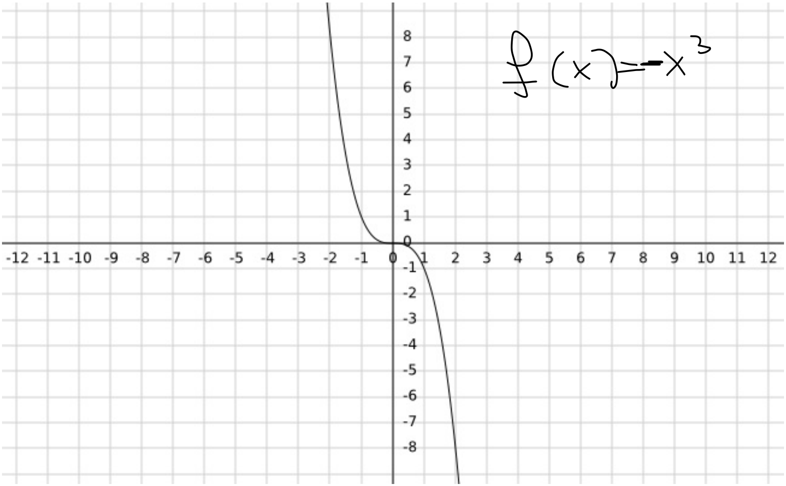
ANALYSIS 1

(ENGLISH)

LECTURE 23

TAYLOR EXPANSION & LOCAL EXTREMA

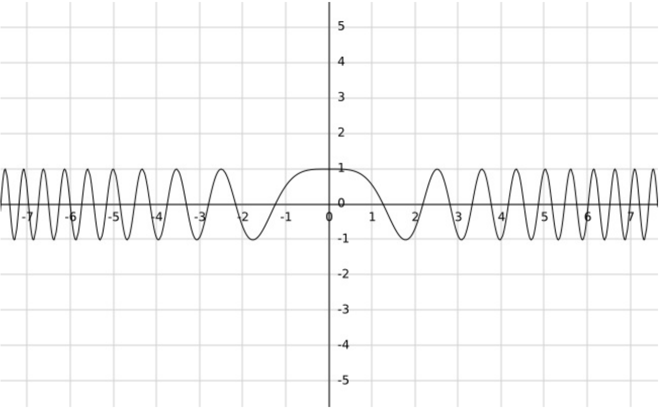


$f(x) = -x^3$

VICEVERSA NOT TRUE

• If f has a local max/min at a pt x_0

$\Longrightarrow f'(x_0) = 0$



$$f(x) = \cos(x^2)$$

0 is a stat. pt

$$f'(x) = -\sin(x^2)(2x)$$

$$f'(0) = 0$$

INTUITION: 0 is a pt at which f has a local max. $\leftarrow$ TRUE around $x_0 = 0$

$$\cos(y) = 1 - \frac{y^2}{2} + \frac{y^4}{4!} - \dots \qquad 0 \le |x^8| \ll |x^4|$$

$$\cos(x^2) = 1 - \frac{x^4}{2} + \frac{x^8}{4!} - \dots$$

$\uparrow$ FOR $x \ll 1$

$f :]a, b[\to \mathbb{R}$, $x_0 \in]a, b[$

ASSUME THAT $f', f^{(2)}, \dots, f^{(n-1)}, f^{(n)}$ EXIST AROUND x_0 & $f'(x_0) = 0 = f^{(2)}(x_0) =$

$= f^{(3)}(x_0) = \dots = f^{(n-1)}(x_0) = 0$

$f^{(n)}(x_0) \neq 0$

$x \to x_0 \to 0$

$$f(x) = f(x_0) + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \underline{\varepsilon_n(x)(x - x_0)^n}$$

goes to 0 faster than $(x - x_0)^n$

$$f(x) \approx f(x_0) + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + (x-x_0)^n \varepsilon_n(x)$$

$\varepsilon_n(x) \to 0$ as $x \to x_0$

1. n EVEN around x_0:

$f^{(n)}(x_0) > 0 \Longrightarrow f(x) \geq f(x_0)$ around x_0

f has a local minimum at x_0

[for all x in some interval of the form $(x_0 - \delta, x_0 + \delta)$]

