ANALYSIS 1

(ENGLISH)

LECTURE 2

LECTURE 1 :

LECTURE 1 :

- MOTIVATIONS

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- PROOFS

CONSTRUCTIVE

BY CONTRADICTION

LECTURE 1 :

- MOTIVATIONS
- PROOFS
  - CONSTRUCTIVE
  - BY CONTRADICTION
- SETS
  - BASIC DEF'S & OPERATIONS
  - IMPORTANT SETS

IMPORTANT SETS

- $\emptyset$ = EMPTY SET

$\rotatebox{90}{$\supseteq$}$

- $\mathbb{N}$ = NATURAL #'s

$\rotatebox{90}{$\supseteq$}$

- $\mathbb{Z}$ = INTEGERS

$\rotatebox{90}{$\supseteq$}$

- $\mathbb{Q}$ = RATIONAL #'s

$\rotatebox{90}{$\supseteq$}$

- $\mathbb{R}$ = REAL #'s

NOT DEF'D YET

HOW DO WE DEFINE REAL #'S ?

INTUITIVELY: $\underbrace{1347\ldots6}_{\text{FINITELY MANY DIGITS}}\,.\,\overbrace{0784321\ldots}^{\text{ANY POSSIBLE } \infty \text{ SEQUENCE OF DIGITS}}$

HOW DO WE DEFINE REAL #'S?

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PROBLEMATIC DEFINITION

$$0.\overline{9} = 1$$

HOW TO DEFINE $\mathbb{R}$?

THERE IS AN ACTUAL SOLUTION TO THIS PROBLEM

RATHER THAN CONSTRUCTING $\mathbb{R}$ WE LIST ITS MOST IMPORTANT PROPERTIES:

TODAY

1. $\mathbb{Q} \subseteq \mathbb{R}$
2. $\mathbb{R}$ IS AN ORDERED FIELD
3. $\mathbb{R}$ SATISFIES "THE INFIMUM AXIOM"

SETS: SOME MORE NOTATION

$$\mathbb{R}^* \doteq \{ x \in \mathbb{R} \mid x \neq 0 \}$$

$$\mathbb{R}_+ \doteq \{ x \in \mathbb{R} \mid x \geq 0 \} \doteq \mathbb{R}_{\geq 0}$$

$$\mathbb{R}_- \doteq \{ x \in \mathbb{R} \mid x \leq 0 \} \doteq \mathbb{R}_{\leq 0}$$

$$\mathbb{R}^*_+ \doteq \mathbb{R}_+ \setminus \{0\} \doteq \mathbb{R}_{>0}$$

$$\mathbb{R}^*_- \doteq \mathbb{R}_- \setminus \{0\} \doteq \mathbb{R}_{<0}$$

BOUNDED INTERVALS: $a, b \in \mathbb{R}$

$a < b$

OPEN BOUNDED INTERVAL:

$$]a, b[\doteq \{ x \in \mathbb{R} \mid a < x < b \}$$

ALTERNATIVE NOTATION: (a, b)

CLOSED B. INTERVAL:

$$[a, b] \doteq \{ x \in \mathbb{R} \mid a \leq x \leq b \}$$

HALF-OPEN INTERVALS

$$[a, b[\doteq \{ x \in \mathbb{R} \mid a \leq x < b \}$$

$$\doteq$$

$$[a, b)$$

$$]a, b] \doteq \{ x \in \mathbb{R} \mid a < x \leq b \}$$

$$\doteq$$

$$(a, b]$$

BOUNDS

DEF LET $S \subseteq \mathbb{R}$ & $a \in \mathbb{R}$. ASSUME THAT $S \neq \emptyset$

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DEF LET $S \subseteq \mathbb{R}$ & $a \in \mathbb{R}$, $S \neq \emptyset$.

