ANALYSIS 1

(ENGLISH)

LECTURE 25

THM [FUNDAMENTAL THM OF CALCULUS]

$$f : [a, b] \longrightarrow \mathbb{R} \quad . \quad C^0$$

THEN $G(\underline{\underline{x}}) = \int_a^{\overline{\overline{x}}} f(t)\,dt$ IS AN ANTI-DERIVATIVE OF f

PROOF $G(x) \overset{\cdot}{\underset{\cdot}{=}} \int_a^x f(x)\,dx$

NEED TO SHOW: $G'(x_0) = f(x_0) \quad \forall x_0 \in \underline{\underline{]a,b[}}$

NEED TO SHOW: $G'(x_0) = f(x_0)\ \forall x_0 \in \,]a,b[$

$$\lim_{x \to \overset{x}{0}} \frac{G(x)-G(x_0)}{x-x_0} = \lim_{x \to \overset{x}{0}} \frac{\int_{x_0}^{x} f(\overset{t}{\cancel{x}})\, d\overset{t}{\cancel{x}}}{x-x_0} =$$

$$\overset{||}{\lim_{h\to 0}} \frac{G(x_0+h)-G(x_0)}{h} \qquad \overset{||}{\lim_{h\to 0}} \boxed{\frac{\int_{x_0}^{x_0+h} f(t)\,dt}{h}} \qquad e_h \in [x_0, x_0+h]$$

MVT for $\int$'s

$$C^0([c,d]) \longrightarrow \frac{\int_c^d f(t)\,dt}{(d-c)} = f(e) \text{ for some } e \in [c,d]$$

$$\lim_{h\to 0} f(e_h) = f(x_0)$$

THM [FUNDAMENTAL THM OF CALCULUS, 2nd EDITION]

$f : [a,b] \longrightarrow \mathbb{R}$. C^0

ASSUME $G : [a,b] \longrightarrow \mathbb{R}$ IS AN ANTI-DERIVATIVE OF f

THEN $\int_a^b f(x)\,dx = G(b) - G(a)$

PROOF

$$\underbrace{G(x)}_{\text{ANTIDER.}} = \underset{\in\mathbb{R}}{C} + \underbrace{\int_a^x f(t)\,dt}_{1^{\text{st}}\text{ VERS}} \leftarrow \text{ANTIDER. of } f$$

$$\Rightarrow \quad G(b) - G(a)$$

$$\cancel{C} + \int_a^b f(t)\,dt \quad - \quad \left[\cancel{C} + \cancel{\int_a^a f(t)\,dt}\right]$$

CONVENTION last time

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NOTATION:

$$\int_a^b f(\underset{\mathbb{R}}{\underset{\cap}{t}})\,dt \leftarrow \text{"DEFINED INTEGRAL"}$$

ANTIDERIVATIVES:

$$\boxed{\int f(x)\,dx} \leftarrow \text{ANTIDERIVATIVE for } f$$

$$\int f(x)\,dx \quad + \quad c$$

$f \in C'(\mathbb{R}) \rightsquigarrow f$ is DIFF. EVERYW.

& f' is $C^0(\mathbb{R})$

f is ANTIDER. of f' over any CLOSED, BOUNDED INTERVAL $[a, b]$

$$f(x) - f(a) \underset{\uparrow}{=} \int_a^x f'(t)\,dt$$

FTC 2nd vers.

IF $f \in C^1(\mathbb{R})$ let's assume

that $\underline{\underline{|f'(x)| \le 1}}$ on $[a, b]$

$$|f(b) - f(a)| = \left|\int_a^b f'(t)\,dt\right|$$

$$\wedge| \rightarrow \left|\int_a^b \quad\right| \le \int_a^b |\quad|$$

$$\int_a^b |f'(t)|\,dt$$

$$\wedge| \longleftarrow 0 \le h \le g \Rightarrow \int_a^b h \le \int_a^b g$$

$$b - a = \int_a^b 1\,dt$$

$$\xRightarrow{|f'(x)| \le 1} \quad |f(b) - f(a)| \le |b - a|$$

IF $|f'(x)| \le \overset{\text{on } [a,b]}{C} \implies$ SAME PROOF SHOWS

THE

$$|f(b) - f(a)| \le C \cdot |b - a|$$

$$(*) \quad |f(x) - f(y)| \le C \cdot |x - y|$$

$$\forall x, y \in [a, b]$$

$\implies f$ IS LIPSCHITZ W/ CONSTANT C on $[a, b]$

IF YOU ASSUME $f \in C^1(\mathbb{R})$

$\rightsquigarrow f' \in C^0(\mathbb{R})$

$$f'|_{\underset{\substack{\uparrow \text{ closed} \\ \text{bounded}}}{[a,b]}} : [a,b] \longrightarrow \mathbb{R}$$