$f^{(n)}(x_0) < 0 \Longrightarrow f(x) \leq f(x_0)$ around x_0

f has a local maximum at x_0

2. n ODD around x_0:

$$(x - x_0)^n \begin{cases} > 0 & \text{when } x > x_0 \\ < 0 & \text{when } x < x_0 \end{cases}$$

INFLECTION PTS

if $f^{(n)}(x_0) > 0$

$f(x) \geq f(x_0)$ $\quad x > x_0$

$f(x) \leq f(x_0)$ $\quad x < x_0$

EXAMPLE

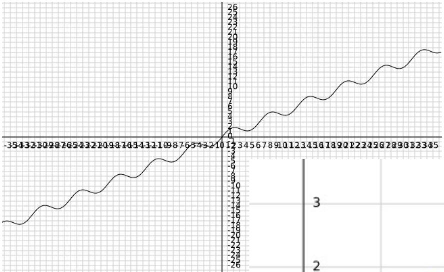
$$f(x) = \sin(x) + \frac{1}{2}x \qquad C^0$$

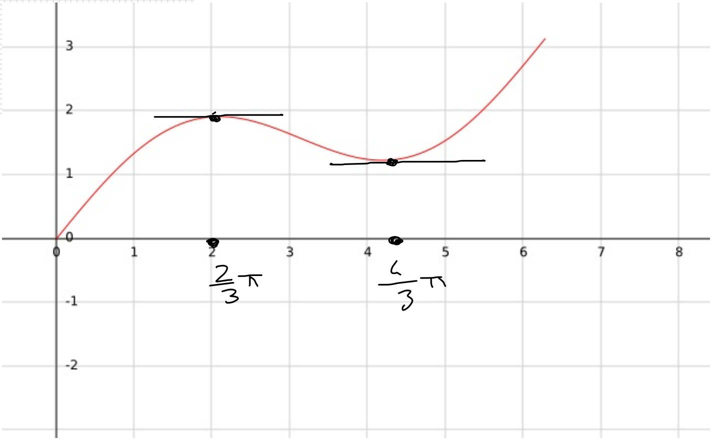
$$x \in [0, 2\pi]$$

pt. min

pt. max

$$R(f|_{[0,2\pi]}) = [m, M]$$





EXAMPLE

$$f(x) = \sin(x) + \frac{1}{2}x \qquad x \in [0, 2\pi]$$

$$M = \max_{[0,2\pi]} f \qquad\qquad m = \min_{[0,2\pi]} f$$

$$f'(x) = \cos(x) + \frac{1}{2}$$

STAT. PTS: $f'(x) = 0$ for $x \in [0, 2\pi]$

$$\cos(x) = -\frac{1}{2} \qquad x = \frac{2}{3}\pi, \frac{4}{3}\pi$$

$$f''(x) = -\sin(x) \implies f''\left(\tfrac{2}{3}\pi\right) < 0 \quad \text{LOCAL MAX}$$

$$f''\left(\tfrac{4}{3}\pi\right) > 0 \quad \text{LOCAL MIN}$$

$$M = \max\left\{f(0), f(2\pi), f(2/3\,\pi)\right\} = \cancel{\max}\left\{\cancel{0}, \pi, \frac{\sqrt{3}}{2} + \frac{1}{3}\pi\right\}$$

≈ 3 $\qquad \frac{\sqrt{3}}{2} \approx$ … $\approx 1.8 < \pi$

DEF $f :]a,b[\to \mathbb{R}$, $x_0 \in D(f)$

x_0 IS AN INFLECTION PT. IF

$\exists \alpha > 0$

FOR ALL $x \in]x_0, x_0+\alpha[\Rightarrow (x, f(x))$ IS ABOVE THE TANGENT LINE TO $(x_0, f(x_0))$

AND FOR ALL $\forall x \in]x_0-\alpha, x_0[\Rightarrow (x, f(x))$ IS BELOW THE TANGENT LINE TO $(x_0, f(x_0))$

[OR VICEVERSA]