① a IS A LOWER BOUND FOR S IF

$$a \leq x \quad , \quad \forall x \in S$$

"FOR ALL"

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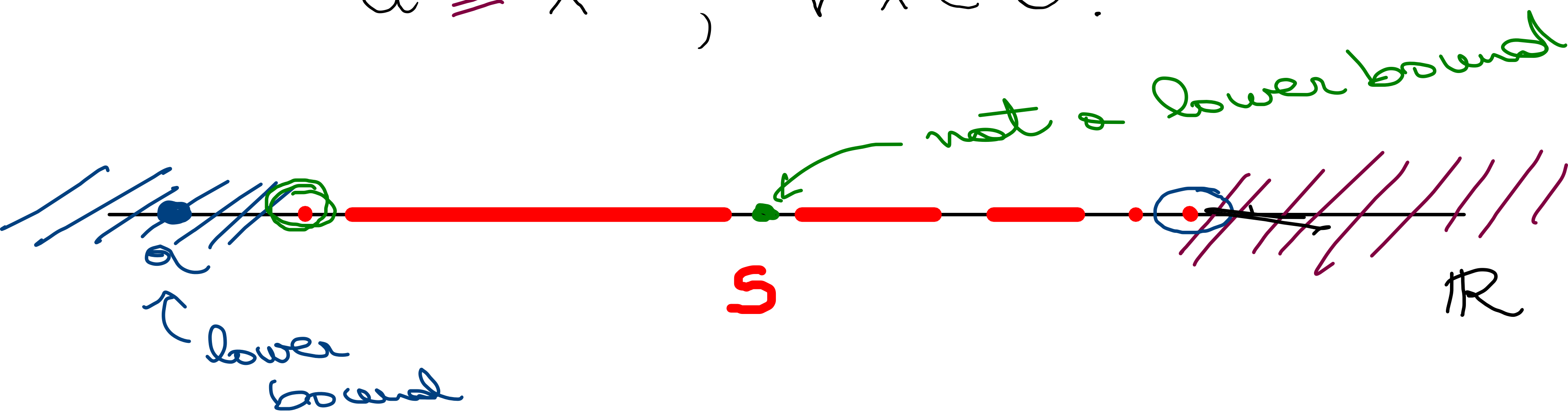

BOUNDS

DEF LET $S \subseteq \mathbb{R}$ & $a \in \mathbb{R}$. $S \neq \emptyset$

① a IS A LOWER BOUND FOR S IF
UPPER BOUND

$$a \underset{\geq}{\leq} x \ , \ \forall x \in S.$$

② S IS BOUNDED FROM BELOW IF
ABOVE

S ADMITS A LOWER BOUND.
UPPER BOUND

③ S IS BOUNDED IF IT IS BOUNDED FROM ABOVE & BELOW.

EXAMPLE $\mathbb{N}$ IS BOUNDED FROM BELOW

$$\mathbb{N} = \{0, 1, 2, 3, \dots, n, n+1, \dots\}$$
$$\cap|$$
$$\mathbb{R}$$

$$\forall n \in \mathbb{N}, \quad n \geq 0$$

$\Longrightarrow 0$ is a lower bound for $\mathbb{N}$,

IS $\mathbb{N}$ BOUNDED FROM ABOVE?

IF IT IS $\Longrightarrow \exists x \in \mathbb{R}$ S.T. x U.B.

$$\forall n \in \mathbb{N} \quad n \leq x.$$

DOES NOT SEEM RIGHT

$\mathbb{N}$ IS NOT BOUNDED FROM ABOVE

ARCHIMIDEAN PROPERTY OF $\mathbb{R}$

$x, y \in \mathbb{R}_+$, $x \neq 0$.

THEN $\exists\, n \in \mathbb{N}$ S.T. SUCH THAT.

$nx > y$.

IN OUR CASE : $x = 1$, $y \in \mathbb{R}_+$

$\exists\, n \in \mathbb{N}$ S.T. $\underset{\mathbb{N}}{n} \cdot 1 > y$.

EXAMPLE ① $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ ARE UNBOUNDED

$T \subseteq S \wedge T$ IS UNBOUNDED

$$\Downarrow$$

S IS ALSO UNBOUNDED.