$$\begin{tikzcd} {|f'(t)| \le C_{a,b}} \arrow[d, equal] & {R(f'|_{[a,b]})} \arrow[l, Rightarrow] \arrow[d, equal] \\ {|f'(t)| \le \max\{|m|, |M|\}} & {[m, M]} \arrow[l, Rightarrow] \end{tikzcd}$$

over $[a,b]$

f is OF CLASS C^1 on $\mathbb{R}$

$\Longrightarrow$ f is LIPSCHITZ

on any $\boxed{\text{closed bounded}}$ interval

$f(x) = x^2$ is not lipschitz

$$\left|f(\overset{a}{n}) - f(\overset{b}{n+1})\right| \boxed{=} \left|n^2 - (n+1)^2\right|$$

$$= \underbrace{\boxed{|n + n + 1|}}_{\infty \leftarrow} \cdot \left|\underset{a}{n} \underset{-}{-} \underset{b}{(n+1)}\right|$$

$f : \mathbb{R} \to \mathbb{R}$ $\boxed{\text{differentiable } [f' \ \exists \text{ everywhere}]}$

$\boxed{\text{if } |f'(x)| \le C}$ $\forall x \in \mathbb{R}$

f is LIPSCHITZ (w/ constant C)

$$\frac{f(a) - f(b)}{a - b} \overset{\underline{\underline{MVT}}}{=} f'(c) \text{ for some } c \in (a,b)$$

$$\Downarrow$$

$$|f(a) - f(b)| = |f'(c)| \, |a - b| \le C|a - b|$$

$f : [0,1] \longrightarrow \mathbb{R}$

let's assume that $f \in C^1$

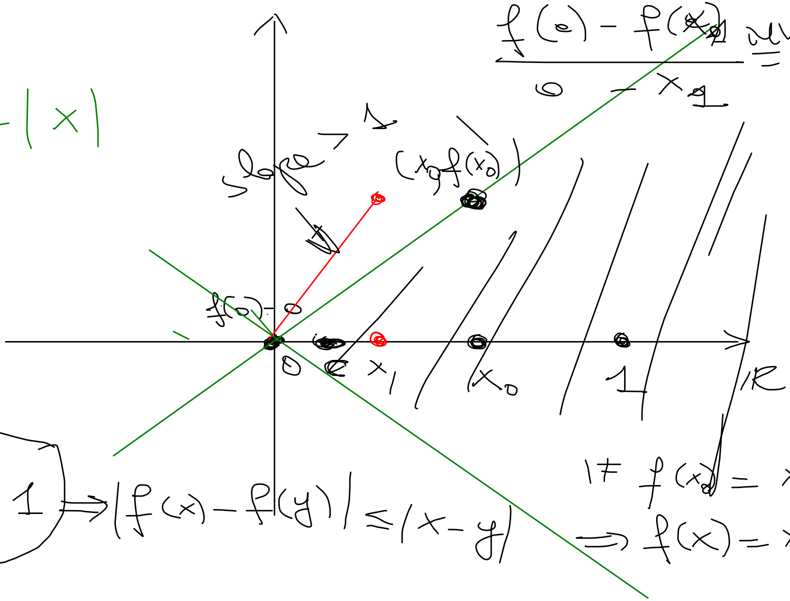
& $|f'(x)| \le 1 \quad \forall x \in (0,1)$

$f(0) = 0$

$$f(x) = f(x) - f(0) = \int_0^x f'(t)\,dt$$

$|f(x)| \le |x|$

$$\frac{f(0)-f(x_1)}{0-x_1} \overset{MVT}{=} f'(c) \le 1$$



$$\boxed{|f'| \le 1} \Rightarrow |f(x)-f(y)| \le |x-y|$$

If $f(x_0) = x_0$

$\Rightarrow f(x) = x$

TECHNIQUES OF COMPUTATION:

