ANALYSIS 1

(ENGLISH)

LECTURE 4

LAST TIME:

- MORE ON MAX/MIN & INF/SUP
- ARCHIMEDES' PRINCIPLE
- INTEGRAL PART OF REAL #'s
- $\mathbb{Q}$ IS DENSE IN $\mathbb{R}$
- ABSOLUTE VALUE FOR REAL #'s

TODAY:

- TRIANGLE INEQUALITY
- EXTENDED REAL LINE
- COMPLEX NUMBERS
  - DEFINITION
  - OPERATIONS
  - REPRESENTING
  - SOLVING EQUATIONS

PROP $\sqrt{3}$ IS A REAL NUMBER.

PROOF $S \doteq \{ x \in \mathbb{R} \mid x^2 \leq 3 \}$

PROP $\sqrt{3}$ IS A REAL NUMBER.

PROOF $S \doteq \{ x \in \mathbb{R} \mid x^2 \leq 3 \} \ni \pm 1$

- $S \neq \emptyset$ $\quad 0 \in S$
- S IS BOUNDED $\quad S \subseteq [-3, 3]$

$0 < a < b \implies b^2 > a^2$

$$\inf S \overset{0}{\vee}, \ \sup S \overset{0}{\wedge} \in \mathbb{R}$$

TO SHOW: $(\sup S)^2 = 3$

$$(\sup S)^2 = \begin{cases} > 3 \\ = 3 \\ < 3 \end{cases}$$

ASSUME THIS

$(\sup S)^2 - 3 > 0$

WTS: $\exists\, r \in \mathbb{R}$ $r < \sup S$ $\wedge$ r IS U.B. FOR S

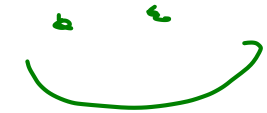
$$\left(\sup S - \frac{1}{n}\right)^2 = (\sup S)^2 + \frac{1}{n^2} - 2\,\frac{\sup S}{n}$$

$n \in \mathbb{N}^*$

$$\vee$$

$$(\sup S)^2 - 2\,\frac{\sup S}{n}$$

IF I CAN SHOW

$$\left(\sup S - \frac{1}{n}\right)^2 > 3 \Longrightarrow$$ 

$$> x^2$$

$$\wedge$$

$$\sup S$$

$$\sup S - \frac{1}{n} \geq x \in S$$

$$\left(\sup S - \frac{1}{n}\right)^2 = \underline{(\sup S)^2 + \frac{1}{n^2} - 2\frac{\sup S}{n}} \qquad n \in \mathbb{N}^*$$

$$\left(\sup S - \frac{1}{n}\right)^2 > 3 \qquad (\sup S)^2 + \frac{1}{n^2} - 2\frac{\sup S}{n} > (\sup S)^2 - 2\frac{\sup S}{n}$$

WE NEED : $\frac{2\sup S}{n} < \underline{(\sup S)^2 - 3}$

TO FIND $n \in \mathbb{N}^*$

$$\Downarrow$$

$$(\sup S)^2 - \frac{2\sup S}{n} > (\sup S)^2 - \left((\sup S)^2 - 3\right) = 3$$

$\exists\, n \in \mathbb{N}^*$ s.t. $n > \frac{2\sup S}{(\sup S)^2 - 3} > 0$

BY THE ARCH. PROP. ☺ $\blacksquare$

$$\left(\sup S - \frac{1}{n}\right)^2 = (\sup S)^2 + \frac{1}{n^2} - 2\,\frac{\sup S}{n}$$

$$\vee$$

$$(\sup S)^2 - 2\,\frac{\sup S}{n}$$

WE NEED:
$$\frac{2\sup S}{n} < (\sup S)^2 - 3$$

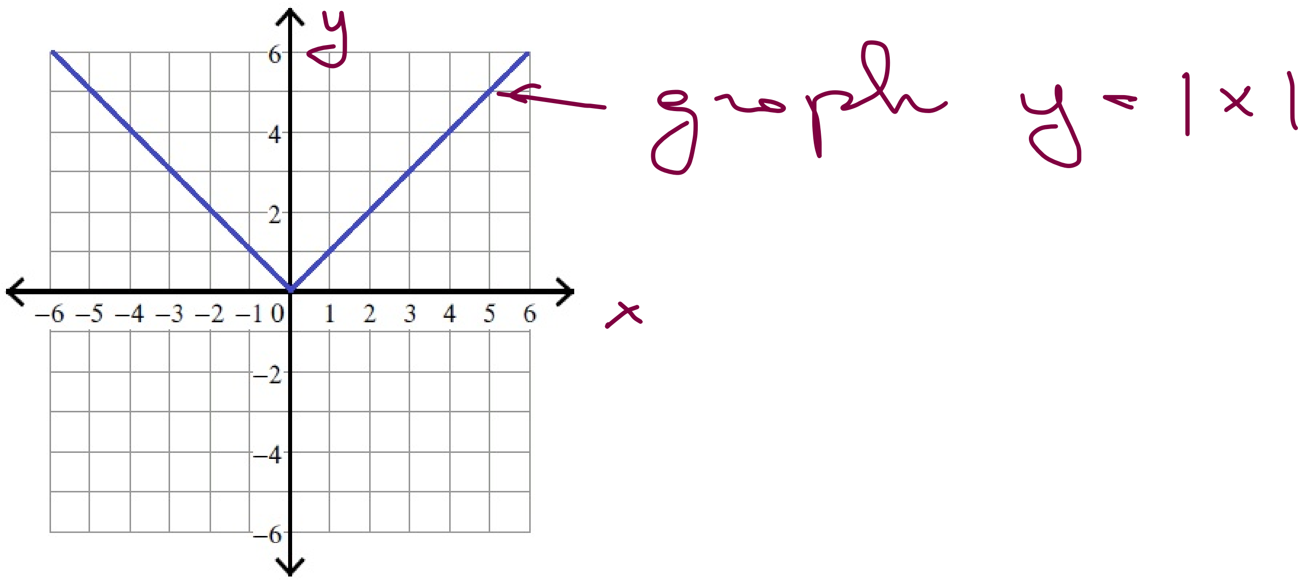
$$\Updownarrow$$

$$\frac{2\sup S}{(\sup S)^2 - 3} < n$$

ABSOLUTE VALUE

DEF $x \in \mathbb{R}$

$$|x| = \begin{cases} x & \text{IF } x \geq 0 \\ -x & \text{IF } x < 0 \end{cases}$$



$$|x| = \begin{cases} x & \text{IF } x \geq 0 \\ -x & \text{IF } x < 0 \end{cases}$$

PROPERTIES:

- $|\cdot|$ RESPECTS MULTIPLICATION

$$\forall x, y \in \mathbb{R}, \quad |x \cdot y| = |x| \cdot |y|$$

$$\forall x, y \in \mathbb{R}, \ \underline{\underline{y \neq 0}}, \quad \left| \frac{x}{y} \right| = \frac{|x|}{|y|}$$

$$|x| = \begin{cases} x & \text{IF } x \geq 0 \\ -x & \text{IF } x < 0 \end{cases}$$

PROPERTIES:

- $|\cdot|$ RESPECTS MULTIPLICATION
- $|\cdot|$ DOES NOT RESPECT ADDITION

$$|(-3)+5| \qquad\qquad |-3| + |5|$$

$$|2| = 2 \neq 8 = 3 + 5$$

$- \ |\cdot|$ DOES NOT RESPECT ADDITION

Q: CAN WE SAY ANYTHING AT ALL?

$$a \geq 0 \ , \ b \geq 0 \implies a+b = |a| + |b|$$

$$a+b = |a+b|$$

$$a \geq 0 \ , \ b < 0 \implies 0 < a + b = |a \pm b| \ \wedge \ |a| + |b|$$

ASSUME $|a| > |b|$

- $|\cdot|$ DOES NOT RESPECT ADDITION

Q: CAN WE SAY ANYTHING AT ALL?

TRIANGLE INEQUALITY:

PROP $a, b \in \mathbb{R}$.

$$|a+b| \leq |a| + |b|$$

TRIANGLE INEQUALITY:

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PROOF

TRIANGLE INEQUALITY :

PROP $a, b \in \mathbb{R}$.

$$|a+b| \leq |a| + |b|$$

PROOF CASE 1: $a+b \geq 0$

TRIANGLE INEQUALITY :

PROP $a, b \in \mathbb{R}$.

$$|a+b| \leq |a| + |b|$$

PROOF CASE 1 : $a+b \geq 0$

$$\Longrightarrow |a+b| = a+b$$

$$\mathbb{R} \ni x: \quad -|x| \leq x \leq |x|$$

TRIANGLE INEQUALITY:

PROP $a, b \in \mathbb{R}$.

$$|a+b| \leq |a| + |b|$$

PROOF CASE 1: $a+b \geq 0$

$$\implies |a+b| = a+b \leq |a| + |b|$$

$$- \; \forall x \in \mathbb{R} \quad -|x| \leq \boxed{x \leq |x|}$$

TRIANGLE INEQUALITY :

PROP $a, b \in \mathbb{R}$.

$$|a+b| \leq |a| + |b|$$

PROOF CASE 1 : $a+b \geq 0$

$$\Longrightarrow |a+b| = a+b \leq |a|+|b|$$

CASE 2 : $a+b<0$

TRIANGLE INEQUALITY :

PROP $a, b \in \mathbb{R}$.

$$|a+b| \leq |a| + |b|$$

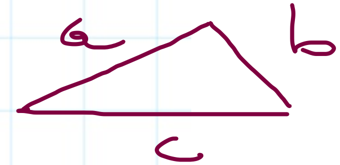
PROOF CASE 1 : $a+b \geq 0$:)

$$\Longrightarrow |a+b| = a+b \leq |a|+|b|$$

CASE 2 : $a+b < 0$

$$\Longrightarrow |a+b| = -a-b$$

TRIANGLE INEQUALITY :



$a + b \geq c$

$b + c \geq a$

$c + a \geq b$

PROP $a, b \in \mathbb{R}$.

$$|a+b| \leq |a| + |b|$$

PROOF CASE 2 : $a - b < 0$

$$\Longrightarrow |a+b| = -a-b \leq |a|+|b|$$

- $\forall x \in \mathbb{R}$ $\boxed{-|x| \leq x} \leq |x|$

$\Longleftrightarrow |x| \geq -x$

$\blacksquare$

TRIANGLE INEQUALITY :

PROP $a, b \in \mathbb{R}$.