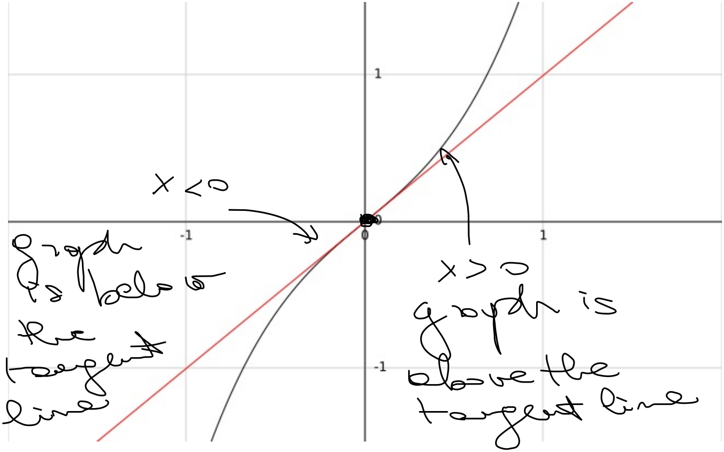
DEF $f :]a, b[\to \mathbb{R}$, $x_0 \in Dom(f)$

x_0 IS AN INFLECTION PT. IF $\exists\, \alpha > 0$

FOR $x \in]x_0, x_0 + \alpha[\Rightarrow (x, f(x))$ IS ~~ABOVE~~ BELOW THE TANGENT LINE TO $(x, f(x))$

AND $x \in]x_0 - \alpha, x_0[\Rightarrow (x, f(x))$ IS ~~BELOW~~ ABOVE THE TANGENT LINE TO $(x, f(x))$

[OR VICEVERSA]



$$f(x) = x + x^3$$

$$x_0 = 0$$

$$f'(0) = 1$$

$$y - f(0) = 1 \cdot (x - 0)$$

tangent line

TANGENT LINE $y = x$

X_0 IS AN INFLECTION PT.

PROP $f :]a,b[\to \mathbb{R}$, $x_0 \in D(f)$

ASSUME f HAS n DERIVATIVES ON $]a,b[$ $[f \in C^n((a,b))]$

ASSUME $f^{(n)}$ IS C^0 ~~AT x_0~~ &

$$f''(x_0) = f'''(x_0) = \dots = f^{(n-1)}(x_0) = 0$$

IF n IS ODD & $f^{(n)}(x_0) \neq 0$

$\Rightarrow$ x_0 is an inflection pt.

PROOF

ON THE RIGHT OF x_0

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \underbrace{(x-x_0)^n\,\varepsilon_n(x)}_{\text{ERR.}}$$

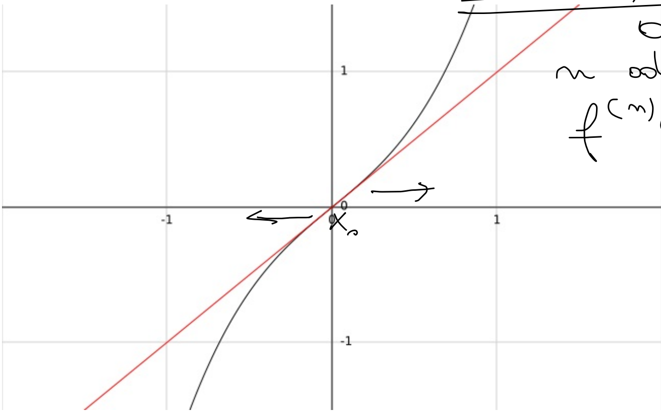
$0 \quad 0 \quad 0 \quad (x \to x_0)$

ON THE LEFT

0

n odd

$f^{(n)}(x_0) > 0$



STATIONARY PT $\left(f'(x_0)=0\right)$

$$n \doteq \min\left\{k \in \mathbb{N}^* \mid f^{(k)}(x_0) \neq 0\right\}$$

n even :