1. INTEGRATION BY PARTS

LEIBNIZ FORMULA: $(fg)' = f'g + g'f$

$$\int_a^b (fg)'(t)\,dt = \int_a^b (f'g)(t)\,dt + \int_a^b (fg')(t)\,dt$$

$$\overset{\text{FTC}}{=}\ \boxed{(f\cdot g)(b) - (f\cdot g)(a)}$$

F(x) IS ANTIDER. FOR F'(x)

$f : \mathbb{R} \to \mathbb{R}$, $f \in \underline{C^1(\mathbb{R})}$

$$\int_a^b \underline{\underline{f'(x)}} \, dx$$

$$\left.\begin{array}{l} f|_{[a,b]} : [a,b] \longrightarrow \mathbb{R} \qquad C^0 \\ f \text{ is DIFF. on } (a,b) \\ f'(x) \end{array}\right\}$$

$$\Downarrow$$

f is AN ANTIDER. for f'

INTEGRATION BY PARTS

$g, f : E \to \mathbb{R}$ DIFF.

$[a, b] \subseteq E$

$$\underline{\underline{\int_a^b f(t)\,g'(t)\,dt}}$$

$$\|$$

$$\underline{\underline{f(b)g(b) - f(a)g(a) - \boxed{\int_a^b f'(t)\,g(t)\,dt}}}$$

NOTATION :

$$\underline{\underline{\int f g'}} = \overset{= fg + C}{\int (fg)'} - \int f' g = \underline{\underline{fg + C - \int f' g}}$$

INTEGRATION BY PARTS

$g, f : E \longrightarrow \mathbb{R}$ DIFF.

$[a, b] \subseteq E$

$$\int_a^b \underline{\underline{f(t)g'(t)\,dt}}$$

$$\parallel$$

$$f(x)\,g(x)\Big|_{x=a}^{x=b} - \int_a^b \underline{\underline{f'(t)g(t)\,dt}}$$

NOTATION:

$$h(x)\Big|_{x=a}^{x=b} := h(b) - h(a)$$

EXAMPLE

$$\int_a^b \underline{\underline{x e}}^x \, dx$$

FORMULA: $\displaystyle\int_a^b f(x)\, g'(x)\, dx = \boxed{(f \cdot g)(x)\Big|_{x=a}^{x=b}} - \int_a^b \underline{\underline{f'(x)\, g(x)\, dx}}$

$f(x) = x$

$f'(x) = 1$

$g'(x) = e^x$

$g(x) = e^x$

$$\int_a^b 1 \cdot e^x \, dx$$

$''$

$e^b - e^a$

EXAMPLE

$$\int_a^b x e^x \, dx$$

FORMULA: $\int_a^b f(x)\, g'(x)\, dx = (f \cdot g)(x) \Big|_{x=a}^{x=b} - \int_a^b f'(x)\, g(x)\, dx$

$f = x \qquad f' = 1$

$g = e^x \qquad g' = e^x$

$$\int_a^b x e^x = x e^x \Big|_{x=a}^{x=b} \ominus \int_a^b \underline{\underline{1 \cdot e^x}}\, dx = b e^b - a e^a - \left(e^b - e^a\right)$$

EXAMPLE

$$\int_a^b \sin(x) e^x \, dx =$$

FORMULA: $$\int_a^b f(x)\, g'(x)\, dx = (f \cdot g)(x) \Big|_{x=a}^{x=b} - \int_a^b f'(x)\, g(x)\, dx$$

$f(x) = \sin(x)$ $\quad g'(x) = e^x$

$f'(x) = \cos(x)$ $\quad g(x) = e^x$

$$\int_a^b \underline{\underline{\sin x\, e^x}} = \sin x\, e^x \Big|_{x=\cancel{b}a}^{x=b} - \int_a^b \underset{f}{\cos x}\ \underset{g'}{e^x}$$

$$\int_a^b \cos x\ e^x = \cos x\, e^x \Big|_{x=a}^{x=b} + \int_a^b \sin x\, e^x$$

$$\underline{\underline{\int_a^b \sin x\, e^x}} = \sin x\, e^x \Big|_a^b - \cos x\, e^x \Big|_a^b \boxed{\boxed{-}\int_a^b \underline{\underline{\sin x\ e^x}}}$$

$$\Downarrow$$

$$\cancel{2}\int_a^b \sin x\, e^x = \frac{\sin x\, e^x \Big|_a^b - \cos x\, e^x \Big|_a^b}{2}$$

