ANALYSIS 1
(ENGLISH)
LECTURE 3

LAST TIME:

- UPPER & LOWER BOUNDS OF SETS
- MAX & MIN OF SETS
- SUP & INF OF SETS

TODAY:

- SOME PROPERTIES OF INF / SUP
- $\mathbb{Q} \subseteq \mathbb{R}$ IS DENSE
- ABSOLUTE VALUE

A NEW DEFINITION OF INF/SUP

RECALL: FOR $S \subseteq \mathbb{R}$, $S \neq \emptyset$.
IF S IS BOUNDED FROM BELOW, THEN
$\inf S$ IS THE LARGEST LOWER BOUND.

:

A NEW DEFINITION OF INF/SUP

RECALL: FOR $S \subseteq \mathbb{R}$, $S \neq \emptyset$.
IF S IS BOUNDED FROM BELOW, THEN
inf S IS THE LARGEST LOWER BOUND.

PROP $S \subseteq \mathbb{R}$, $a \in \mathbb{R}$, $S \neq \emptyset$.

a IS THE INFIMUM OF S IFF:

1. a IS A LOWER BOUND FOR S
2. $\forall \varepsilon > 0$, THERE EXISTS $s_\varepsilon \in S$
   SUCH THAT $s_\varepsilon < a + \varepsilon$

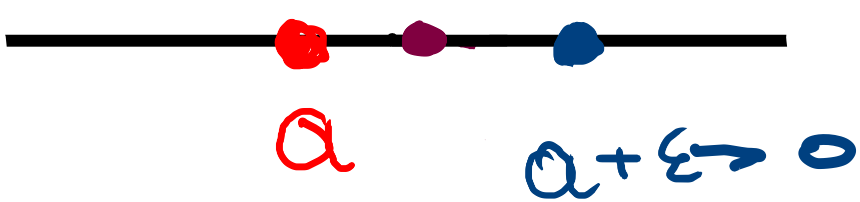
A NEW DEFINITION OF INF/SUP

PROP $S \subseteq \mathbb{R}$, $a \in \mathbb{R}$, $S \neq \emptyset$.

a IS THE INFIMUM OF S IFF

1. a IS A LOWER BOUND FOR S
2. $\forall \varepsilon > 0$, THERE EXISTS $s_\varepsilon \in S$ SUCH THAT $a \leq s_\varepsilon < a + \varepsilon$

$s_\varepsilon \in S$



EXAMPLE

$$S = \left\{ \frac{1}{n} \;\middle|\; n \in \mathbb{N}^* \right\}.$$

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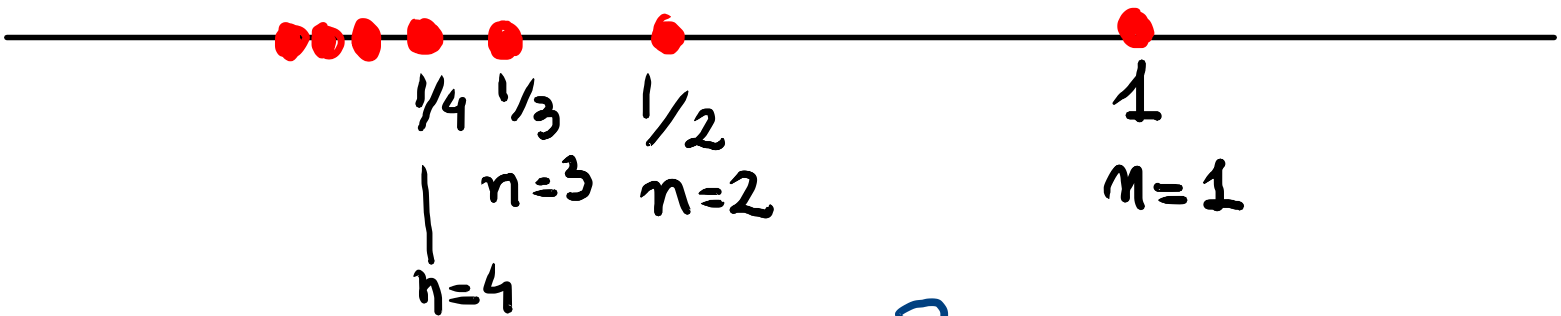


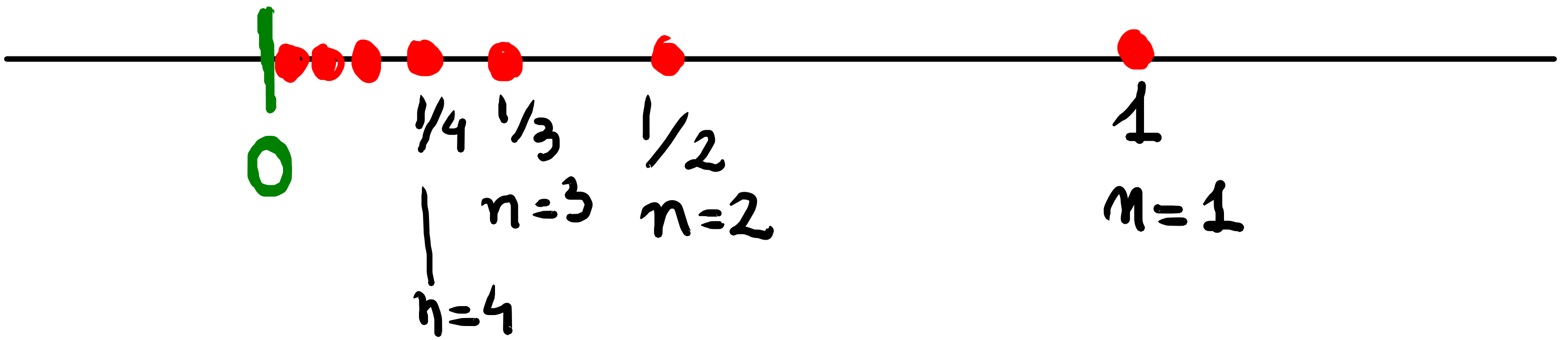
$\inf S = ?$

[POLL]