n : odd

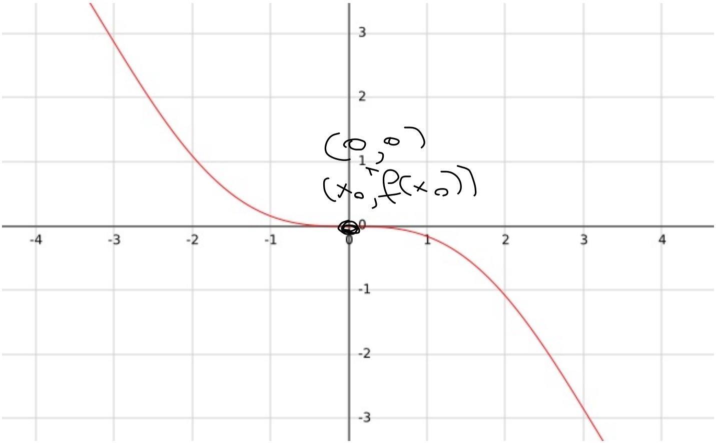
EXAMPLE

$f(x) = \sin(x) - x$ $[-3,3]$

$f \in C^{\infty}(\mathbb{R})$

$f'(x) = \cos(x) - 1 \Rightarrow f'(0) = 0$

STAT PT



WE EXPECT $x_0 = 0$ TO BE AN INFLECTION

$f''(x) = -\sin(x)$

$f''(0) = 0$

$f'''(x) = -\cos(x)$

$f'''(0) = -1 \neq 0$

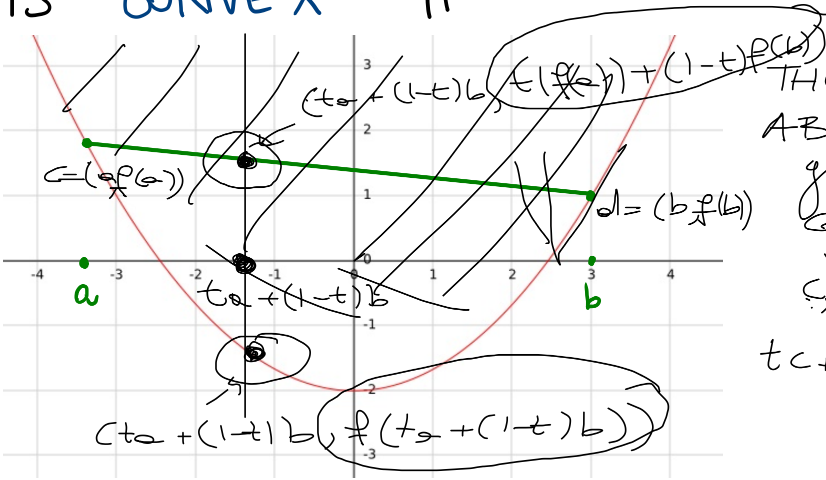
$\Downarrow$

INFL. PT.

CONVEX & CONCAVE FCTS

DEF $f : E \to \mathbb{R}$, E INTERVAL

IS CONVEX IF



THE AREA ABOVE graph(f) CONVEX

$\Updownarrow$

$c, d \in ////$

$tc+(1-t)d$

CONVEX & CONCAVE FCTS

DEF $f : E \to \mathbb{R}$, E INTERVAL

IS CONVEX IF $\forall a, b \in E$,

$a < b$,

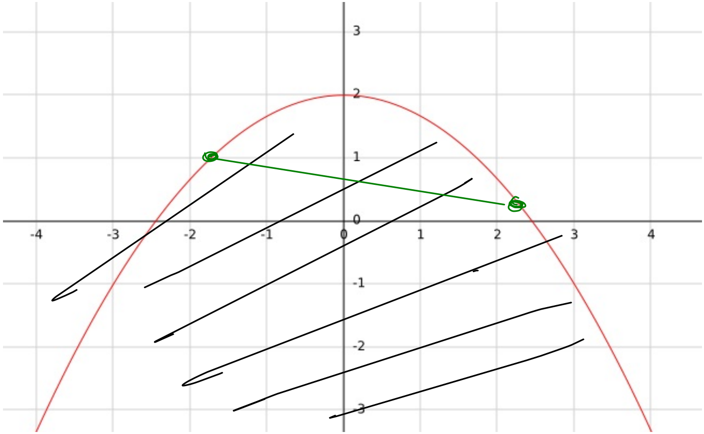
$$t f(a) + (1-t) f(b) \geq f(t \cdot a + (1-t) b)$$

