{N}$

EXAMPLE

- $P(K)$: "$(K+1)^2 \geq 2K$"

- $P(K)$: "$1+2+3+\dots+K = \frac{(K+1)\cdot K}{2}$"

$P(1)$: "$1 = 1$";

INDUCTION

$P(K)$ = PROPERTY [STATEMENT WE WANT TO PROVE IS TRUE FOR $K \in \mathbb{N}$]

$\mathbb{N}$

EXAMPLE

- $P(K)$: "$(K+1)^2 \geq 2K$"

- $P(K)$: "$1+2+3+\ldots+K = \dfrac{(K+1)\cdot K}{2}$"

$P(1)$: "$1=1$"; $P(2)$: "$1+2 = \dfrac{2\cdot 3}{2}$"

INDUCTION

$P(K)$ = PROPERTY [STATEMENT WE WANT TO PROVE IS TRUE FOR $K \in \mathbb{N}$]

$K \in \mathbb{N}$

EXAMPLE

- $P(K)$: "$(K+1)^2 \geq 2K$"
- $P(K)$: "$1+2+3+\ldots+K = \dfrac{(K+1)\cdot K}{2}$"

$P(1)$: "$1=1$"; $P(2)$: "$1+2 = \dfrac{2\cdot 3}{2}$"; $P(50)$: $1+2+3+\ldots+50 = 25\cdot 51 =$

INDUCTION

$P(K)$ = PROPERTY [STATEMENT WE WANT TO PROVE IS TRUE FOR $K \in \mathbb{N}$]

$\cap$

$\mathbb{N}$

HOW TO PROVE $P(K)$ IS TRUE ?

$\forall K \in \mathbb{N}$

2-STEP RECIPE

INDUCTION

$P(K)$ = PROPERTY [STATEMENT WE WANT TO PROVE IS TRUE FOR $K \in \mathbb{N}$]

$\cap$

$\mathbb{N}$

HOW TO PROVE $P(K)$ IS TRUE $\forall K \in \mathbb{N}$?

2-STEP RECIPE

① STARTING POINT : $P(0)$ IS TRUE

INDUCTION

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HOW TO PROVE $P(K)$ IS TRUE $\forall K \in \mathbb{N}$?

2 - STEP RECIPE

1. STARTING POINT : $P(0)$ IS TRUE
2. INDUCTION STEP :

$P(n)$ TRUE $\Longrightarrow$ $P(n+1)$ TRUE

2-STEP RECIPE

① STARTING POINT : $P(0)$ IS TRUE

② INDUCTION STEP :

$P(n)$ TRUE $\Longrightarrow$ $P(n+1)$ TRUE

EXAMPLE $\quad P(K): (K+1)^2 \geq K+1$

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① STARTING POINT : $P(0)$ IS TRUE

② INDUCTION STEP :

$P(n)$ TRUE $\Longrightarrow$ $P(n+1)$ TRUE

EXAMPLE $\quad P(K) : (K+1)^2 \geq K+1$

① $P(0) : (0+1)^2 \geq 0+1$

2 - STEP RECIPE

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② INDUCTION STEP :

$P(n)$ TRUE $\Longrightarrow$ $P(n+1)$ TRUE

EXAMPLE $\quad P(K) : (K+1)^2 \geq K+1$

① $P(0) : (0+1)^2 \geq 0+1$ TRUE

② ASSUME $(n+1)^2 \geq (n+1)$ TRUE

2 - STEP RECIPE

① STARTING POINT : $P(0)$ IS TRUE

② INDUCTION STEP :

$$P(n) \text{ TRUE } \Longrightarrow P(n+1) \text{ TRUE}$$

<u>EXAMPLE</u> $P(K): (K+1)^2 \geq K+1$

① $P(0): (0+1)^2 \geq 0+1$ TRUE

② ASSUME $(n+1)^2 \geq (n+1)$ TRUE

NEED TO PROVE :

2-STEP RECIPE

① STARTING POINT : $P(0)$ IS TRUE

② INDUCTION STEP :

$P(n)$ TRUE $\Longrightarrow$ $P(n+1)$ TRUE

EXAMPLE $P(K): (K+1)^2 \geq K+1$

① $P(0): (0+1)^2 \geq 0+1$ TRUE

② ASSUME $(n+1)^2 \geq (n+1)$ TRUE

NEED TO PROVE : $(\qquad +1)^2 \geq \qquad +1$

2-STEP RECIPE

① STARTING POINT : $P(0)$ IS TRUE

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EXAMPLE $\quad P(K): (K+1)^2 \geq K+1$