EXAMPLE · $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ ARE UNBOUNDED

② $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ NOT BOUNDED FROM ABOVE / BELOW

$\mathbb{Z}$ IS NOT BOUNDED FROM BELOW

BY THE ARCH. PROPERTY

BY CONTR.: if $\exists\, \bar{y} \in \mathbb{R}$ A LOWER BOUND FOR $\mathbb{Z}$

$\overset{DEF}{\Longleftrightarrow}$ $\forall z \in \mathbb{Z}$, $\bar{y} \leq z$

MULT. BY (-1): $\forall z \in \mathbb{Z}$, $-\bar{y} \geq -z$.

$-\bar{y}$ IS AN UPPER BOUND FOR $\mathbb{Z}$.

EXAMPLE • $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ ARE UNBOUNDED

• $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ NOT BOUNDED FROM ABOVE / BELOW

③ FOR $a < b \in \mathbb{R}$, CONSIDER

$$[a, b] \doteq \{ x \in \mathbb{R} \mid a \leq x \leq b \}$$

$\ell \in \mathbb{R}$ IS A LOW.-B. FOR $[a, b]$

$$\Updownarrow$$

$$\ell \leq x \quad \forall x \in [a, b]$$

$$\Updownarrow$$

$$\ell \leq a \leq x \quad \forall x \in [a, b]$$

EXAMPLE • $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ ARE UNBOUNDED

• $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ NOT BOUNDED FROM ABOVE / BELOW

• $a < b \in \mathbb{R}$ $[a,b], [a,b[,]a,b],$ $]a,b[$ ARE ALL BOUNDED.

④ DEFINE $S \doteq [0,\pi] \cap \mathbb{Q}$. IS IT BOUNDED? [POLL]

YES

$S' = [0,\pi]$ BOUNDED

$\cup$

S

0 IS A L.B. FOR S' $\Rightarrow$ 0 IS A L.B FOR S

π IS U.B FOR S' $\Rightarrow$ π IS AN U.B FOR S

MAX/MIN & INF/SUP

DEF $S \subseteq \mathbb{R}$, $a \in \mathbb{R}$.

(1) a IS THE MINIMUM FOR S IF

$a \in S$ AND a IS AN LOWER BOUND FOR S

(1) a IS THE MAXIMUM FOR S IF

$a \in S$ AND a IS AN UPPER BOUND FOR S

MAX/MIN & INF/SUP

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NOTATION: $\min(S)$, $\max(S)$

$\min S$ $\qquad \max S$

EXAMPLE $S \doteq [0,1[$

lower bounds of $S =]-\infty, 0] \doteq \mathbb{R}_-$

upper ———"——— $= [1, +\infty[$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ "

$\{x \in \mathbb{R} \mid x \geq 1\}$

max S DOES NOT EXIST.

min $S = 0$

PROPOSITION

PROP $S \subseteq \mathbb{R}$, $S \neq \emptyset$.

- IF min S & max S EXIST, THEY ARE UNIQUE.
- IF S IS NOT BOUNDED FROM ABOVE, THEN max S DOES NOT EXISTS. no upper bounds
- IF S IS NOT BOUNDED FROM BELOW, THEN min S DOES NOT EXISTS.
- IF S IS A BOUNDED INTERVAL AND $a < b$ ARE THE EXTREMES, THEN