$$|a+b| \leq |a| + |b|$$

FANCIER VERSION :

$$\big||a| - |b|\big| \leq |a \pm b| \leq |a| + |b|$$

EXERCISE IN WEEK 3

EXTENDED REAL LINE

$$\overline{\mathbb{R}} \doteq \mathbb{R} \cup \{+\infty, -\infty\}$$

EXTENDED REAL LINE

$$\overline{\mathbb{R}} \doteq \mathbb{R} \cup \{+\infty, -\infty\}$$

THESE ARE NOT NUMBERS

OPERATIONS ARE NOT DEFINED

EXTENDED REAL LINE

$$\mathbb{R} \cup \{+\infty, -\infty\}$$

THESE ARE NOT NUMBERS

OPERATIONS ARE NOT DEFINED

$$\text{"}+\infty\text{"} + \text{"}-\infty\text{"} = ?$$

EXTENDED REAL LINE

$$\overline{\mathbb{R}} \doteq \mathbb{R} \cup \{+\infty, -\infty\}$$

USED FOR

EXTENDED REAL LINE

$$\mathbb{R} \cup \{+\infty, -\infty\}$$

USED FOR

$$[a, +\infty[\doteq \{ x \in \mathbb{R} \mid x \geq a \}$$

$$]a, +\infty[$$

EXTENDED REAL LINE

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$$

USED FOR

$$[a, +\infty[\doteq \{ x \in \mathbb{R} \mid x \geq a \}$$

$$]-\infty, b[\doteq \{ x \in \mathbb{R} \mid x < b \}$$

COMPLEX NUMBERS

ANALYSIS $\longrightarrow$ $f : \mathbb{R} \longrightarrow \mathbb{R}$

COMPLEX NUMBERS

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FUNCTION

TAKES REAL #'s

COMPLEX NUMBERS

ANALYSIS $\longrightarrow$ $f : \mathbb{R} \to \mathbb{R}$

FUNCTION

TAKES REAL #'s

RETURNS REAL #'s

EXAMPLE

COMPLEX NUMBERS

ANALYSIS $\longrightarrow$ $f : \mathbb{R} \to \mathbb{R}$

FUNCTION

TAKES REAL #'s

RETURNS REAL #'s

EXAMPLE

time

$P(t) =$

COMPLEX NUMBERS

ANALYSIS $\longrightarrow$ $f : \mathbb{R} \to \mathbb{R}$

FUNCTION

TAKES REAL #'s

RETURNS REAL #'s

EXAMPLE

$p(t)$ = POSITION AT TIME t

for some particle

pointwise particle

COMPLEX NUMBERS

ANALYSIS $\longrightarrow$ $f : \mathbb{R} \rightarrow \mathbb{R}$

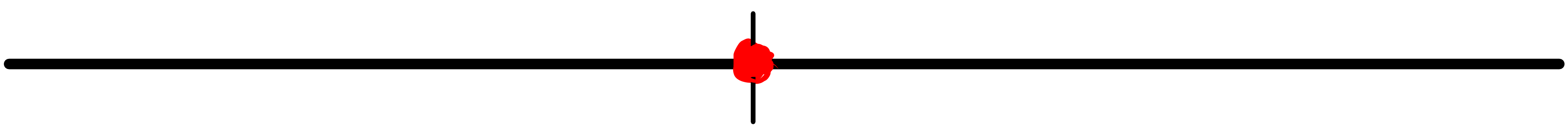
FUNCTION

TAKES REAL #'s

RETURNS REAL #'s

EXAMPLE

$p(t)$ = POSITION AT TIME t



0 AT $t=0$

COMPLEX NUMBERS

ANALYSIS $\longrightarrow$ $f : \mathbb{R} \to \mathbb{R}$

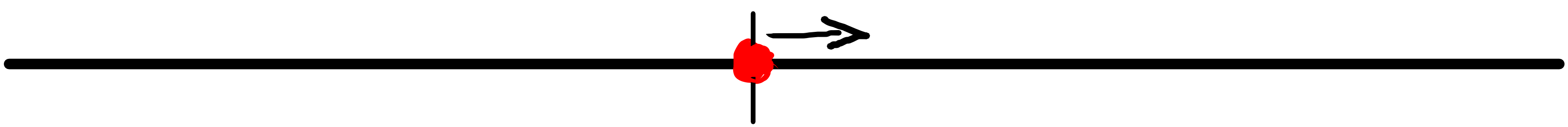
FUNCTION

TAKES REAL #'s

RETURNS REAL #'s

EXAMPLE

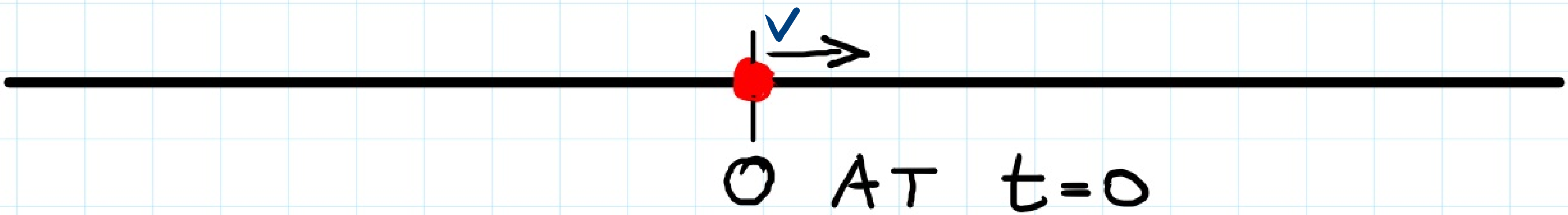
$p(t)$ = POSITION AT TIME t



0 AT $t=0$

EXAMPLE

$p(t)$ = POSITION AT TIME t

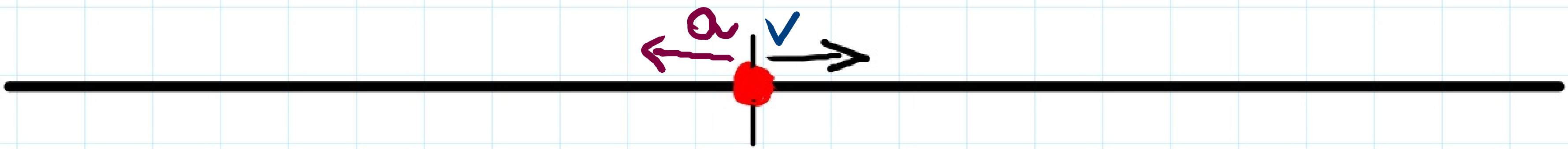


v IS VELOCITY AT $t=0$

- NO FORCE IS APPLIED $\Rightarrow p(t) = v \cdot t$

EXAMPLE

$p(t)$ = POSITION AT TIME t



0 AT $t=0$

v IS VELOCITY AT $t=0$

- NO FORCE IS APPLIED $\Rightarrow p(t) = v \cdot t$

APPLY CONSTANT ACCELERATION $-a$

$$p(t) = -\frac{1}{2} a t^2 + vt$$

FIXED VALUES

QUESTION: HOW TO FIND SOLUTIONS

UNKNOWNS

OF $-\frac{1}{2}ax^2 + vx = c$

DOES THE PARTICLE GO THRU $c \in \mathbb{R}$? IF SO, WHEN?

FIXED

$p(t) = c$

QUESTION: HOW TO FIND SOLUTIONS

OF $\quad ax^2 - 2vx + 2c = 0$

unknown

Known

$$\text{SOL'S} = \begin{cases} \dfrac{2v + \sqrt{4v^2 - 8ac}}{2a} \\[2ex] \dfrac{2v - \sqrt{4v^2 - 8ac}}{2a} \end{cases}$$

WE NEED ≥ 0

ISSUE: $\cancel{\text{No}}\ \nexists \sqrt{c}$

$\uparrow$

$\mathbb{R}$

WHEN $c < 0$

COMPLEX NUMBERS

$i^2 = -1$ $\longrightarrow$ SOL'S TO EQN

$x^2 = -1$

SOL'S $= \{ i , -i \}$

COMPLEX NUMBERS

$$i^2 = -1 \qquad [\, i \notin \mathbb{R} \,]$$

COMPLEX NUMBERS

$$i^2 = -1 \qquad [\, i \notin \mathbb{R} \,]$$

COMPLEX NUMBERS: $\underset{\substack{\uparrow \\ \mathbb{R}}}{a} + \underset{\substack{\uparrow \\ \mathbb{R}}}{b}\, i$

COMPLEX NUMBERS

$$i^2 = -1 \qquad [\, i \notin \mathbb{R} \,]$$

COMPLEX NUMBER $\qquad a + bi$

REAL #'s

SET OF COMPLEX #'s

$$\mathbb{C} \supseteq \mathbb{R} = \{ \underset{\underset{a+bi}{\shortparallel}}{z} \in \mathbb{C} \mid b = 0 \}$$