1. 1
2. 0
3. -1
4. $1/2$

EXAMPLE

$$S = \left\{ \frac{1}{n} \;\middle|\; n \in \mathbb{N}^* \right\}$$



$$\inf S = 0$$

A NEW DEFINITION OF INF/SUP

PROP $S \subseteq \mathbb{R}$, $a \in \mathbb{R}$.

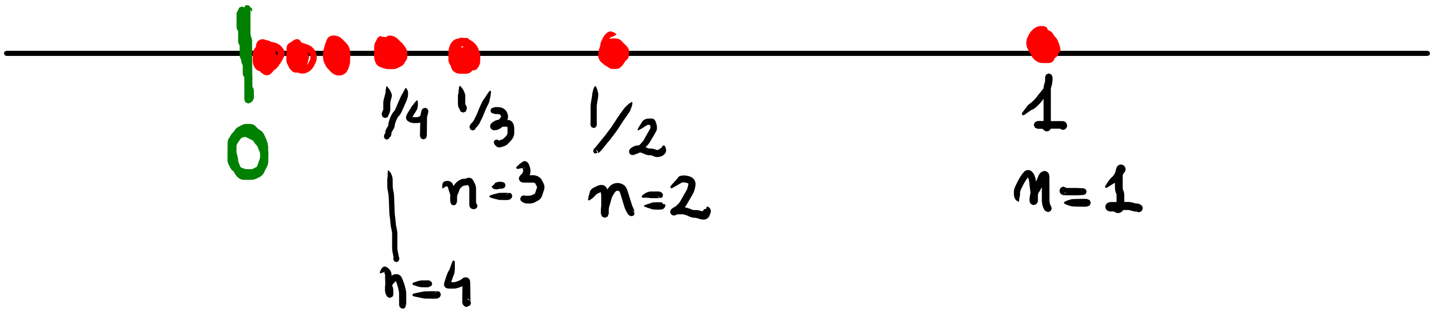
a IS THE INFIMUM OF S IFF:

$\boxed{\text{(1) } a \text{ IS A LOWER BOUND FOR } S}$

(2) $\forall \varepsilon > 0$, THERE EXISTS $s_\varepsilon \in S$ SUCH THAT $s_\varepsilon < a + \varepsilon$

EXAMPLE

$$S = \left\{ \frac{1}{n} \;\middle|\; n \in \mathbb{N} \setminus \{0\} \right\}$$

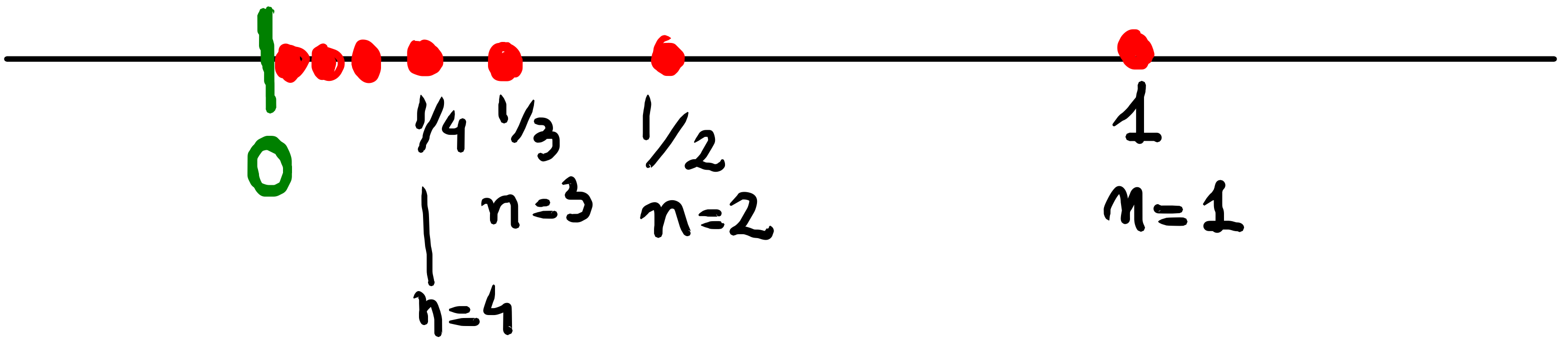


$$\inf S = 0$$

(1) $0 \leq \frac{1}{n}, \quad \forall n \in \mathbb{N}$ 0 is a L.B.

EXAMPLE

$$S = \left\{ \frac{1}{n} \;\middle|\; n \in \mathbb{N} \setminus \{0\} \right\}$$



$$\inf S = 0$$

① $0 \leq \frac{1}{n}, \quad \forall n \in \mathbb{N}$ LOWER BOUND FOR S

A NEW DEFINITION OF INF/SUP

PROP $S \subseteq \mathbb{R}$, $a \in \mathbb{R}$.