$$\max S = b \overset{\text{iff}}{\iff} b \in S$$

$$\min S = a \overset{\text{iff}}{\iff} a \in S.$$

$[0, +\infty[\; =: S$

0 is a lower bound

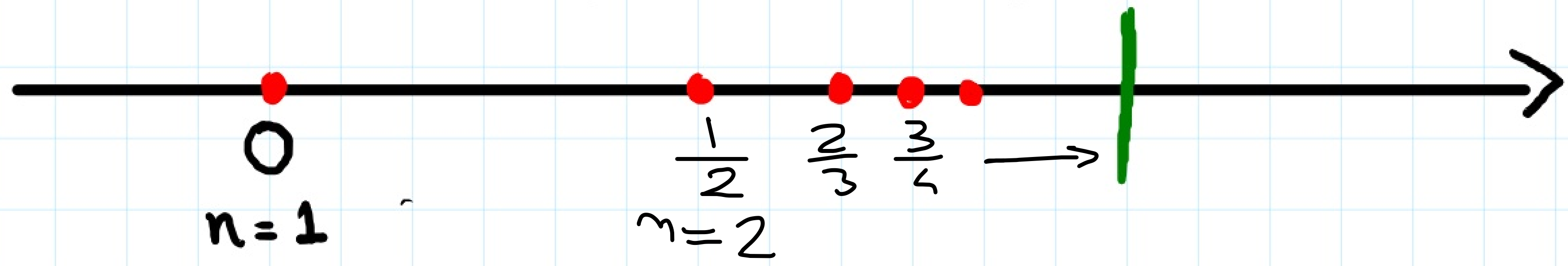
$\min S = 0$

S not bounded from above

$\Longrightarrow$ max S does not $\exists$

$\mathbb{R}$

③ $S = \left\{ 1 - \frac{1}{n} \mid n \in \mathbb{N}^* \right\}$



$$\frac{n-1}{n} < 1 \iff n-1 < n$$

S IS BOUNDED

$\forall s \in S \quad \boxed{0 \leq s < 1}$

L.B. U.P

max S does not exist.

$\min S = 0$

Suppose $y \in S$. $y = \max S$

$$y = \frac{\bar{n} - 1}{\bar{n}} , \ \exists\, \bar{n} \in \mathbb{N}^*$$

$$S \ni \frac{(\bar{n}+1) - 1}{(\bar{n}+1)} > \frac{\bar{n} - 1}{\bar{n}} = y$$

↯

WHAT CAN WE SAY IF $\nexists$ MAX/MIN?

DEF $S \subseteq \mathbb{R}$, $a \in \mathbb{R}$. $S \neq \emptyset$.

① a IS THE SUPREMUM OF S IF

a IS THE SMALLEST UPPER BOUND FOR S.

② a IS THE INFIMUM OF S IF

a IS THE LARGEST LOWER BOUND FOR S.

$\{x \in \mathbb{R} \mid x \text{ IS AN U.B. FOR } S\}$

$\sup S =$ smallest element

$\inf S =$ largest element of $\{x \in \mathbb{R} \mid x \text{ IS A L.B FOR } S\}$

WHAT CAN WE SAY IF $\nexists$ MAX / MIN?

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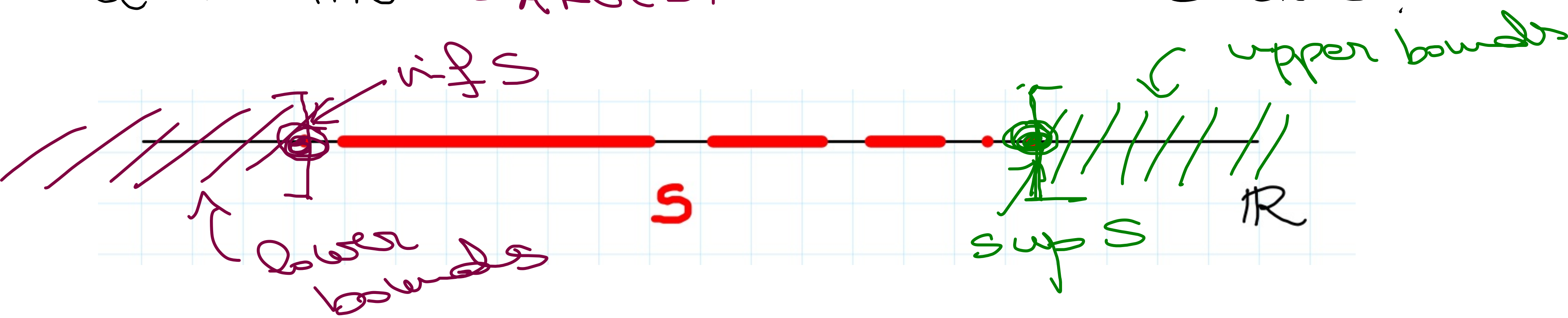


WHAT CAN WE SAY IF $\nexists$ MAX / MIN ?