COMPLEX NUMBERS

$$i^2 = -1 \qquad [\, i \notin \mathbb{R} \,]$$

COMPLEX NUMBER

real part — imaginary part

$$a + bi$$

← imaginary unit

REAL #'S

$$\mathbb{C} \supseteq \mathbb{R} = \{ z = a + bi \in \mathbb{C} \mid b = 0 \}$$

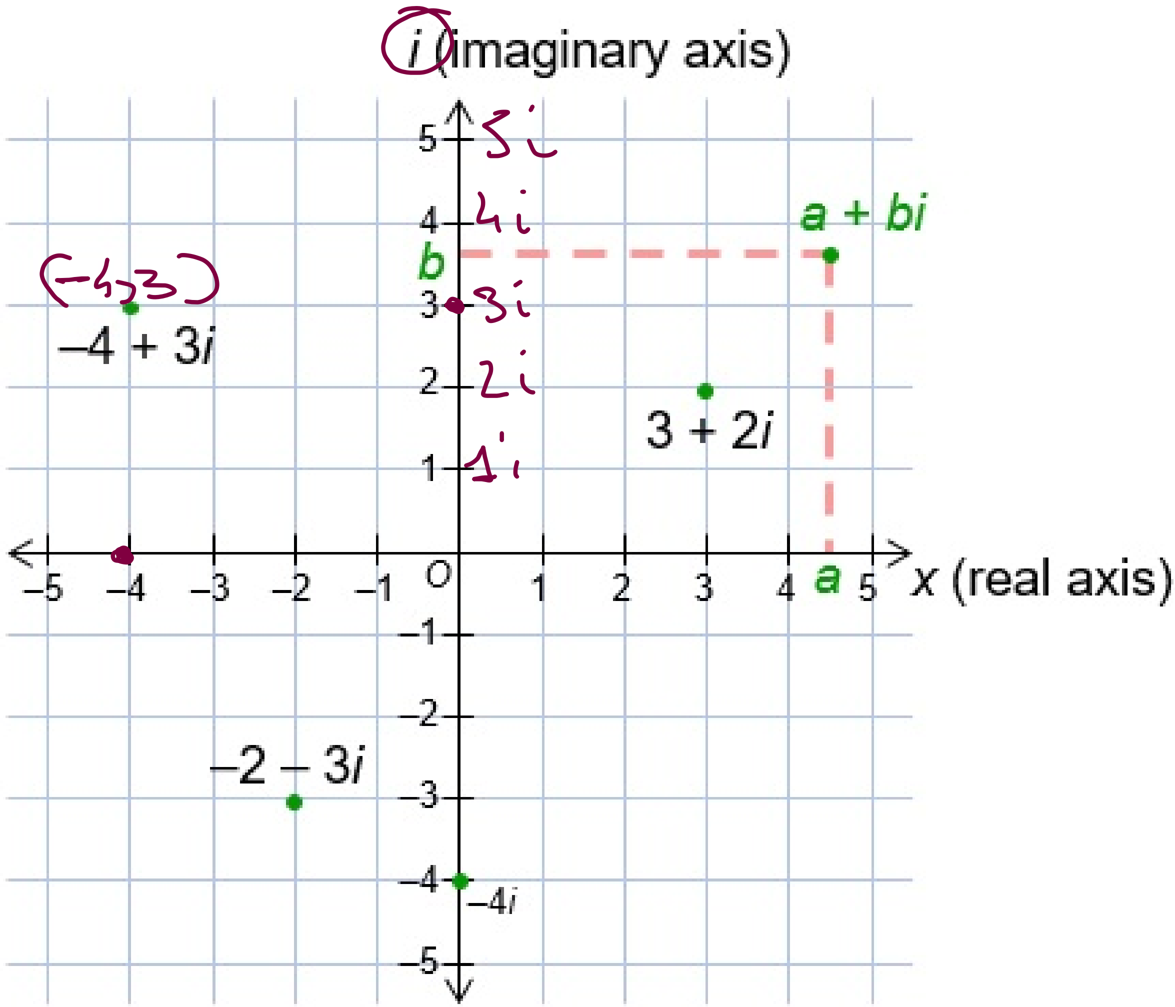
EXAMPLE $z = 1 + \sqrt{2}\, i$, $-4i$, $3\sqrt{3}$

$Re(z) = 1$ $\quad$ $Re = 0$ $\quad$ $Re = 3\sqrt{3}$

$Im(z) = \sqrt{2}$ $\quad$ $Im = -4$ $\quad$ $Im = 0$

COMPLEX PLANE

GRAPHIC REPRESENTATION OF COMPLEX #'S



OPERATIONS:

- $\underline{\underline{(4+i)}} + (4+3i) =$

$$\shortparallel$$

$$(4+4) + (1+3)\cdot i$$

$$\shortparallel$$

$$8 + 4\cdot i$$

$$Re(z_1 + z_2) \qquad Im(z_1 + z_2)$$

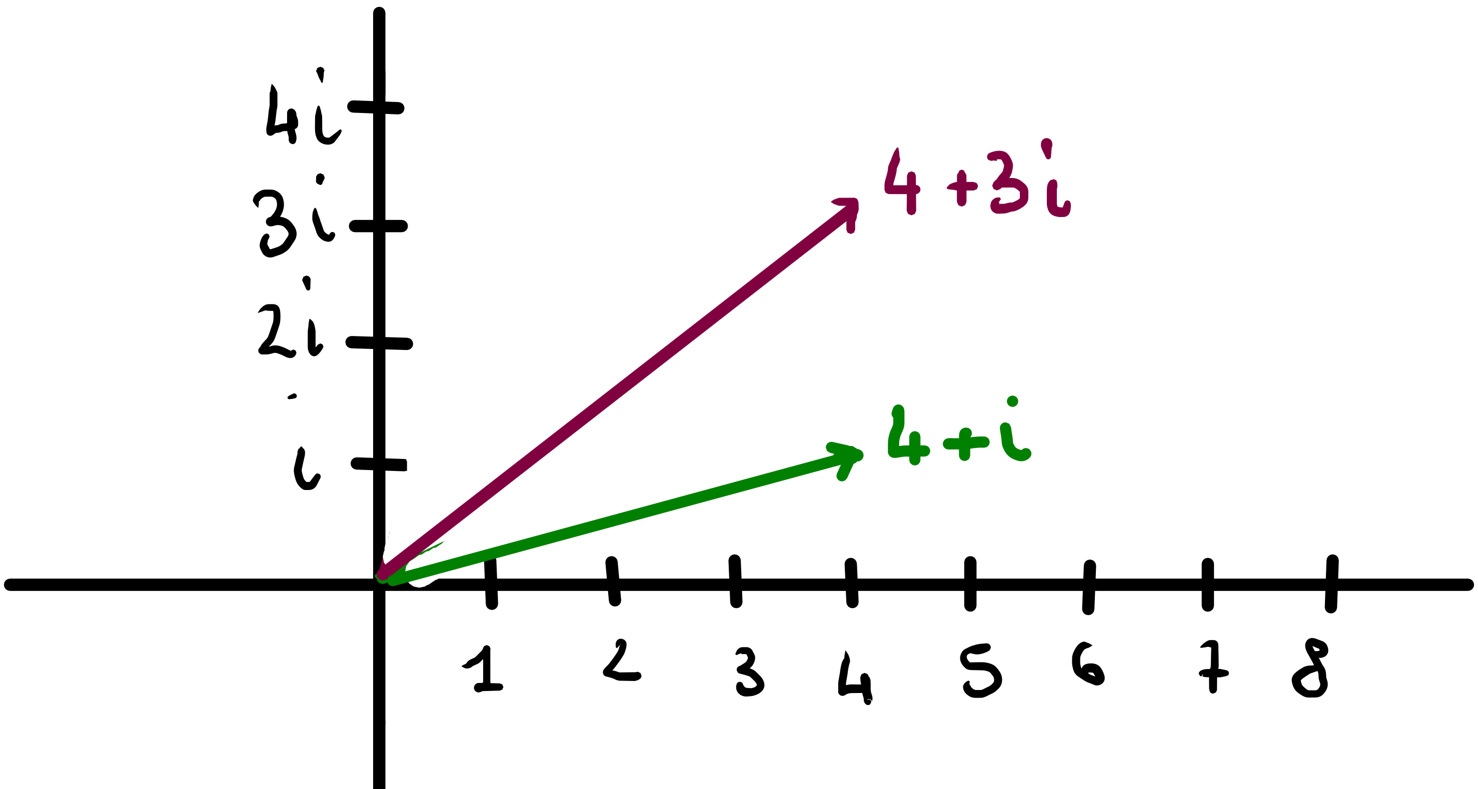
$$\shortparallel \qquad\qquad\qquad \shortparallel$$

$$Re(z_1) + Re(z_2) \qquad Im(z_1 + z_2)$$

OPERATIONS:

- $(4+i)+(4+3i)$

$$= (4+4) + (1+3)i = 8+4i$$

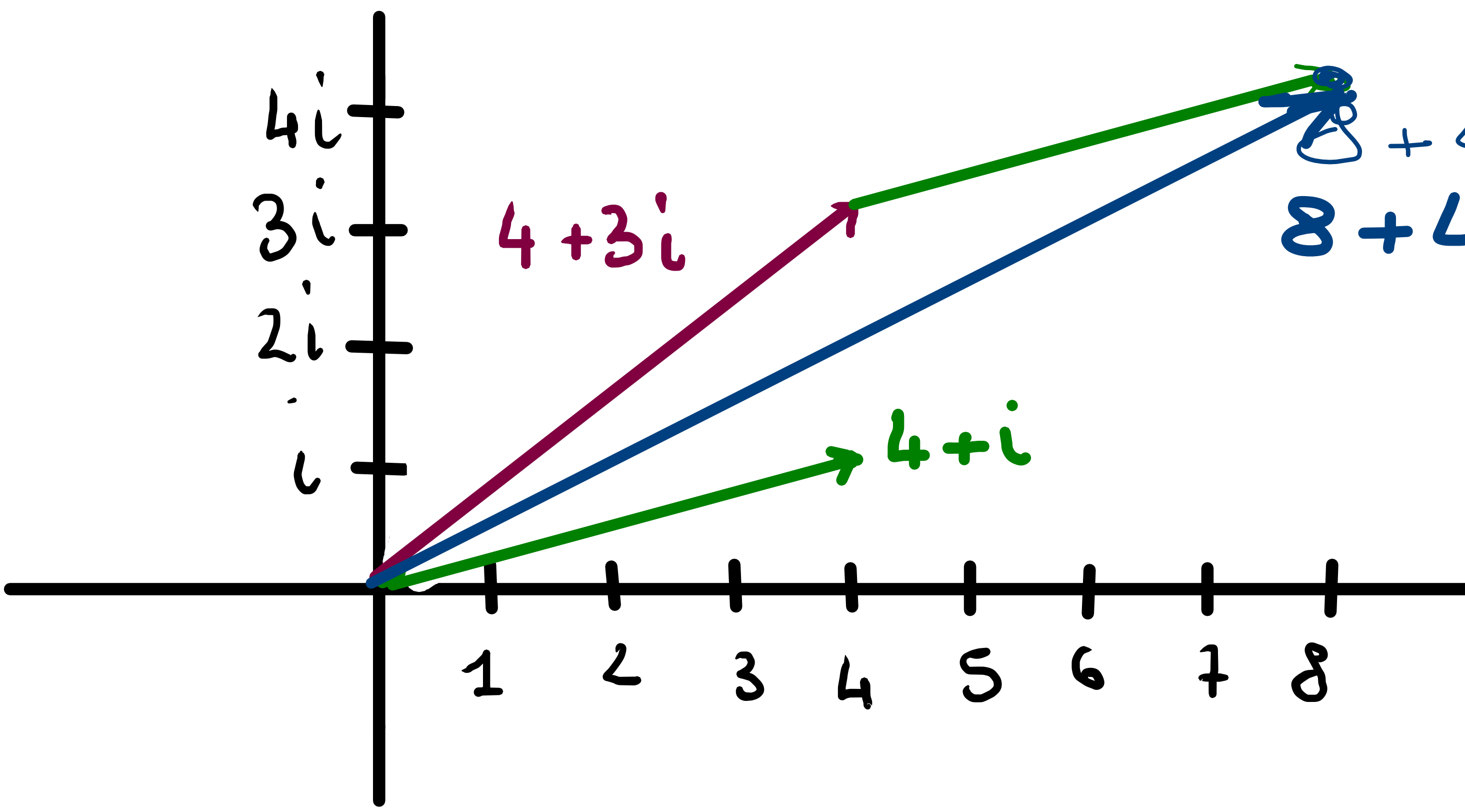


OPERATIONS:

- $(4+i) + (4+3i)$

$$\| $$

$$(4+4) + (1+3)i = 8+4i$$



OPERATIONS:

- $(4+i)\cdot(4+3i)$ $\qquad i^2=-1$

$$=$$

$$16+12i+4i+3i^2$$

$$=$$

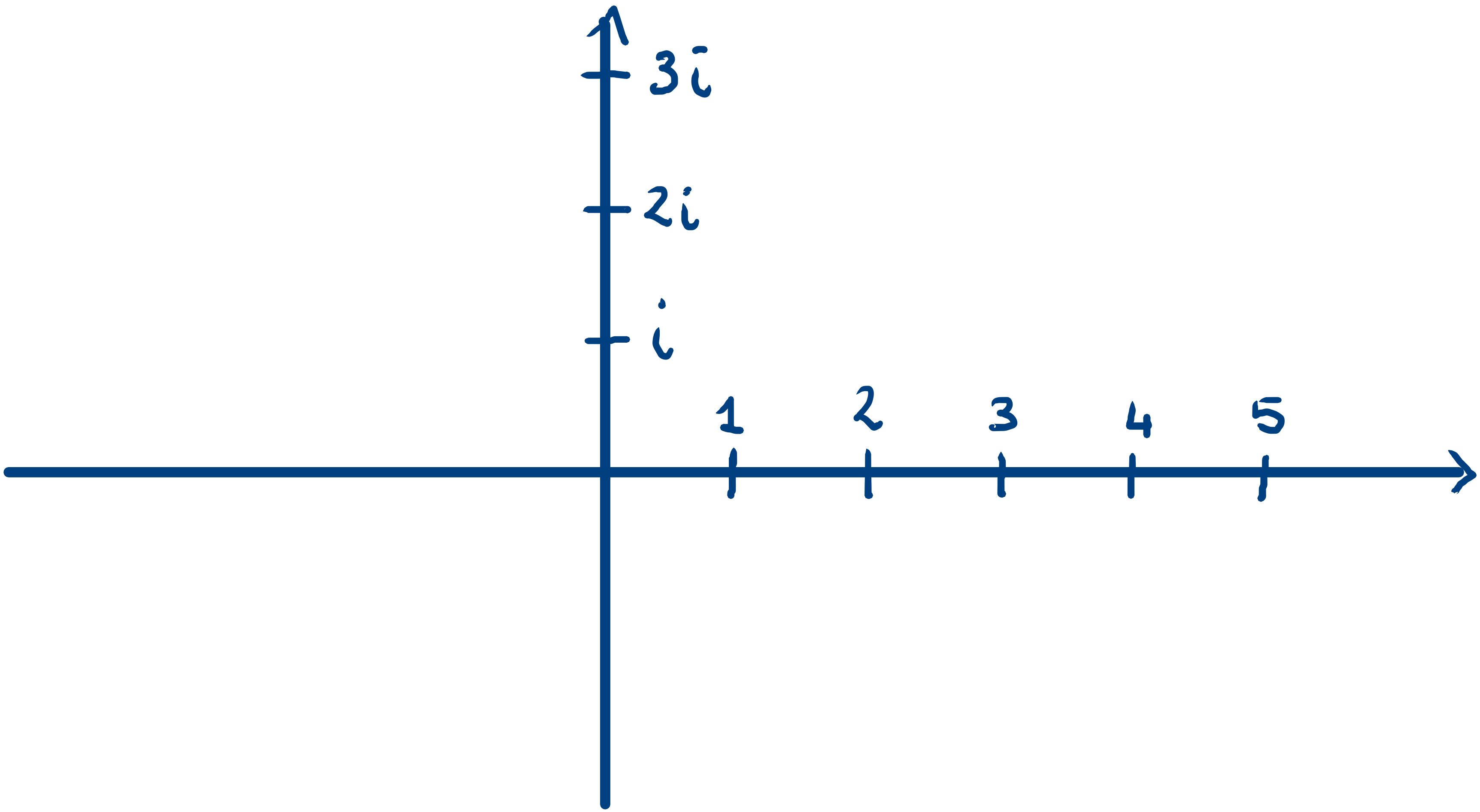
$$(16-3)+(12+4)\cdot i$$

$$=$$

$$13+16i$$

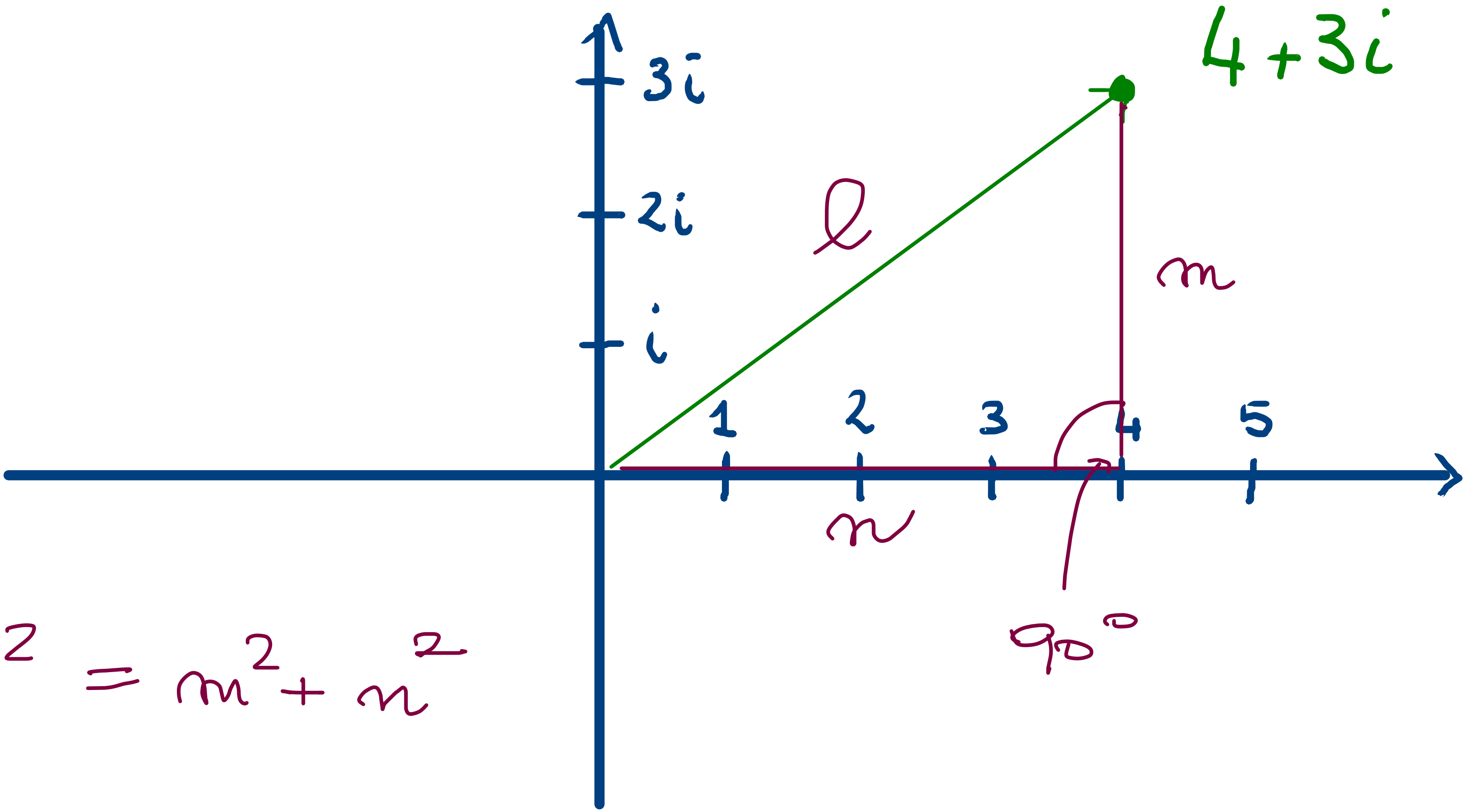
MODULUS (LENGTH)

PYTHAGORA'S THM FOR COMPLEX #'S



MODULUS (LENGTH)

PYTHAGORA'S THM FOR COMPLEX #'S

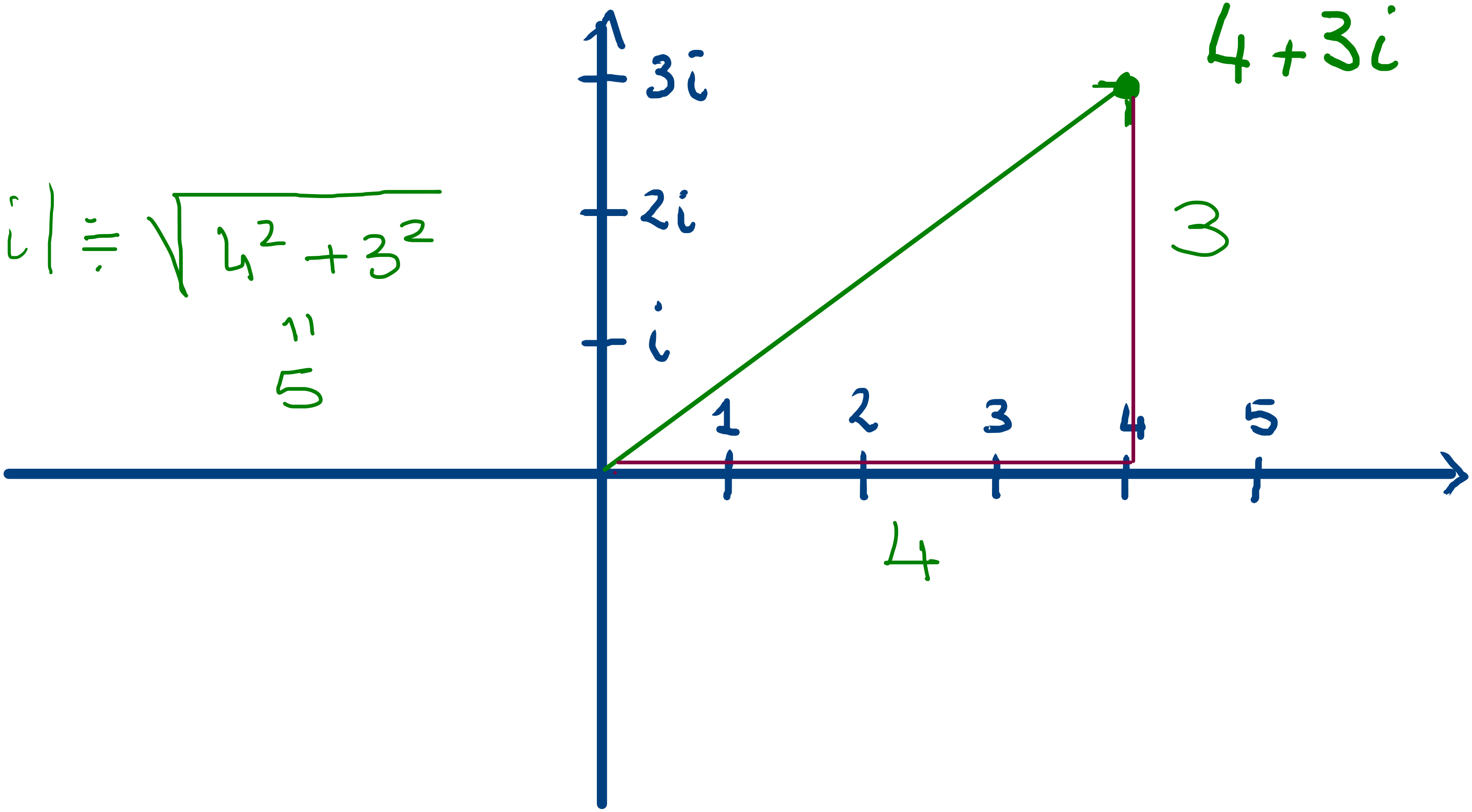


$$l^2 = m^2 + n^2$$

MODULUS (LENGTH)