$$\forall t \in [0,1]$$

CONVEX & CONCAVE FCTS

DEF $f : E \to \mathbb{R}$, E INTERVAL

IS CONCAVE IF



THE AREA BELOW THE GRAPH IS CONVEX

CONVEX & CONCAVE FCTS

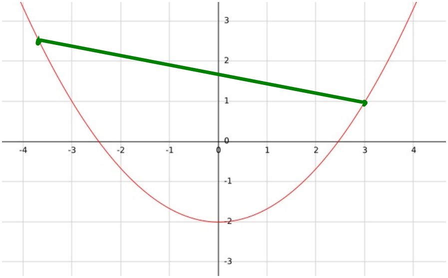
DEF $f : E \to \mathbb{R}$, E INTERVAL

IS CONCAVE IF $\forall a, b \in E$,

$a < b$,

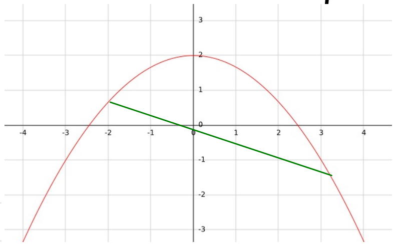
$$t f(a) + (1-t) f(b) \leq f(t \cdot a + (1-t) b)$$

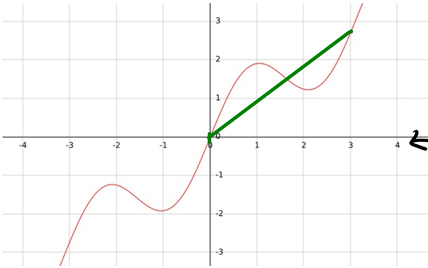
$$\forall t \in [0,1]$$



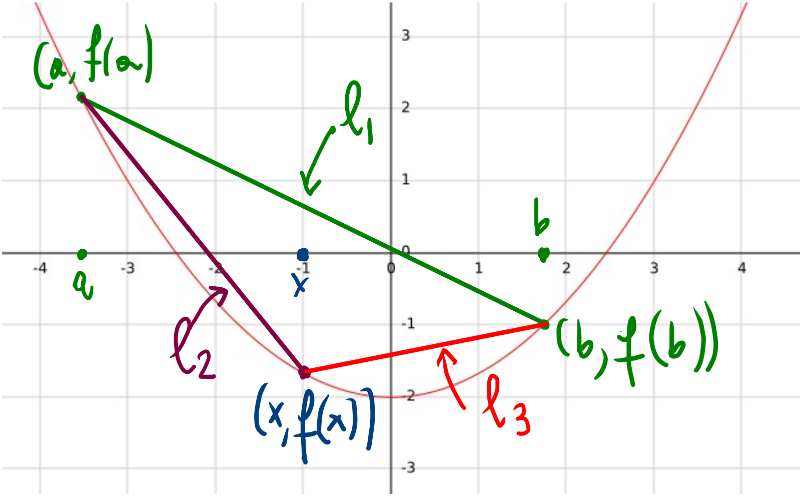
CONVEX

CONCAVE





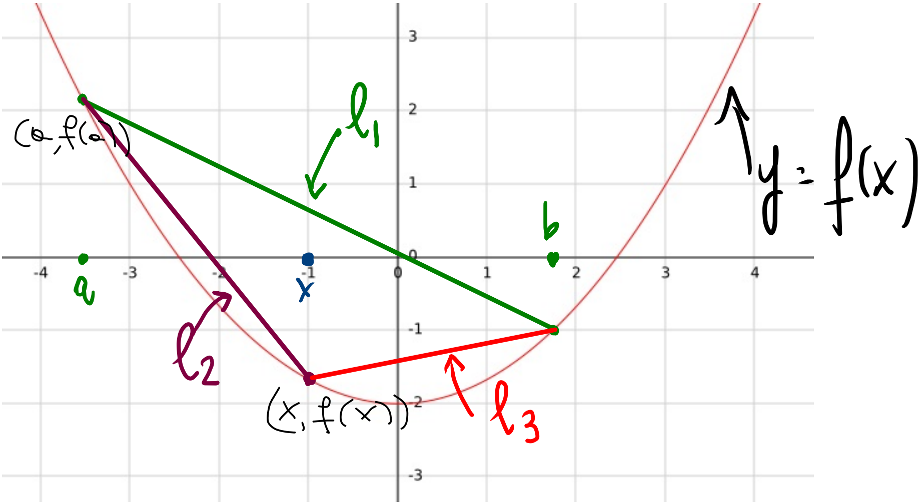
NEITHER CONC.
NOR CONV.