DEF $S \subseteq \mathbb{R}$, $a \in \mathbb{R}$.

a IS THE SUPREMUM / INFIMUM OF S IF

a IS THE SMALLEST UPPER / LARGEST LOWER BOUND FOR S.

EXAMPLE ① $]a, b] = S$ $\quad a < b$

$\inf S = a$ $\quad$ min S IS NOT ~~DEF~~ DEF.

$\sup S = b = \max S$

lower bounds of $S =]-\infty, a]$

EXAMPLE

$$S \doteq \left\{ \frac{n-1}{n} \;\middle|\; n \in \mathbb{N}^* \right\}$$

$$\frac{n-1}{n} = 1 - \frac{1}{n}$$

$$\min S = 0$$

$$\text{lower bounds of } S = \left]-\infty, 0\right]$$

$$\sup S = 1$$

$\sup S = 1$ ⟶ TO PROVE THIS WNP THAT $\forall x < 1$ x IS NOT AN U.B FOR S

EARLIER: 1 IS AN U.B

$$\{\text{upper bounds of } S\} \supseteq [1, +\infty[$$

$x < 1$

$1 - x = c > 0$

Want to show that $\exists\, n \in \mathbb{N}^*$ s.t.

$$x < \underbrace{1 - \frac{1}{n}}_{\in S} < 1$$

$$1 - \frac{1}{n} > x \iff \frac{1}{n} < 1 - x = c > 0$$

$$\Updownarrow$$

TRUE BY ARCH. PROP. $\longrightarrow$ $n > \frac{1}{c} > 0$

PROP $S \subseteq \mathbb{R}$, $S \neq \emptyset$.

- IF $\inf S$ OR $\sup S$ EXISTS, IT IS UNIQUE.
- IF S IS NOT BOUNDED FROM ABOVE / BELOW THEN $\sup S$ / $\inf S$ DOES NOT EXISTS.
- $\max S$ EXISTS $\iff$ $\sup S$ EXISTS & $\sup S \in S$

  $\max S = \sup S$
- $\min S$ EXISTS $\iff$ $\inf S$ EXISTS & $\inf S \in S$

  $\min S = \inf S$
- IF S IS A BOUNDED INTERVAL AND $a < b$ ARE THE EXTREMES, THEN

  $\sup S = b$, $\inf S = a$

NOTATION: $S \neq \emptyset$

IF $\sup S$ DOES NOT EXIST

WE WRITE $\sup S = +\infty$

IF $\inf S$ DOES NOT EXIST

$\inf S = -\infty$

DOES THE INFIMUM EXIST?

DOES THE INFIMUM EXIST?

ANSWER: INFIMUM AXIOM GUARANTEES ITS EXISTENCE

DOES THE INFIMUM EXIST?

ANSWER: INFIMUM AXIOM GUARANTEES ITS EXISTENCE

INFIMUM AXIOM

IF $S \subseteq \mathbb{R}^{*}_{+}$, THEN $\inf S$ EXISTS

DOES THE INFIMUM EXIST?

ANSWER: INFIMUM AXIOM GUARANTEES ITS EXISTENCE

INFIMUM AXIOM

IF $S \subseteq \mathbb{R}_+^*$ THEN $\inf S$ EXISTS

IMPORTANT : AXIOM NOT TRUE FOR $\mathbb{Q}_+^*$

$S \doteq \left]-\sqrt{3}, \sqrt{3}\right] \cap \mathbb{Q} \subseteq \mathbb{Q}$

WE CANNOT FIND $\inf S$, $\sup S$ in $\mathbb{Q}$

COR LET $S \subseteq \mathbb{R}$, $S \neq \emptyset$.

COROLLARY OF THE INFIMUM AXIOM.

① IF S IS BOUNDED FROM BELOW THEN inf S EXISTS IN $\mathbb{R}$.