PYTHAGORA'S THM FOR COMPLEX #'S

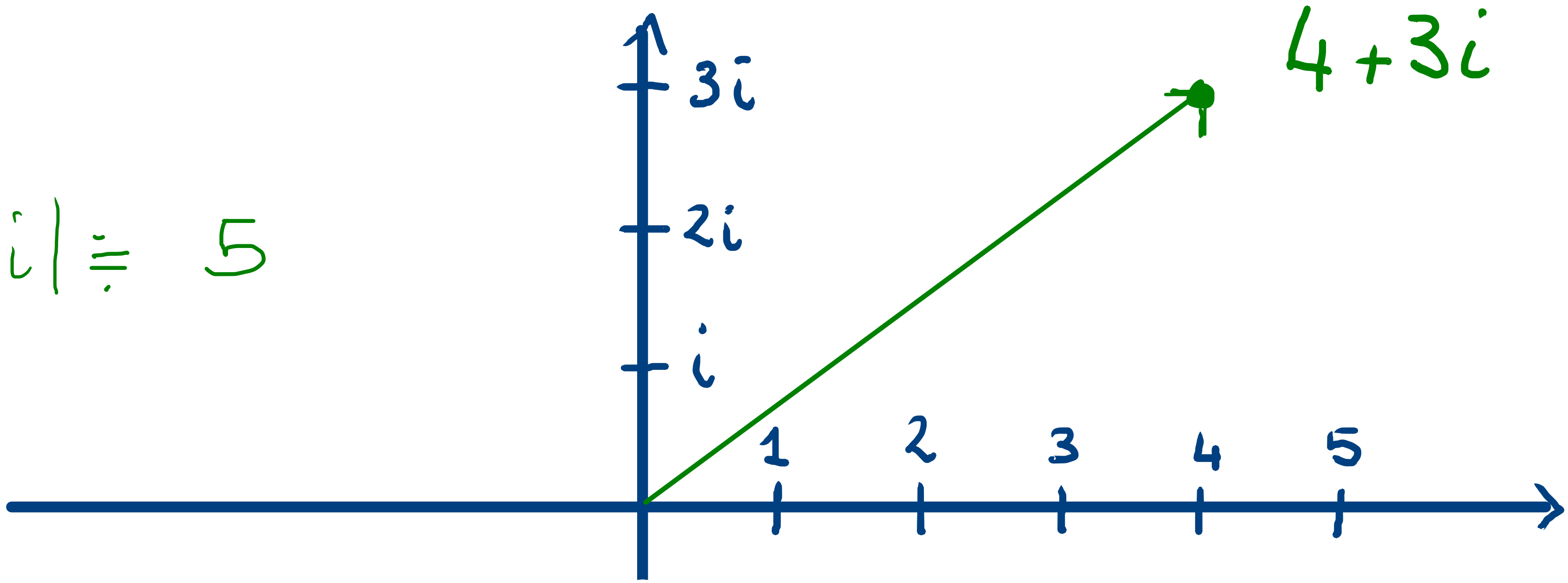
$$|4+3i| = \sqrt{4^2+3^2} = 5$$



MODULUS (LENGTH)