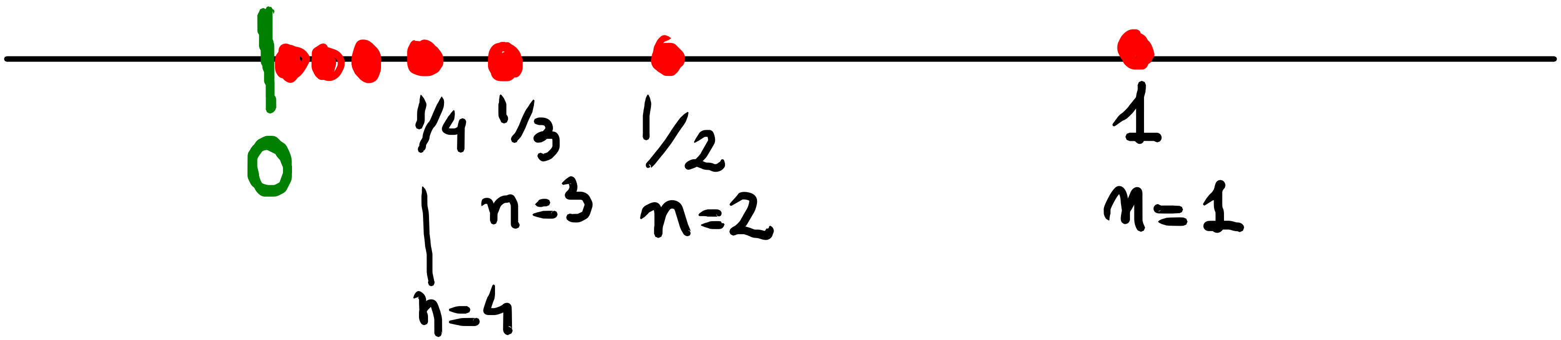
a IS THE INFIMUM OF S IFF:

① a IS A LOWER BOUND FOR S

$$\boxed{\text{② } \forall \varepsilon > 0,\ \text{THERE EXISTS } s_\varepsilon \in S \text{ SUCH THAT } \underbrace{a}_{=0} \le s_\varepsilon < \underbrace{a+\varepsilon}_{=\varepsilon}}$$

EXAMPLE

$$S = \left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{N} \setminus \{0\} \right\}$$



$$\inf S = 0$$

① $0 \le \frac{1}{n}, \quad \forall n \in \mathbb{N}$ LOWER BOUND FOR S

② TAKE $\varepsilon > 0$:

EXAMPLE

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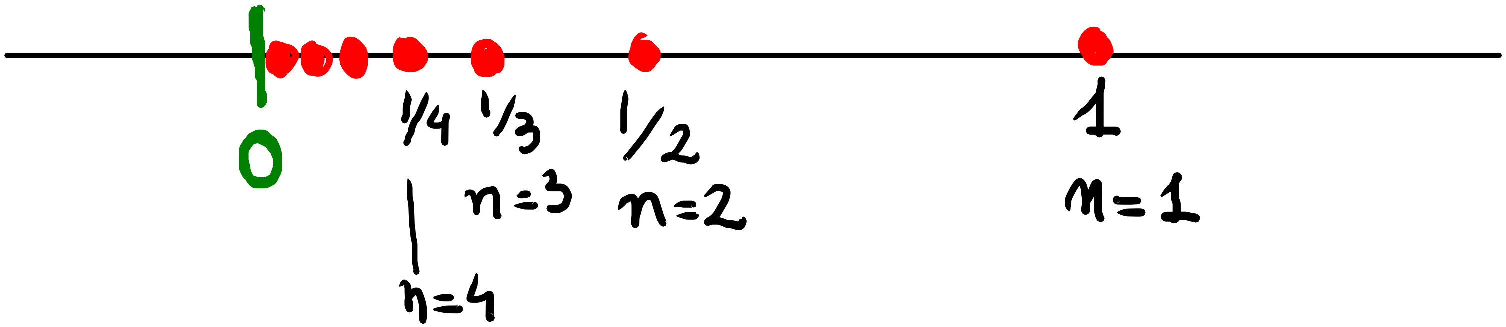

$$\inf S = 0$$

① $0 \leq \frac{1}{n}$, $\forall n \in \mathbb{N}$

② TAKE $\varepsilon > 0$: IF $\varepsilon > 1 \Longrightarrow s_\varepsilon := 1$