EXAMPLE

$$\int_a^b \underset{\underset{g'}{\uparrow}}{\sin(x)} \; \underset{\underset{f}{\uparrow}}{x} \; dx =$$

FORMULA: $$\int_a^b f(x)\, g'(x)\, dx = (f \cdot g)(x)\Big|_{x=a}^{x=b} - \underline{\underline{\int_a^b \overset{\overset{1}{\parallel}}{f'(x)}\, \overset{(-\cos(x))}{g(x)}\, dx}}$$

$$\int_a^b x \sin x = -x\cos(x)\Big|_{x=a}^{x=b} + \int_a^b \cos(x)$$

$$\parallel$$

$$-x\cos(x)\Big|_{x=a}^{x=b} + \sin(b) - \sin(a)$$

HARD PART: HOW DO WE CHOOSE f, g' ?

RECIPE THAT WORKS OFTEN:

$\underset{g'}{e^x}\;\underset{f}{\sin x}$

- EXPONENTIALS
- TRIGONOMETRIC FCTS
- POLYNOMIALS & RAT'L FCTS
- LOGARITHMS
- INVERSE FCTS OF TRIGONOMETRIC FCTS

TECHNIQUES OF COMPUTATION:

2. SUBSTITUTION

$$\left(F\left(\psi(t)\right)\right)' = F'\left(\psi(t)\right)\cdot\psi'(t)$$

$$\int_0^1 \tan(t)\,dt = -\int_0^1 \frac{(-\sin(t))}{\cos(t)}\,dt$$

$$F'\cancel{\ln}(\cos t)\cdot(\cos(t))'$$

$$\cos(t)' = -\sin(t)$$

$$F'\cancel{\ln}(x) = \frac{1}{x}$$

$$F'(x) = \frac{1}{x} \implies F(x) = \log(|x|)$$

$$\frac{-\sin(t)}{\cos(t)} = F'(\cos(t)) \cdot (\cos'(t))$$

$$-\int_0^1 \left(\log(|\cos(t)|)\right)' \, dt \overset{\text{FTC}}{=} -\left(\log(|\cos(1)|) - \log(|\cos(0)|)\right)$$

$$\parallel$$

$$\int_0^1 \tan(t) \, dt$$

THM $f : [a, b] \longrightarrow \mathbb{R}$ C^0

← INTERVAL

ASSUME $\phi : E \longrightarrow \mathbb{R}$ $C^1(E)$

TAKE $\alpha < \beta \in E$ S.T. $\phi([\alpha, \beta]) \subseteq [a, b]$

THM

$f : [a, b] \to \mathbb{R}$ $\quad C^0$

$\varphi : E \to \mathbb{R}$ $\quad C^1$

TAKE $\alpha < \beta \in E$ S.T. $\varphi([\alpha, \beta]) \subseteq [a, b]$

THEN

$$\int_{\alpha}^{\beta} f(\varphi(x)) \boxed{\varphi'(x)\, dx} = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(t) \boxed{dt}$$

$t = \varphi(x)$

PROOF

LET G BE AN ANTIDERIVATIVE OF f

$$\int_{\alpha}^{\beta} f(\varphi(x))\,\varphi'(x)\,dx = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(t)\,dt$$

EXAMPLE

$$\int_0^1 e^{\frac{3}{2}x}\,dx$$

$$\int e^t = e^t + C$$

$$\underline{\underline{t = \frac{3}{2}x = \varphi(x)}}$$

$$dt = \varphi'(x)\,dx$$

$$\downarrow$$

$$= \frac{3}{2}\,dx$$

$$* \ \frac{2}{3}\,dt = \boxed{dx}$$

$$\int_{x=0}^{x=1} e^{\frac{3}{2}x}\,dx = \int_{\varphi(0)=0}^{\varphi(1)=\frac{3}{2}} e^t \,\frac{2}{3}\,dt$$