② ASSUME $(n+1)^2 \geq (n+1)$ TRUE

NEED TO PROVE : $(n+1+1)^2 \geq n+2$

$$(n+1)^2 + 1 + 2(n+1) \geq n+1+1$$

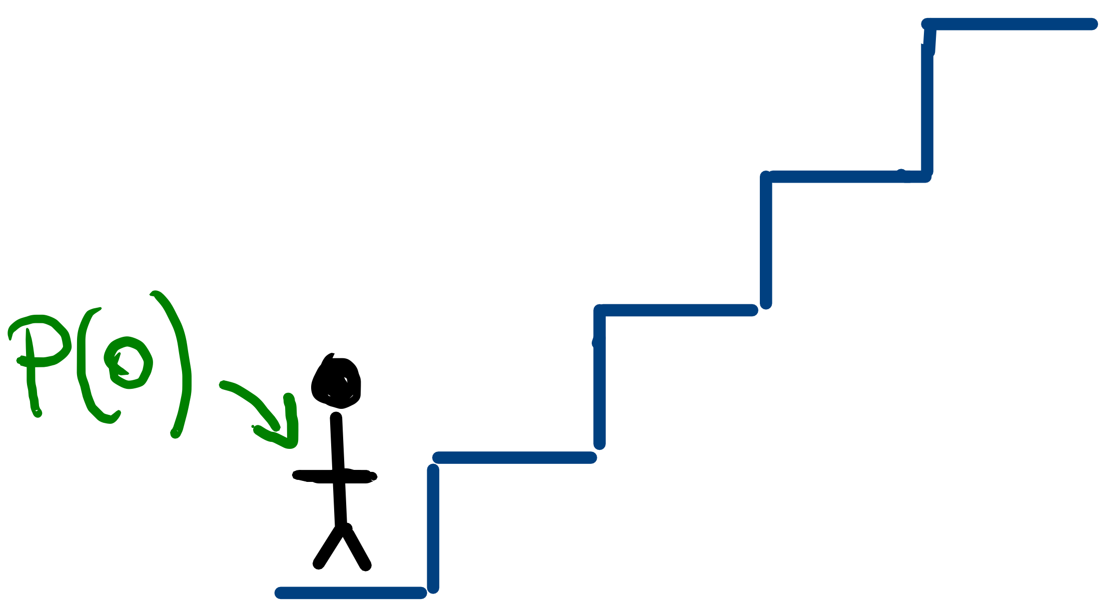
2-STEP RECIPE

① STARTING POINT : $P(0)$ IS TRUE

② INDUCTION STEP :

$P(n)$ TRUE $\Longrightarrow$ $P(n+1)$ TRUE

"LIKE A STAIR"



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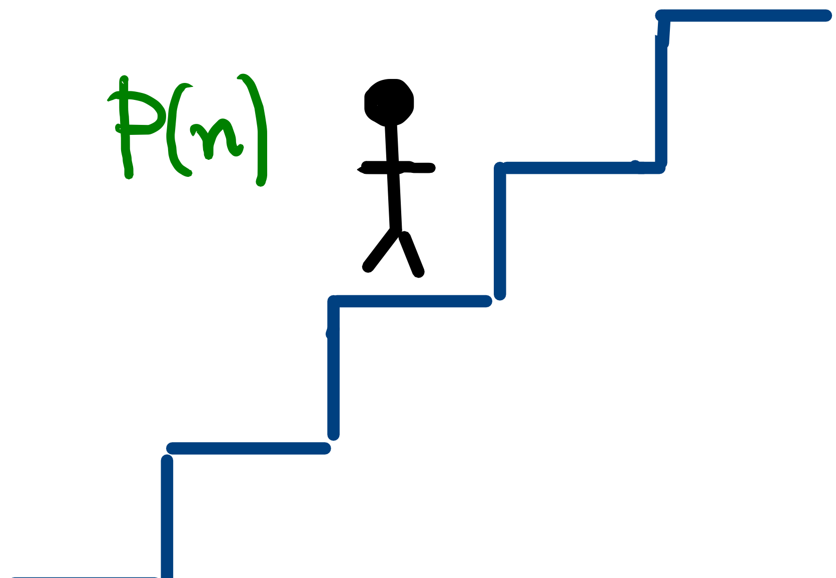

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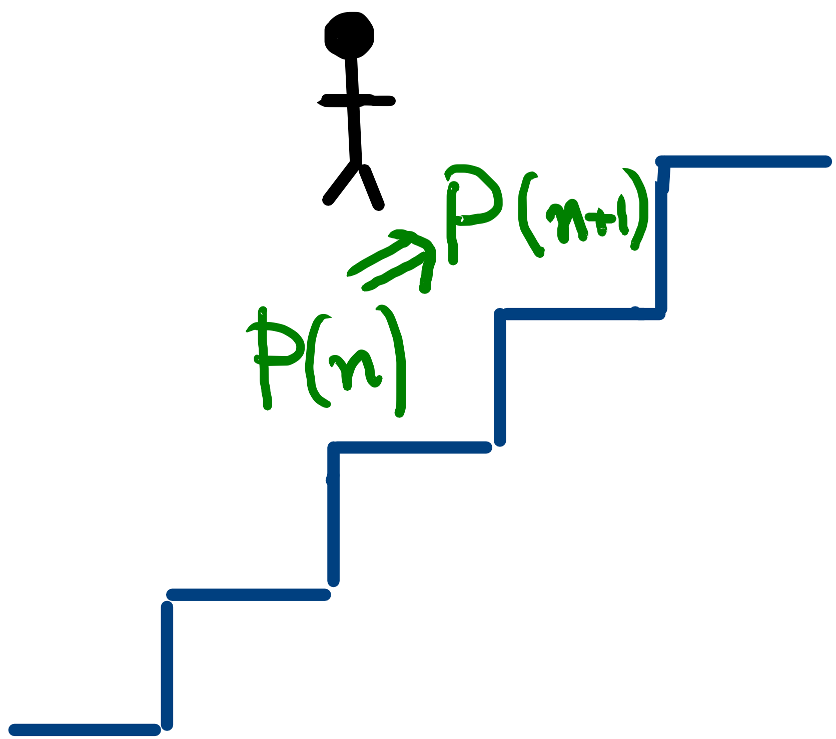

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