$$0 < \underset{\in S}{1} < \varepsilon$$

EXAMPLE

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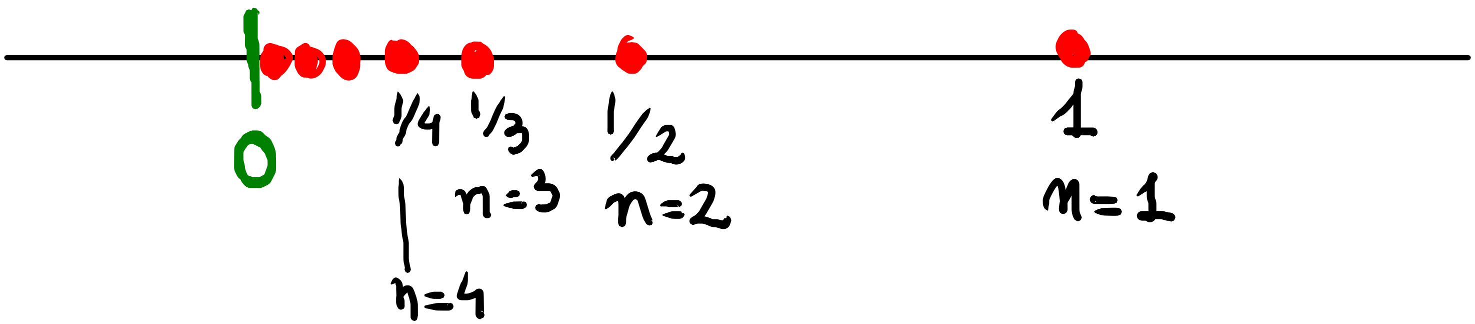

$$\inf S = 0$$

1. $0 \le \frac{1}{n}, \quad \forall n \in \mathbb{N}$

2. TAKE $\varepsilon > 0$: IF $\varepsilon > 1 \Longrightarrow s_\varepsilon := 1$

$0 < \varepsilon \le 1$ WE HAVE TO FIND

$n \in \mathbb{N}^* \mid s_\varepsilon := \frac{1}{n} < \varepsilon \iff \frac{1}{\varepsilon} < n$ BY ARCH. PROP. $\exists n$

A NEW DEFINITION OF INF/SUP

ANALOGOUS NEW DEF. FOR SUP.

PROP $S \subseteq \mathbb{R}$, $\alpha \in \mathbb{R}$, $S \neq \emptyset$.

α IS THE SUPREMUM OF S IFF

1. α IS AN UPPER BOUND FOR S
2. $\forall \varepsilon > 0$, $\exists s'_\varepsilon \in S$ S.T.
$$\alpha - \varepsilon < s'_\varepsilon \leq \alpha$$

PROP (ARCHIMIDEAN PROPERTY)

LET $x, y \in \mathbb{R}_+$, $x > 0$.

THEN $\exists\, n \in \mathbb{N}^*$ S.T. $nx > y$.

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PROOF BY CONTRADICTION:

WE ASSUME $nx \leq y \quad \forall n \in \mathbb{N}^*$.

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$S \doteq \{ nx \mid n \in \mathbb{N}^* \}$

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PROOF BY CONTRADICTION:

WE ASSUME $nx \leq y \quad \forall n \in \mathbb{N}^*$.

$S \doteq \{ nx \mid n \in \mathbb{N}^* \} \implies S \neq \emptyset$ & S IS BOUNDED FROM ABOVE.

$\cup$

x

(y IS AN U.B FOR S)

· sup(S) EXISTS

$(n+1)x \leq \sup S \quad n \geq 1$

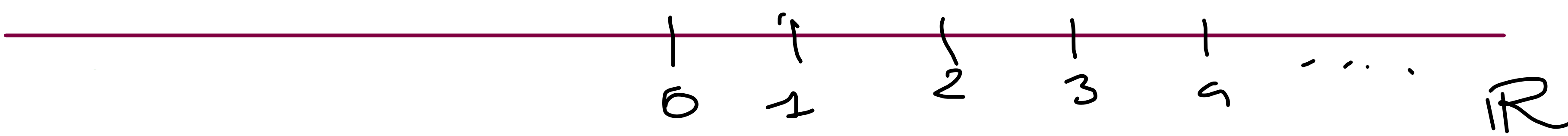
$S \ni nx \leq \sup S - x < \sup S$

PROP LET $\emptyset \neq S \subseteq \mathbb{N} \subseteq \mathbb{R}$.

THEN $\inf S$ EXISTS IN $\mathbb{R}$

AND IT IS A MINIMUM.

$$\left(\overset{\Updownarrow}{\inf S} \in S \right)$$



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PROOF S BOUNDED FROM BELOW

$\xRightarrow{\text{INFIMUM AXIOM}}$ $a \doteq \inf S \in \mathbb{R}$

PROP LET $S \subseteq \mathbb{N} \subseteq \mathbb{R}$.

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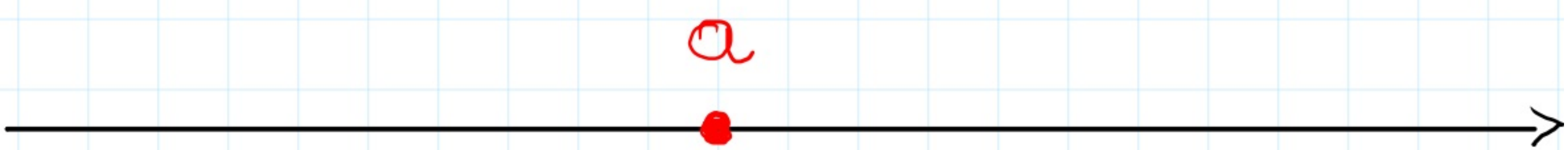
AND IT IS A MINIMUM.

PROOF S BOUNDED FROM BELOW

$$\xRightarrow[\text{INFIMUM AXIOM}]{} \quad a \doteq \inf S \in \mathbb{R}$$

IF $a \in S \Longrightarrow a = \min S$

IF $a \notin S$



A NEW DEFINITION OF INF/SUP

PROP $S \subseteq \mathbb{R}$, $a \in \mathbb{R}$, $S \neq \emptyset$.