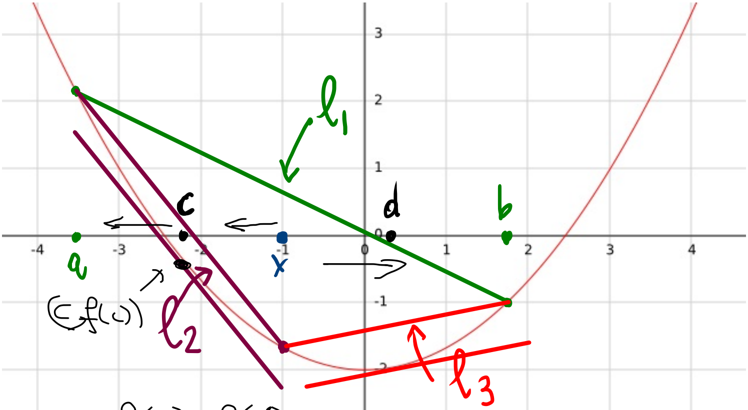
f convex

$f \in C^2$

$a < x < b$



$$\text{slope } \ell_2 \leq \text{slope } \ell_1 \leq \text{slope } \ell_3$$



f is C^2

$a < b$

$\Downarrow$

$f'(a) \le f'(b)$

$$\frac{f(a) - f(x)}{a - x} \qquad\qquad \frac{f(x) - f(b)}{x - b}$$

$$\text{slope } l_2 \le \text{slope } l_1 \le \text{slope } l_3$$

$\| \text{ MVT} \qquad\qquad \| \text{ MVT}$

$f'(c), \quad a < c < x \qquad\qquad f'(d), \quad x < d < b$

$c \to a, \quad x \to a \qquad\qquad x \to b, \quad d \to b$

$f'(c) \to f'(a) \qquad\qquad f'(d) \to f'(b)$

$$f'(a) \le \text{slope } l_1 \le f'(b)$$

PROP $f: E \to \mathbb{R}$, E OPEN INTERVAL

ASSUME THAT f IS DIFFERENTIABLE 2 TIMES $[f \in C^2(E)]$

THEN f IS CONVEX

PREVIOUS $\Downarrow$ $\Updownarrow$ $\Uparrow$

MONOTONE

f' IS INCREASING

$\Updownarrow$ THM DERIV. + MONOT.

$f'' \geq 0$

PROP $f : E \to \mathbb{R}$, E OPEN INTERVAL

ASSUME THAT f IS DIFFERENTIABLE 2 TIMES

THEN f IS CONCAVE

$$\Updownarrow$$

f' IS DECREASING

$$\Updownarrow$$

$$f'' \leq 0$$

IN GENERAL:

- f IS CONVEX $\iff -f$ IS CONCAVE

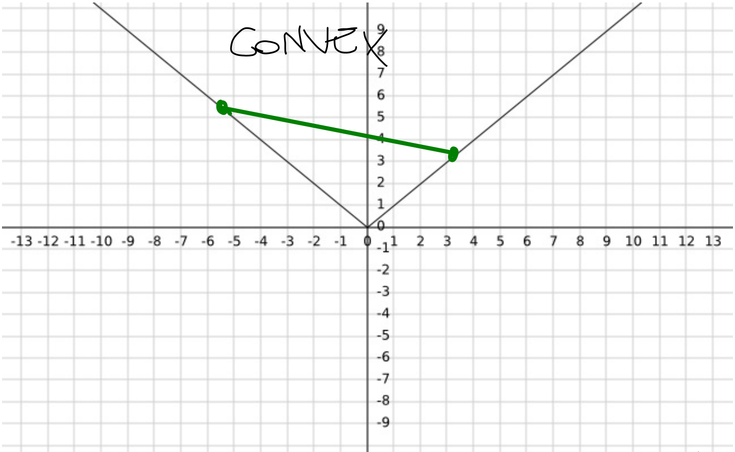
- f CONVEX / CONCAVE $\Rightarrow$ f IS C^0

- f CONVEX / CONCAVE $\Rightarrow$

  THE LEFT & THE RIGHT DERIV. EXIST AT ANY POINT.

$$\underset{\text{CONVEX}}{f'_+(x) \geq f'_-(x)} \quad \forall x \in \mathrm{Dom}(f)$$