PYTHAGORA'S THM FOR COMPLEX #'S

$|4+3i| \doteq 5$



GENERAL: $|a+ib| \doteq \sqrt{a^2+b^2}$

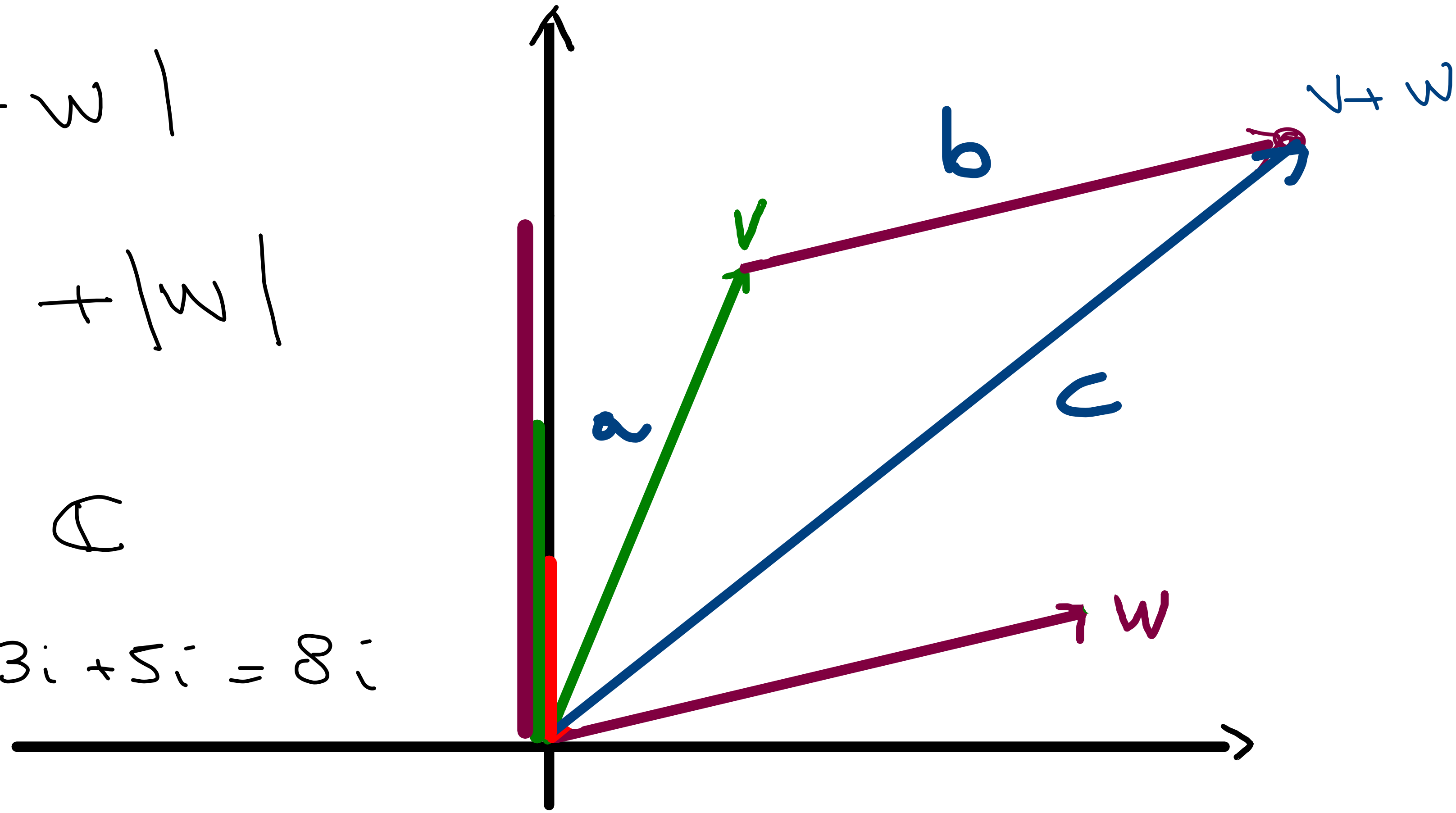
TRIANGLE INEQUALITY FOR COMPLEX NUMBERS

$|v+w|$

$|v|+|w|$

$v, w \in \mathbb{C}$

$3i+5i=8i$



$a+b \geq c \iff |v|+|w| \geq |v+w|$

$$3i + 5i = 8i$$

$$|3i| = 3$$

$$|5i| = 5$$

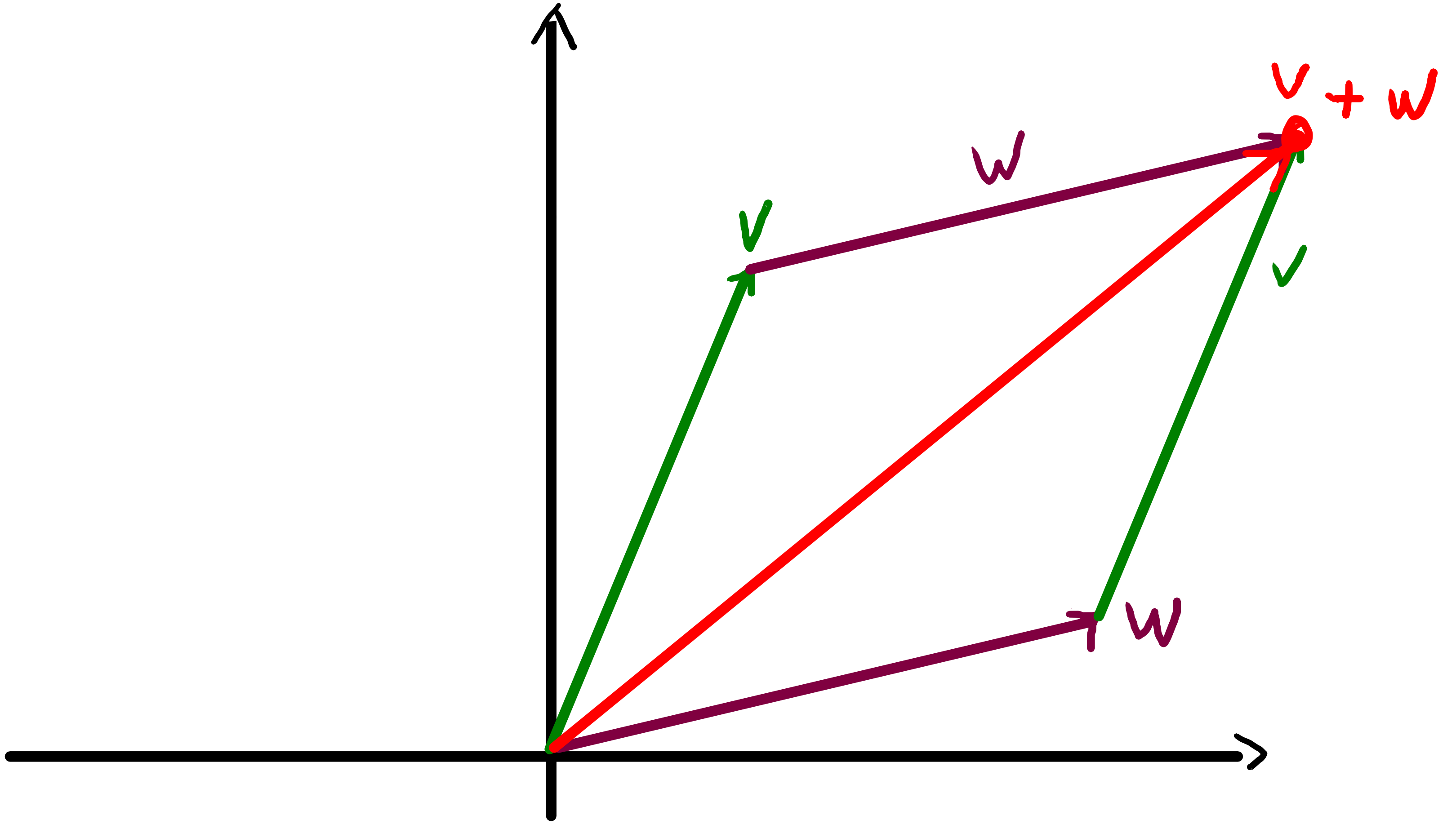
$$|8i| = 8$$

$$\geq$$

$$3 + 5 = 8$$

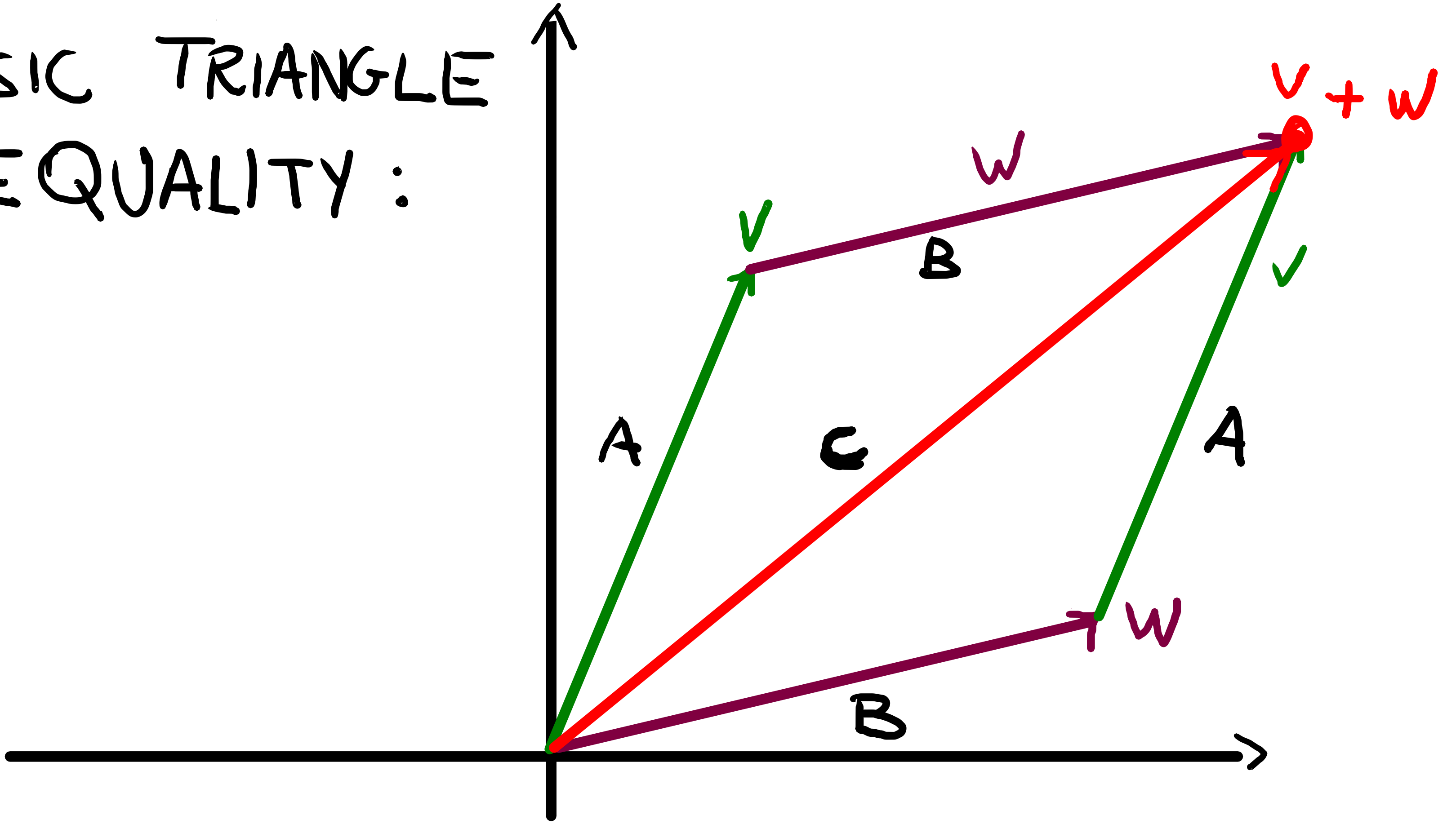
$$|3i + 5i| = |3i| + |5i|$$

TRIANGLE INEQUALITY FOR COMPLEX NUMBERS



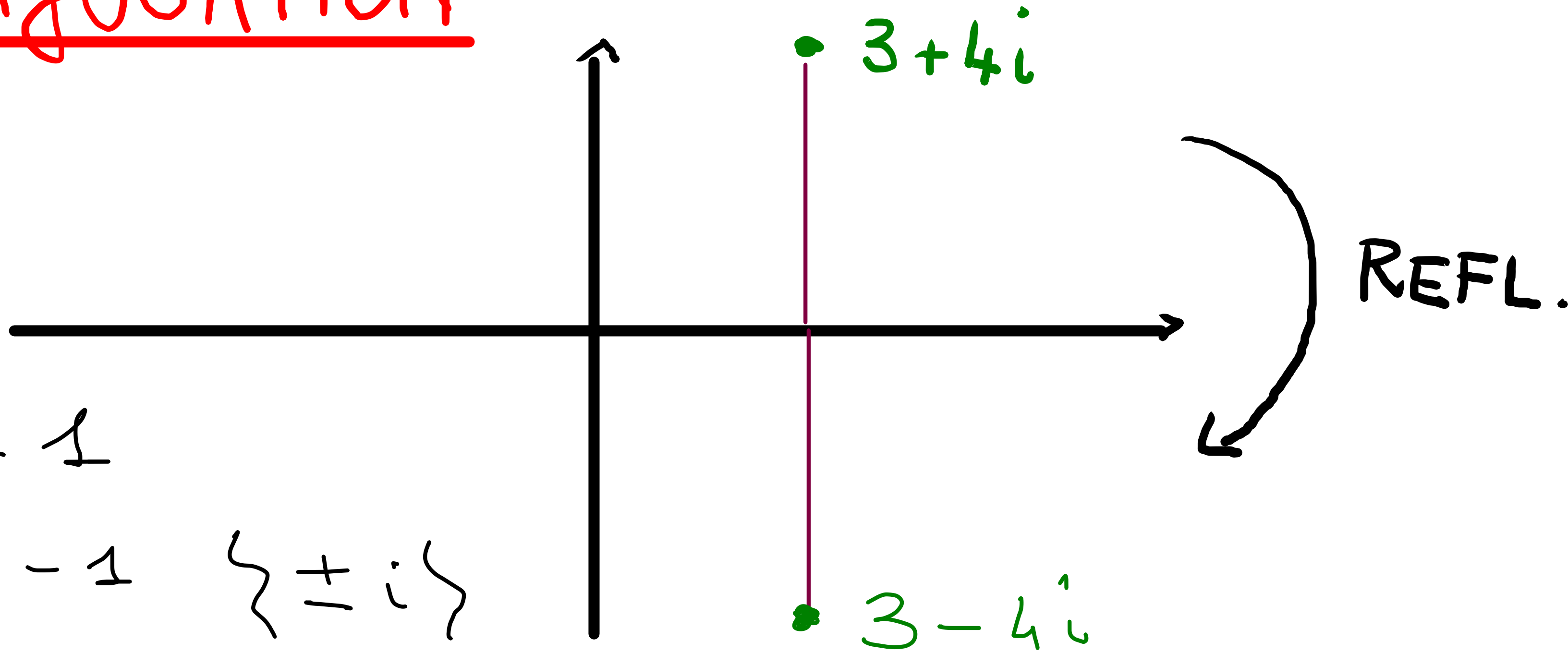
TRIANGLE INEQUALITY FOR COMPLEX NUMBERS

CLASSIC TRIANGLE INEQUALITY:



CONJUGATION

CONJUGATION



$i^2 = -1$

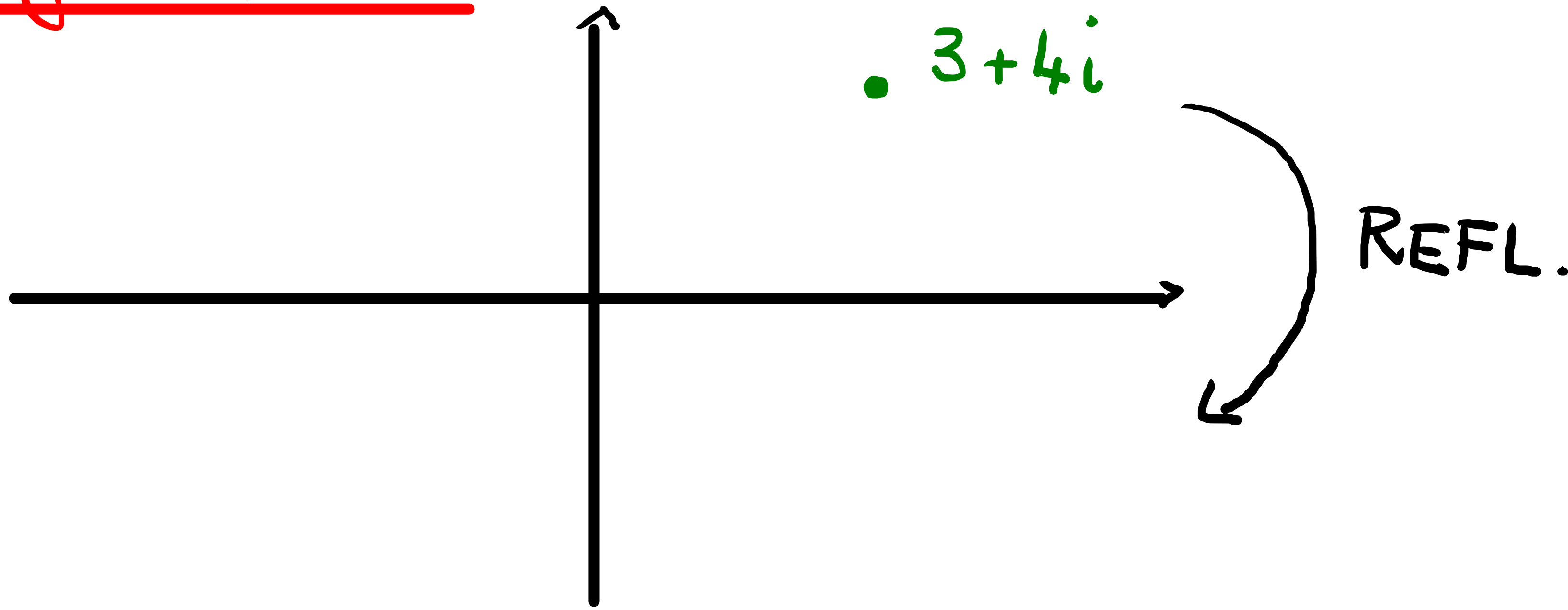
$x^2 = -1 \quad \{\pm i\}$

$\overline{3+4i} = 3-4i$

the complex conjugate of $3+4i$

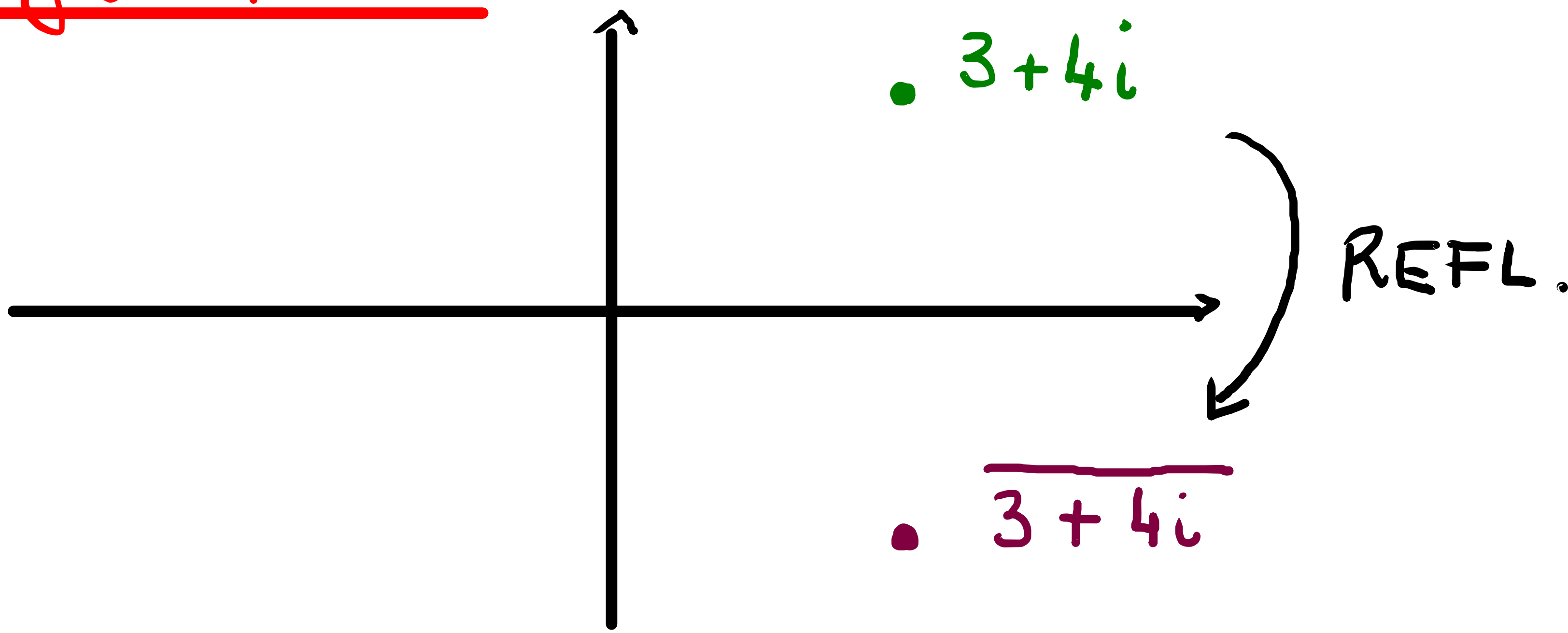
$\overline{a+bi} = a-bi$

CONJUGATION



$\overline{3+4i} \doteqdot$

CONJUGATION



$$\overline{3+4i} \doteqdot 3-4i$$

GENERAL: $\overline{a+bi} \doteqdot$

$$z \cdot \bar{z} = |z|^2$$

$$= (a+bi)(a-bi)$$

$$= a^2 - \cancel{abi} + \cancel{abi} + b^2 \cancel{i^2} = a^2 + b^2$$

$$z \cdot \overline{z} = |z|^2 \Longrightarrow \text{IF } z \neq 0 \ (\Longleftrightarrow |z| \neq 0) \text{ THEN}$$

$$z \cdot \frac{\overline{z}}{|z|^2} = 1$$

$$z^{-1} = \frac{\overline{z}}{|z|^2} \Longrightarrow \mathbb{C} \text{ IS A FIELD}$$

$$z \cdot \bar{z} = |z|^2 \implies \text{IF } z \neq 0$$

THEN

$$z^{-1} = \frac{\bar{z}}{|z|^2} \impliedby z \cdot \frac{\bar{z}}{|z|^2} = 1$$

$$z \cdot \bar{z} = |z|^2 \Longrightarrow \text{IF } z \neq 0$$

THEN

$$z^{-1} = \frac{\bar{z}}{|z|^2} \Longleftarrow \frac{z \cdot \bar{z}}{|z|^2} = 1$$

EXAMPLE

$$\frac{1}{4+3i} = \frac{4-3i}{25} = \frac{4}{25} - \frac{3}{25} \cdot i$$

$$\frac{1}{4+3i} = (4+3i)^{-1}$$

CONJUGATION & REAL \ IMAG. PART

CONJUGATION & REAL \ IMAG. PART

EXAMPLE $4+3i$

$$\overline{4+3i} = 4-3i$$

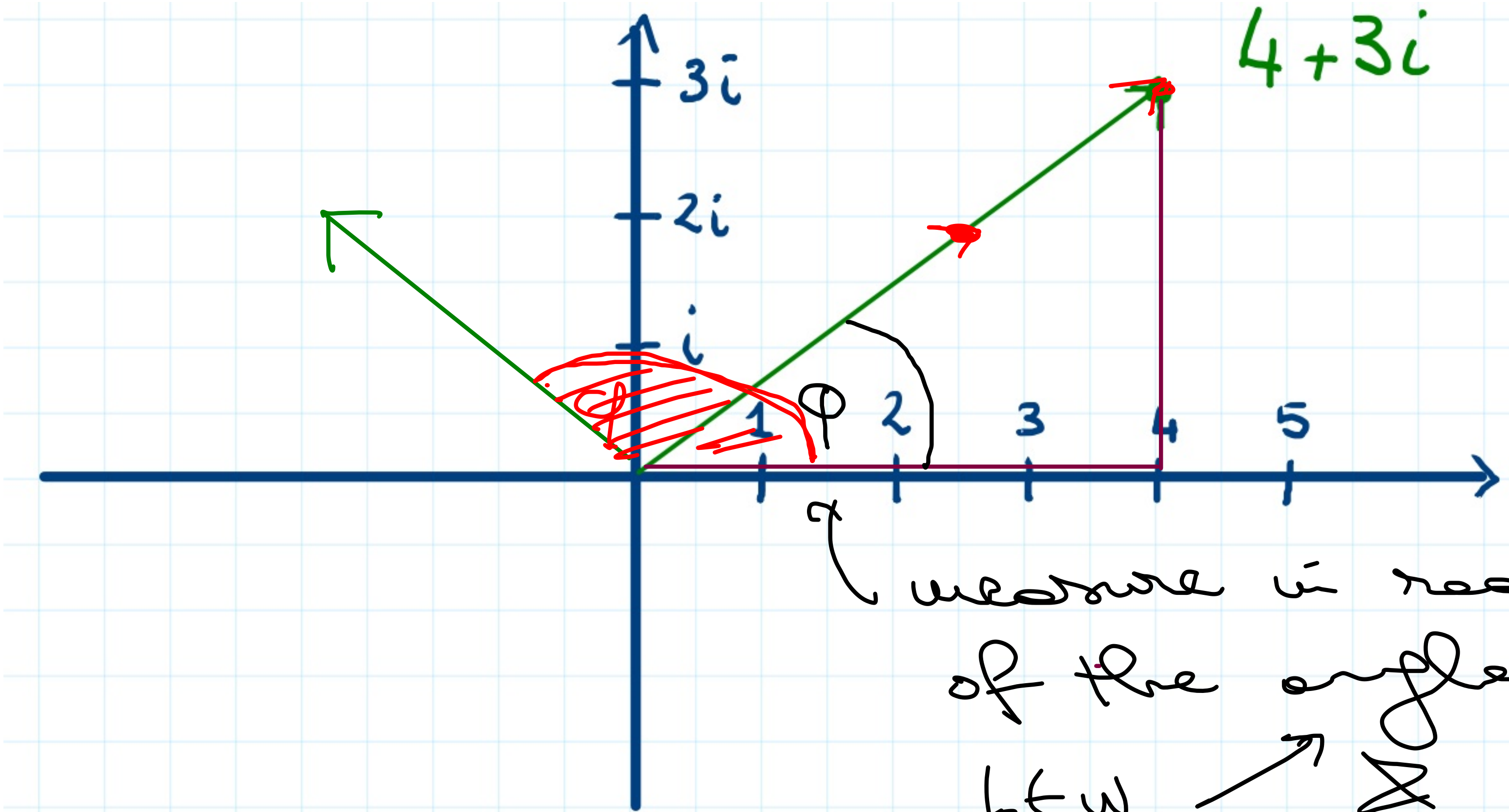
$$4+3i+\left(\overline{4+3i}\right) = 8 = 2\cdot \mathrm{Re}(4+3i)$$