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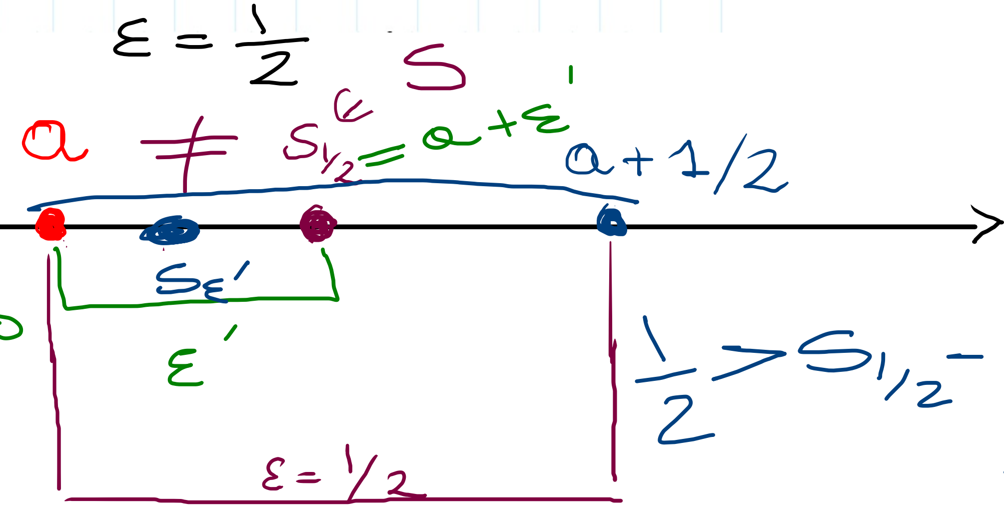
PROOF S BOUNDED FROM BELOW

$\xRightarrow[\text{INFIMUM AXIOM}]{}$ $a \doteq \inf S \in \mathbb{R}$

IF $a \notin S$



$\varepsilon' \doteq s_{1/2} - a > 0$

$\frac{1}{2} > s_{1/2} - s_{\varepsilon'} > 0$

EXAMPLE

$$S = \{ n \in \mathbb{N} \mid n^2 \geq 2 \}$$

WHAT IS MIN S ?

EXAMPLE

$$S = \{ n \in \mathbb{N} \mid n^2 \geq 2 \}$$

WHAT IS $\min S$? [POLL]

~~①~~ $\min S$ DOES NOT EXIST BUT INF S DOES

~~②~~ $1 \notin S \Rightarrow 1^2 = 1 < 2$

③ 2 ☺ $\in S \Rightarrow 2^2 = 4 \geq 2$

~~④~~ NEITHER $\min S$ NOR $\inf S$ EXIST

PROP LET $S \subseteq \mathbb{N} \subseteq \mathbb{R}$.
THEN $\inf S$ EXISTS IN $\mathbb{R}$
AND IT IS A MINIMUM.

GENERALIZATION:

PROP LET $\emptyset \neq S \subseteq \mathbb{Z} \subseteq \mathbb{R}$.
IF S IS BOUNDED FROM ABOVE / BELOW
THEN $\sup S$ / $\inf S$ EXISTS &
IT IS A MAXIMUM / MINIMUM.

$S = (a, b)$ $\qquad$ $\sup S = b$

$\mathbb{Q}$ VS $\mathbb{R}$

$x \in \mathbb{R}$

ROUND DOWN (FLOOR) : $\lfloor x \rfloor \doteq$ LARGEST INT'R $\leq x$

$\mathbb{R} \supseteq \mathbb{Z} \supseteq \{ n \in \mathbb{Z} \mid n \leq x \}$ BOUNDED FROM ABOVE

ROUND UP (CEILING) : $\lceil x \rceil \doteq$ MINIMAL INT'R $\geq x$

B.F.B.

$\{ n \in \mathbb{Z} \mid n \geq x \} \subseteq \mathbb{Z}$

INTEGRAL PART : $[x] \doteq \begin{cases} \lfloor x \rfloor & \text{for } x \geq 0 \\ \lceil x \rceil & \text{for } x \leq 0 \end{cases}$

$\mathbb{Q}$ VS $\mathbb{R}$

$x \in \mathbb{R}$

$\lfloor x \rfloor \in \mathbb{Z} \leftarrow$ MAXIMUM INT'R $\leq x$

$\lceil x \rceil \in \mathbb{Z} \leftarrow$ MININIMUM INT'R $\geq x$

$$[x] \in \mathbb{Z} \doteq \begin{cases} \lfloor x \rfloor & \text{FOR } x \geq 0 \\ \lceil x \rceil & \text{FOR } x \leq 0 \end{cases}$$

EXAMPLE: $\sqrt{2} = 1.41\ldots$

$\lfloor \sqrt{2} \rfloor = 1 = [\sqrt{2}]$ $\qquad \lfloor -\sqrt{2} \rfloor = -2$

$\lceil \sqrt{2} \rceil = 2$ $\qquad \lceil -\sqrt{2} \rceil = -1$

$\quad //$

$[-\sqrt{2}]$

$x = \underline{12345}.6789\ldots$

$[x] = 12345$

$[-x] = -12345$

$[0.\overline{9}] = 1$

$[4.\overline{9}] = 5$

$4.1\overline{9} = 4.2$

$\lfloor 4.1\overline{9} \rfloor = 4$

$\lceil \ \rceil = 5$

$\mathbb{Q}$ VS $\mathbb{R}$

$x \in \mathbb{R}$

$\lfloor x \rfloor \in \mathbb{Z} \leftarrow$ MAXIMUM INT'R $\leq x$

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$$[x] \in \mathbb{Z} \doteqdot \begin{cases} \lfloor x \rfloor & \text{FOR } x \geq 0 \\ \lceil x \rceil & \text{FOR } x \leq 0 \end{cases}$$

WHY IS IT WELL DEFINED?