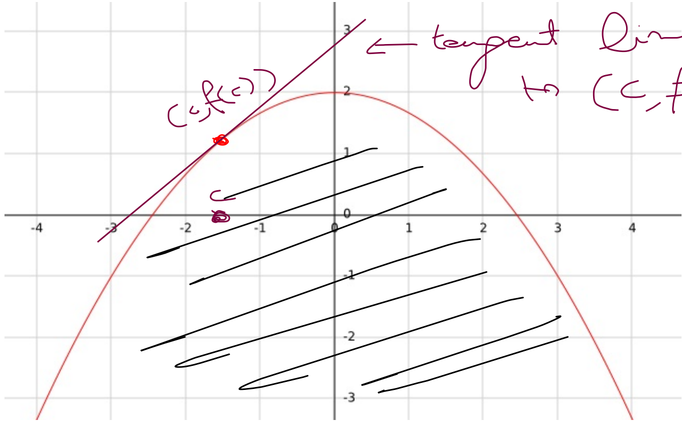
$f(x) = |x|$ CONVEX



f NOT DIFF. AT 0 $\Rightarrow \lim\limits_{x \to 0^+} \dfrac{f(x) - f(0)}{x - 0} = 1$

$x \to 0^-$ $\qquad = -1$

f is concave $\Longleftrightarrow$ graph is everywhere below the tangent line



concave

ASYMPTOTES

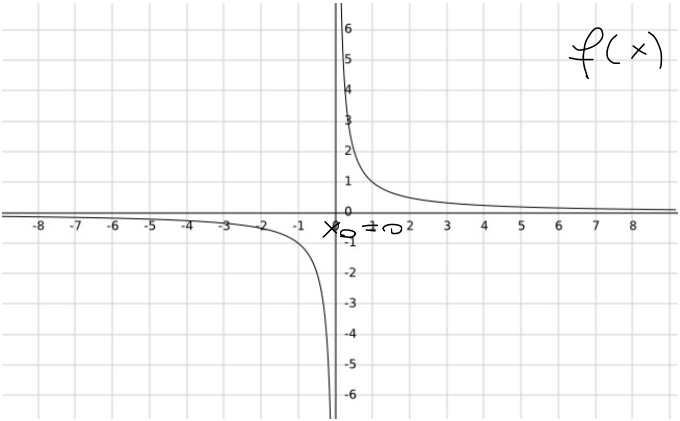
$f : E \to \mathbb{R}$

VERTICAL ASYMPTOTE :

ASYMPTOTES

VERTICAL ASYMPTOTE: $f : E \to \mathbb{R}$, $x_0 \in \mathbb{R}$

$$\lim_{x \to x_0^+} f(x) = \pm\infty = \lim_{x \to x_0^-} f(x)$$



$$f(x) = \frac{1}{x}$$

$$\lim_{x \to 0^+} f(x) = +\infty$$

$$\lim_{x \to 0^-} f(x) = -\infty$$

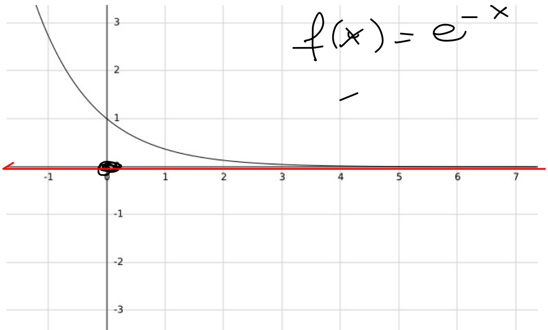
ASYMPTOTES

VERTICAL ASYMPTOTE: $f: E \to \mathbb{R}$, $x_0 \in \mathbb{R}$

$$\lim_{x \to x_0^+} f(x) = \pm\infty = \lim_{x \to x_0^-} f(x)$$

HORIZONTAL ASYMPTOTE: $f: E \to \mathbb{R}$

$$\lim_{x \to +\infty} f(x) = \underbrace{C \quad \text{OR} \quad \lim_{x \to -\infty} f(x) = D}_{\in \mathbb{R}}$$



$f(x) = e^{-x}$

$$\lim_{x \to +\infty} e^{-x} = 0$$

ASYMPTOTES

VERTICAL ASYMPTOTE: $f: E \to \mathbb{R}$, $x_0 \in \mathbb{R}$

$$\lim_{x \to x_0^+} f(x) = \pm\infty = \lim_{x \to x_0^-} f(x)$$

HORIZONTAL ASYMPTOTE:

$$\lim_{x \to +\infty} f(x) = C \quad \text{OR} \quad \lim_{x \to -\infty} f(x) = D$$