$$\mathrm{Re}(z) = \frac{z+\bar{z}}{2}$$

$$4+3i-\left(\overline{4+3i}\right) = 6i = 2\cdot \mathrm{Im}(4+3i)\cdot i$$

$$\mathrm{Im}(z) = \frac{z-\bar{z}}{2i}$$

POLAR FORM

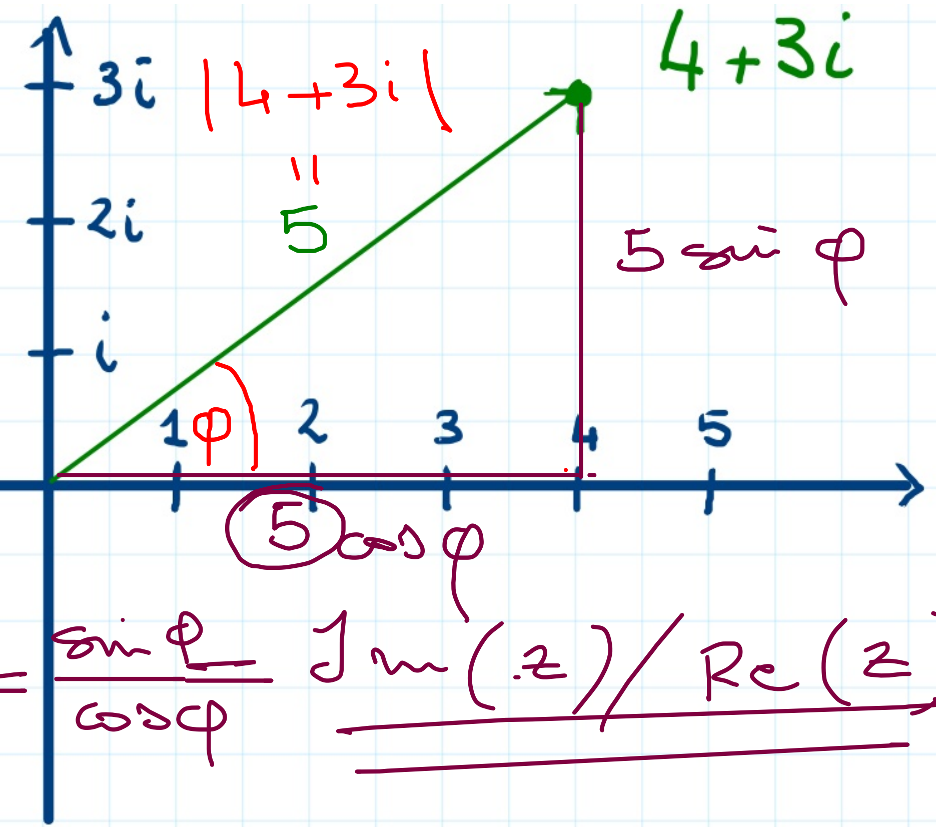


measure in radiants of the angle btw → & Real axis

POLAR FORM

Argument

$\varphi = \arg(z) = \tan^{-1}(\tan(\varphi)) = \tan^{-1}\left(\frac{\operatorname{Im}(z)}{\operatorname{Re}(z)}\right)$

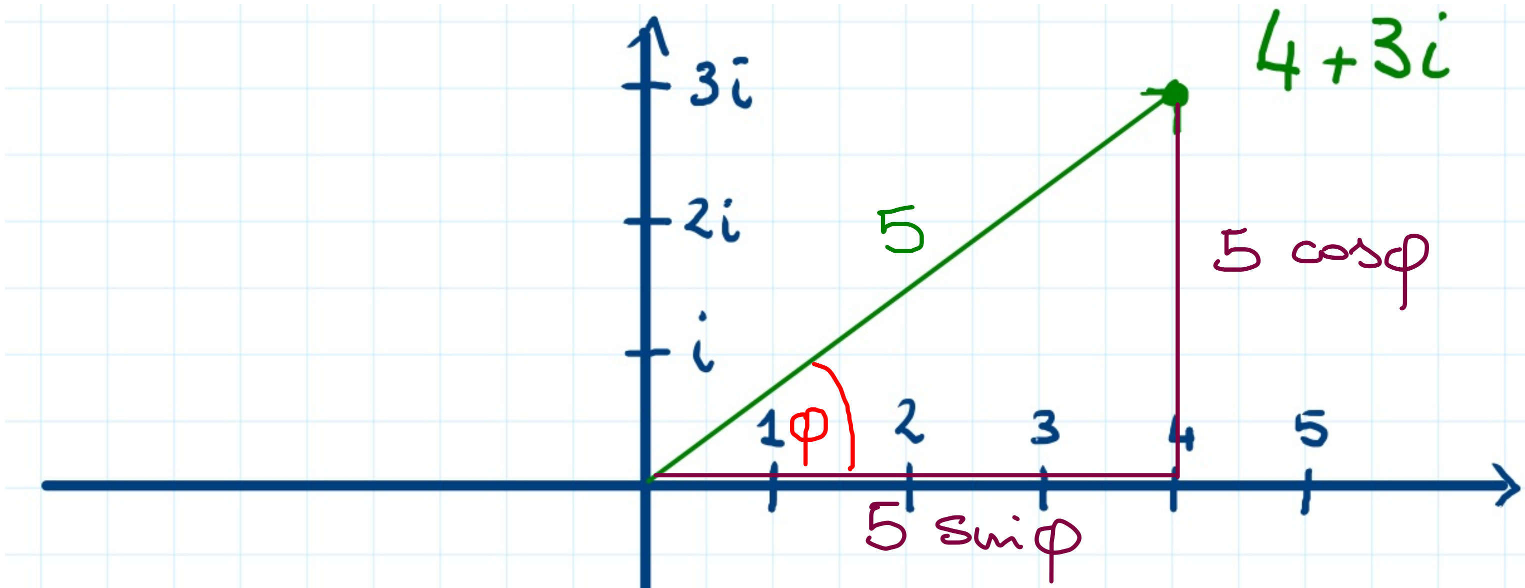


$\tan\varphi = \frac{\sin\varphi}{\cos\varphi} = \operatorname{Im}(z) / \operatorname{Re}(z)$

$4 + 3i = 5(\cos\varphi + i\sin\varphi) = \underbrace{|4+3i|}_{\mathbb{R}}(\cos\varphi + \sin\varphi\, i)$

$\varphi = \arg(4+3i)$

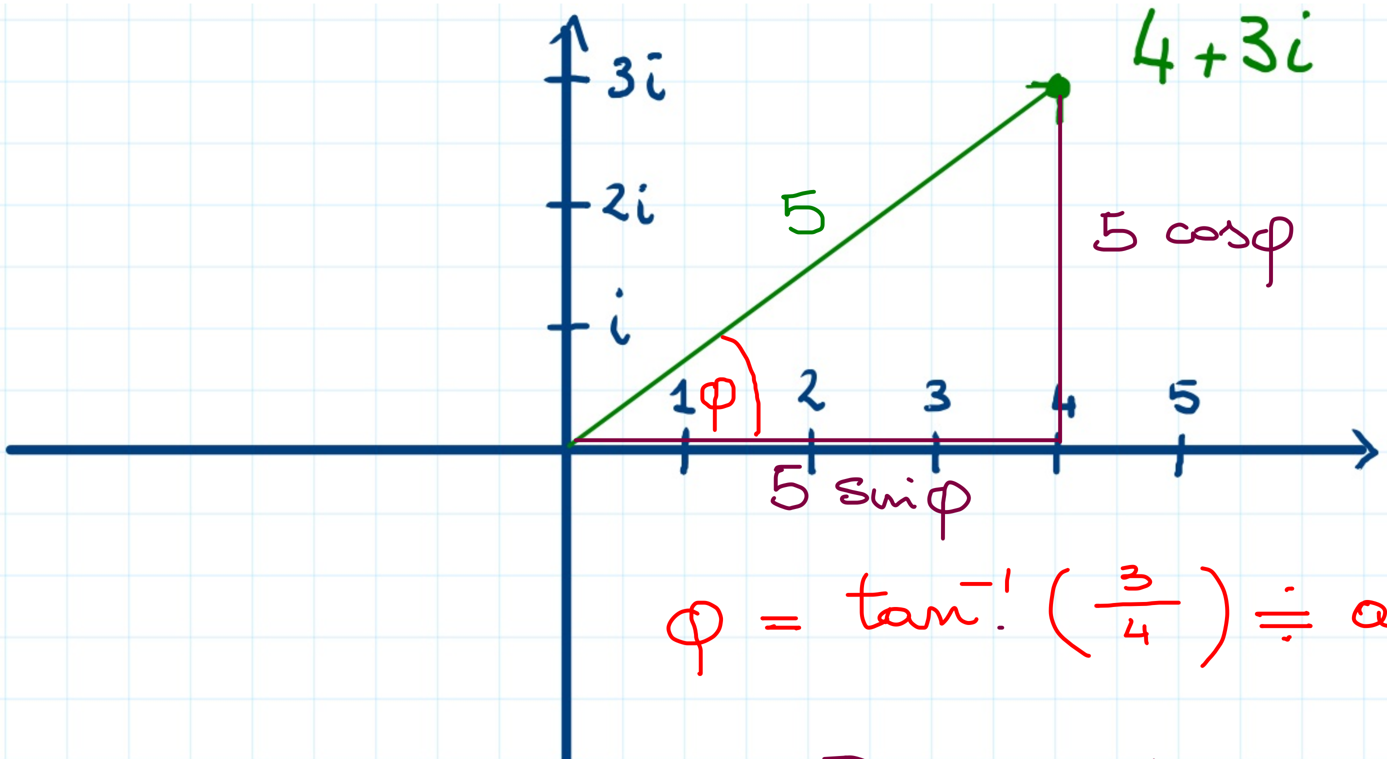
POLAR FORM



$$\varphi = \tan^{-1}\left(\frac{3}{4}\right)$$

$$4+3i = 5\left(\cos\varphi + i\sin\varphi\right) = |4+3i|\left(\cos\varphi + \sin\varphi\, i\right)$$

POLAR FORM



$$\varphi = \tan^{-1}\left(\frac{3}{4}\right) \doteq \arg(4+3i)$$

IN GENERAL: POLAR FORM of $z \in \mathbb{C}$

$$z \doteq |z|\left(\cos(\arg z) + \sin(\arg z)\, i\right)$$

$$\arg(z) = \tan^{-1}\left(\frac{\operatorname{Im}(z)}{\operatorname{Re}(z)}\right)$$