PROPERTIES $\forall x \in \mathbb{R}$

- $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$ $\quad$ = holds IFF $x \in \mathbb{Z}$

- $\lceil x \rceil - 1 < x \leq \lceil x \rceil$ $\quad$ = holds IFF $x \in \mathbb{Z}$

- $\lceil x \rceil = -\lfloor -x \rfloor$

$$\lfloor 0.2 \rfloor = 0$$

$$\lceil -0.2 \rceil = 0$$

EVEN STRONGER RESULT

PROP $a, b \in \mathbb{R}$, $a < b$.

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$\mathbb{Q}$ IS DENSE IN $\mathbb{R}$: APPROXIMATE ANY REAL # BY MEANS OF RATIONAL #'S (WITH ARBITRARY PRECISION)

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$\mathbb{Q}$ IS DENSE IN $\mathbb{R}$: APPROXIMATE ANY REAL # BY MEANS OF RATIONAL #'S

$$\sqrt{2} = 1.\overbrace{4142135}^{n\text{-digits}}\ldots \quad \Rightarrow x \qquad \sqrt{2} - x < \frac{1}{10^n}$$

$$\sqrt{2} - 1 = 0.414\ldots < 1$$

$$\sqrt{2} - 1.414 = 0.0002135 < \frac{1}{10^3}$$

EVEN STRONGER RESULT

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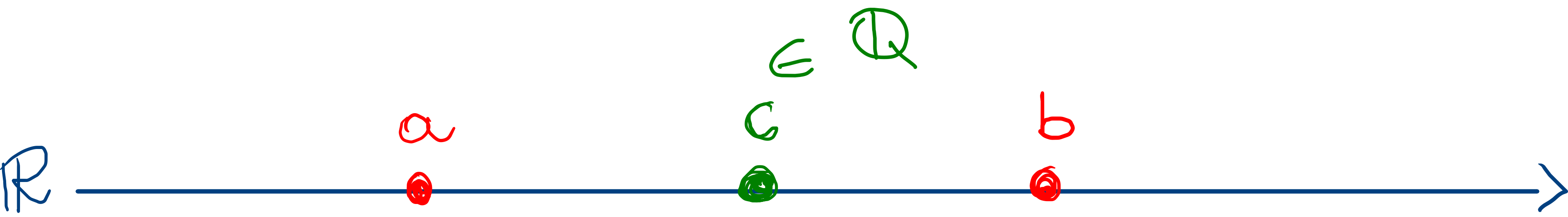

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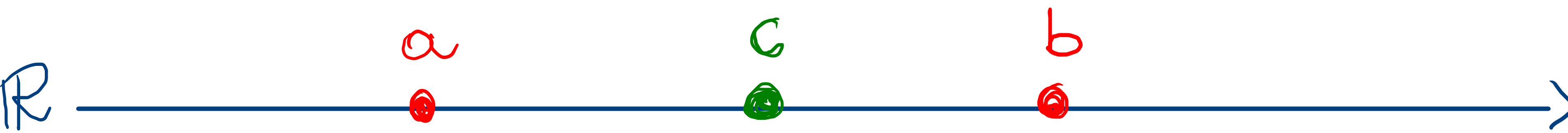
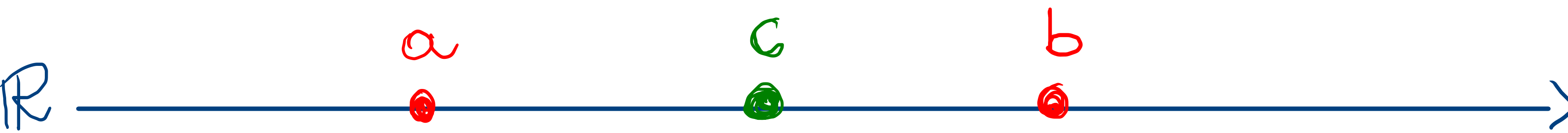


EXAMPLE TAKE $a = 0$, $b > 0$.

EVEN STRONGER RESULT

PROP $a, b \in \mathbb{R}$, $a < b$.
$\exists\, c \in \mathbb{Q}$ S.T. $a < c < b$.



EXAMPLE TAKE $a = 0$ $b > 0$.

$c = \,?$

$0 < c < b \iff \frac{1}{b} < \overset{\mathbb{Q}}{\overset{\cup}{\frac{1}{c}}}$

PROPERTIES

$$\forall x \in \mathbb{R} \qquad \lfloor x \rfloor \leq x \cancel{\leq} \underbrace{\lfloor x \rfloor}_{\mathbb{Z}} + 1$$

EVEN STRONGER RESULT

PROP $a, b \in \mathbb{R}$, $a < b$.