$$= \frac{2}{3}\left(e^{\frac{3}{2}} - e^0\right)$$

EXAMPLE

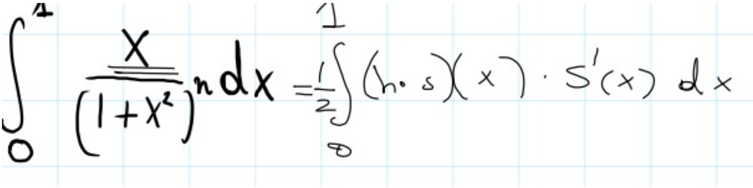
$$\int_0^1 \frac{x}{(1+x^2)^n}\,dx = \frac{1}{2}\int_0^1 (h\circ s)(x)\cdot s'(x)\,dx$$

$$\frac{1}{2}\,\frac{2x}{(1+x^2)^n} \qquad \frac{1}{(1+x^2)^n}$$

$$(h\circ s)(x)$$

$$s(x) = 1+x^2 \qquad 2x = s'(x)$$

$$h(t) = \frac{1}{t^n}$$



$$\int_0^1 \frac{X}{(1+X^2)^n} dx = \frac{1}{2}\int_0^1 (h\circ s)(x)\cdot s'(x)\, dx$$

$$s(x) = 1 + x^2 \qquad h(t) = \frac{1}{t^n}$$

$$t = s(x)$$

$$\int_0^1 \underbrace{\frac{x}{(1+x^2)^n}}_{h\circ s(x)\cdot s'(x)} dx = \frac{1}{2}\int_{\varphi(0)=1}^{\varphi(1)=2} t^{-n}\, dt = -\frac{1}{2(n-1)} t^{-(n-1)}\Big|_1^2$$

ANTIDER $t^{-n} = \frac{-1}{n-1} \; t^{-(n-1)}$

EXAMPLE

$$\int_{\varphi(\alpha)=0}^{\varphi(\beta)\,1} \sqrt{\underset{\substack{\| \\ f(t)}}{\underline{\underline{1-t^2}}}}\,dt = \int_{\pi/2}^{0} f(\varphi(x))\,\varphi'(x)\,dx$$

$$\cos^2(x) + \sin^2(x) = 1$$

$$t = \varphi(x) = \cos(x) \qquad \varphi'(x) = -\sin(x)$$

$$= \int_{\alpha=\pi/2}^{\beta=0} \underset{\|}{\boxed{\sqrt{1-\cos^2(x)}}}\,(-\sin(x))\,dx$$

$$\sqrt{\sin^2(x)} = |\sin(x)| = \sin(x) \quad \text{on } [0, \pi/2]$$

$$= \int_0^{\pi/2} \sin^2(x)\,dx = \int_0^{\pi/2} \frac{1-\cos(2x)}{2}\,dx$$

$$\cos(2x) = \underset{\substack{\| \\ 1-\sin^2(x)}}{\cos^2(x)} - \sin^2(x) = 1 - 2\sin^2(x)$$

$$\int^{\pi/2} \frac{1-\cos(2x)}{2}\,dx$$

〃

$$\int_0^{\pi/2} \frac{1}{2}\,dX \;\boxed{-}\; \int_{x=0}^{x=\pi/2} \frac{\cos(2x)}{2}\,dx$$

$$\|$$

$$\frac{1}{2}\cdot\frac{\pi}{2}$$

$$w = 2x$$

$$\frac{1}{2}\,dw = \cancel{2}\,dx$$

$$= \frac{1}{\cancel{2}4}\int_0^{\pi} \cos(w)\,\cancel{\frac{1}{2}}\,dw$$

EXAMPLE

$$\int_0^1 \sqrt{1+t^2}\,dt$$

$\cosh^2 - \sinh^2 = 1$

$1 + \sinh^2 = \cosh^2$

$t = 0 \Longleftarrow x = 0$

$t = \varphi(x) = \sinh(x) \qquad \varphi'(x) = \cosh(x)$

$dt = \cosh(x)\,dx$

$$= \int_0^{\sinh^{-1}(1)} \sqrt{1+\sinh^2(x)}\,\cosh(x)\,dx$$

$\parallel$

$$\downarrow \qquad \swarrow$$

$$\int (\cosh x)^2 \cancel{\cosh}\,dx = \int \frac{e^{2x}+e^{-2x}+2}{4}\,dx$$

$\parallel$

$$\frac{e^x+e^{-x}}{2}$$