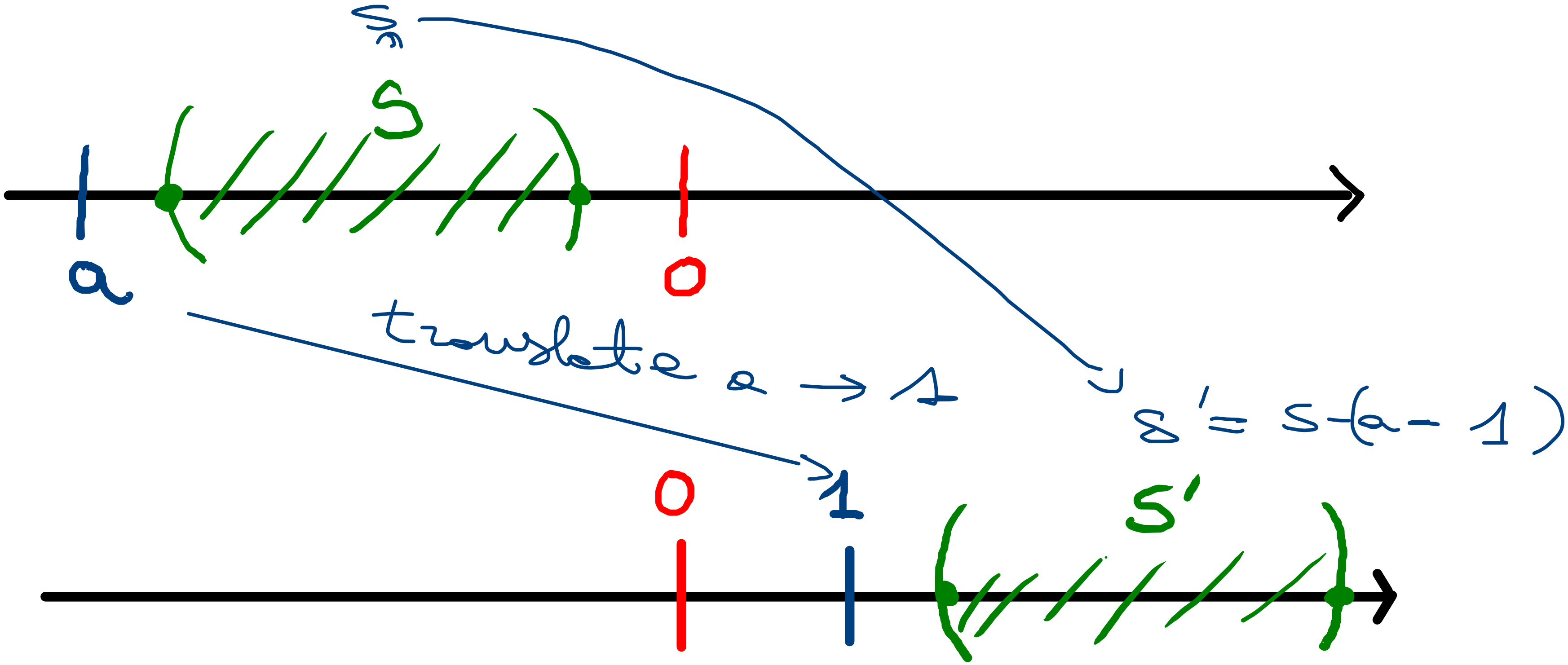
② IF S IS BOUNDED FROM ABOVE THEN sup S EXISTS IN $\mathbb{R}$.

$$S \xrightarrow{\cdot(-1)} S'$$

$$s \longmapsto -s$$

BOUNDED BELOW.

$$\Longrightarrow \inf S' \; \exists \qquad \sup(S) = -\inf(S')$$



IF $a > 0 \implies S \subseteq \mathbb{R}_+^* \xRightarrow{\text{l.A.}}$:)

$S' \subseteq \mathbb{R}_+^* \xRightarrow{\text{l.A.}} \inf(S')$ exists

$\inf(S)$ exists $= \inf(S') + (a-1)$