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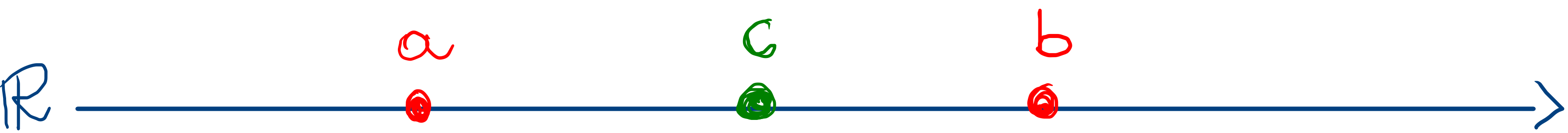

EXAMPLE TAKE $a = 0$ $\quad b > 0$.

$$c = \left(\left\lfloor \frac{1}{b} \right\rfloor + 1 \right)^{-1}$$

$$0 < c < b \iff \frac{1}{b} < \frac{1}{c} = \overset{\mathbb{N}^* \ni}{\left\lfloor \frac{1}{b} \right\rfloor + 1} > 0$$

EVEN STRONGER RESULT

PROP $a, b \in \mathbb{R}$, $a < b$.

$\exists\, c \in \mathbb{Q}$ s.t. $a < c < b$.

PROOF

EVEN STRONGER RESULT

PROP $a, b \in \mathbb{R}$, $a < b$.

$\exists\, c \in \mathbb{Q}$ s.t. $a < c < b$.

PROOF $b - a > 0$

EVEN STRONGER RESULT

PROP $a, b \in \mathbb{R}$, $a < b$.

$\exists c \in \mathbb{Q}$ s.t. $a < c < b$.

PROOF $b - a > 0$ $\mathbb{N}^* \ni n := \left[(b-a)^{-1} \right] + 1$

EVEN STRONGER RESULT

PROP $a, b \in \mathbb{R}$, $a < b$.

$\exists\, c \in \mathbb{Q}$ s.t. $a < c < b$.

PROOF $b - a > 0$ $\quad n \doteq \left[(b-a)^{-1} \right] + 1$

EVEN STRONGER RESULT

PROP $a, b \in \mathbb{R}, a < b.$

$\exists\, c \in \mathbb{Q}$ s.t. $a < c < b$.

PROOF $b - a > 0$ $\quad n \doteq \lfloor (b-a)^{-1} \rfloor + 1 > (b-a)^{-1}$

$\Rightarrow n(b-a) > 1$

EVEN STRONGER RESULT

PROP $a, b \in \mathbb{R}$, $a < b$.

$\exists\, c \in \mathbb{Q}$ s.t. $a < c < b$.

PROOF $b - a > 0$ $\qquad n \doteq \left[(b-a)^{-1} \right] + 1$

$$\Rightarrow n(b-a) > 1$$

$$\Downarrow$$

$$b > \frac{na+1}{n} = a + \frac{1}{n} > a$$

EVEN STRONGER RESULT

PROP $a, b \in \mathbb{R}$, $a < b$.

$\exists\, c \in \mathbb{Q}$ s.t. $a < c < b$.

PROOF $b - a > 0$ $\qquad n \doteq \left\lfloor (b-a)^{-1} \right\rfloor + 1$

$\Rightarrow n(b-a) > 1$

$\Downarrow$

$\in \mathbb{Q}$

$$b > \frac{na+1}{n} > \frac{\lfloor na+1 \rfloor}{n} > a$$

$$\frac{\cancel{n}a}{\cancel{n}} < \frac{\lfloor na+1 \rfloor}{n} \leq \frac{na+1}{n}$$

RAT'L

EVEN STRONGER RESULT

PROP $a, b \in \mathbb{R}$, $a < b$.

$\exists\, c \in \mathbb{Q}$ s.t. $a < c < b$.

PROP $a, b \in \mathbb{R}$, $a < b$

EVEN STRONGER RESULT

PROP $a, b \in \mathbb{R}$, $a < b$.

$\exists\, c \in \mathbb{Q}$ s.t. $a < c < b$.

PROP $a, b \in \mathbb{R}$, $a < b$

$\exists\, d \in \mathbb{R} \setminus \mathbb{Q}$ s.t. $a < d < b$

IRRATIONAL #'S

PROP $a, b \in \mathbb{R}$, $a < b$

$\exists\, d \in \mathbb{R} \setminus \mathbb{Q}$ s.t. $a < d < b$

PROOF

PROP $a, b \in \mathbb{R}$, $a < b$

$\exists\, d \in \mathbb{R} \setminus \mathbb{Q}$ s.t. $a < d < b$

PROOF OBS: $c \in \mathbb{Q}^{*}$, $s \in \mathbb{R} \setminus \mathbb{Q}$

$$\Downarrow$$

$$c \cdot s \notin \mathbb{Q}$$

IF $\overbrace{c \cdot s}^{\in\, \mathbb{Q}} \cdot\, c^{-1} = s$

PROP $a, b \in \mathbb{R}$, $a < b$

$\exists\, d \in \mathbb{R} \setminus \mathbb{Q}$ s.t. $a < d < b$

PROOF OBS: $c \in \mathbb{Q}$, $s \in \mathbb{R} \setminus \mathbb{Q}$

$\Longrightarrow c \cdot s \notin \mathbb{Q}$

$\sqrt{3} \notin \mathbb{Q}$

PROP $a, b \in \mathbb{R}$, $a < b$

$\exists\, d \in \mathbb{R} \setminus \mathbb{Q}$ s.t. $a < d < b$

PROOF OBS: $c \in \mathbb{Q}$, $s \in \mathbb{R} \setminus \mathbb{Q}$

$$\Longrightarrow \quad c \cdot s \notin \mathbb{Q}$$

$$\sqrt{3} \notin \mathbb{Q} \qquad \qquad \mathbb{Q}^*$$

$$\frac{a}{\sqrt{3}} < c < \frac{b}{\sqrt{3}}$$

$$\Downarrow \cdot \sqrt{3}$$

$$a < d \doteq c \cdot \sqrt{3} \in \mathbb{R} \setminus \mathbb{Q} < b$$

$\blacksquare$

PROP $a, b \in \mathbb{R}$, $a < b$

$\exists\, d \in \mathbb{R} \setminus \mathbb{Q}$ s.t. $a < d < b$

PROOF OBS: $c \in \mathbb{Q}$, $s \in \mathbb{R} \setminus \mathbb{Q}$

$$\Longrightarrow \quad c \cdot s \notin \mathbb{Q}$$

$$\sqrt{2} \notin \mathbb{Q}$$

$\exists\, c \in \mathbb{Q}$ s.t. $\dfrac{a}{\sqrt{2}} < c < \dfrac{b}{\sqrt{2}}$

PROP $a, b \in \mathbb{R}$, $a < b$

$\exists\, d \in \mathbb{R} \setminus \mathbb{Q}$ s.t. $a < d < b$

PROOF OBS: $c \in \mathbb{Q}$, $s \in \mathbb{R} \setminus \mathbb{Q}$

$$\Longrightarrow \quad c \cdot s \notin \mathbb{Q}$$

$\exists\, c \in \mathbb{Q}$ s.t. $\dfrac{a}{\sqrt{2}} < c < \dfrac{b}{\sqrt{2}}$

$$\Updownarrow$$

$\sqrt{2} \notin \mathbb{Q}$

TO CONCLUDE :

- RATIONAL #'s ARE DENSE
- IRRATIONAL #'s ARE DENSE

REAL #'s ARE COMPLETE :

THERE ARE NO GAPS !!

[MORE ON THIS LATER]

PROP $\sqrt{3}$ IS A REAL NUMBER.

$\sqrt{3} \doteq$ "SOLUTION" TO THE EQN

$$x^2 - 3 = 0$$

PROP $\sqrt{3}$ IS A REAL NUMBER.

PROOF $S \doteq \{ x \in \mathbb{R} \mid x^2 \leq 3 \}$

PROP $\sqrt{3}$ IS A REAL NUMBER.

PROOF $S \doteq \{ x \in \mathbb{R} \mid x^2 \leq 3 \}$

- $S \neq \emptyset$ $\quad 0 \in S$
- S IS BOUNDED $\quad S \subseteq [-3, 3]$

$$0 < a < b \Rightarrow a^2 < b^2$$

$$0 < \inf S, \ \sup S \in \mathbb{R}$$

TO SHOW : $(\sup S)^2 = 3$

$$(\sup S)^2 = \begin{cases} > 3 \\ = 3 \\ < 3 \end{cases}$$

ASSUME THAT $(\sup S)^2 > 3$

ASSUME THAT $(\sup S)^2 > 3$

$\Rightarrow$ CALL $x = \sup S$

$$\Updownarrow$$

$$x^2 > 3$$

$$\Updownarrow$$

$$x > \frac{3}{x}$$

ASSUME THAT $(\sup S)^2 > 3$

$\Rightarrow$ CALL $x = \sup S$

$$x^2 > 3$$

$$\Updownarrow$$

$$x > \frac{3}{x}$$

LET $c := \frac{\left(x + \frac{3}{x}\right)}{2} \Longrightarrow$

ASSUME THAT $(\sup S)^2 > 3$

TAKING $\varepsilon = \frac{1}{n}$ $\exists$ $s_{1/n} \in S$ S.T.

$$\sup(S) - \frac{1}{n} < s_{1/n} \leq \sup(S)$$

$$\left(\sup(S) - \frac{1}{n}\right)^2 = (\sup S)^2 - \frac{2 \sup S}{n} + \frac{1}{n^2}$$

$$\vee$$

$$(\sup S)^2 - \frac{2 \sup S}{n}$$

TAKE $y \doteq (\underset{\substack{\vee \\ 0}}{\sup S})^2 - 3$ & $\bar{n} \in \mathbb{N}$ s.t. $\bar{n} \frac{1}{2 \sup S} > \frac{1}{y}$

$$\left(\sup(S)-\frac{1}{n}\right)^2=(\sup S)^2-\frac{2\sup S}{n}+\frac{1}{n^2}$$

$$\vee$$

$$(\sup S)^2-\frac{2\sup S}{n}$$

TAKE $y \doteq \underset{\substack{\vee \\ 0}}{(\sup S)^2-3}$ & $n\in\mathbb{N}$ s.t. ny

THEN $(\sup S)^2-\frac{2\sup S}{n} > \sup(S)^2-y>3$

$$\Longrightarrow \underset{\substack{\vee\!| \\ 3}}{\left(\sup S-\frac{1}{n}\right)^2} \underset{\angle}{<} \left(s_{1/n}\right)^2 < (\sup S)^2$$

since $s_{1/n}\in S$.