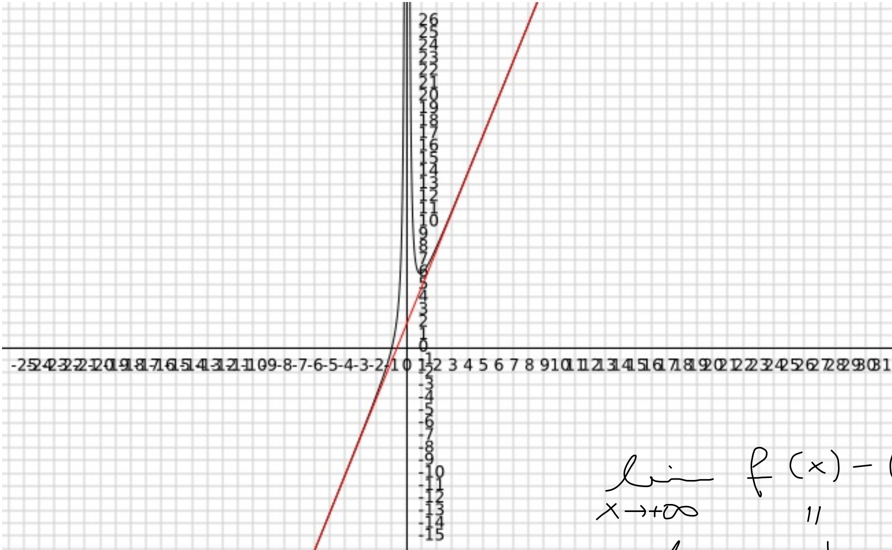
(OBLIQUE) ASYMPTOTE:

ASYMPTOTE

$$y = ax + b$$

$$\lim_{x \to +\infty} f(x) - ax - b = 0 \quad \text{OR}$$

$$\lim_{x \to -\infty} f(x) - ax - b = 0$$



$$y = 3x + 2$$

$$\lim_{x \to +\infty} f(x) - (3x+2) =$$

$$\parallel$$

$$\lim_{x \to +\infty} \frac{1}{x^2} = 0$$

$$y = f(x) \doteq 2 + 3x + \frac{1}{x^2}$$

$$x_n = e^{\frac{1}{n}} - 1$$

DOES $\sum_{n=1}^{\infty} x_n$ CONVERGE ?

$$f(x) = e^x - 1$$

$$x_n = f\left(\frac{1}{n}\right) \qquad \frac{1}{n} \xrightarrow{n\to\infty} 0$$

$$f(x) = f(0) + f'(0)(x-0) + \underbrace{\varepsilon_1(x)}_{x\to 0\ \to\ 0}(x-0)^2$$

$$f\left(\frac{1}{n}\right) = 0 + 1 \cdot \frac{1}{n} + \underbrace{\varepsilon_1\left(\frac{1}{n}\right)}_{\substack{n\to\infty \\ 0}}\left(\frac{1}{n}\right)$$

$$f\left(\frac{1}{n}\right) = 0 + \underline{\underline{1 \cdot \frac{1}{n}}} + \underbrace{\varepsilon_1\left(\frac{1}{n}\right)}_{\substack{\downarrow n\to\infty \\ 0}} \underline{\underline{\left(\frac{1}{n}\right)}}$$

$$f\left(\frac{1}{n}\right) = \frac{1}{n} + \varepsilon_1\left(\frac{1}{n}\right) \cdot \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{f\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty} 1 + \underbrace{\varepsilon_1\left(\frac{1}{n}\right)}_{\substack{\downarrow \\ 0}} = 1$$

$$\sum f\left(\frac{1}{n}\right) \quad \text{DIVERGES}$$

$$\frac{f\left(\frac{1}{n}\right)}{\frac{1}{n}} \longrightarrow 1 \qquad \frac{f\left(\frac{1}{n}\right)}{\frac{1}{n}} \geq \frac{1}{2} \quad \forall n \gg 0$$

$$f\left(\frac{1}{n}\right) \geq \frac{1}{2}\cdot\frac{1}{n} \geq 0$$

$$\sum \frac{1}{n} = +\infty$$

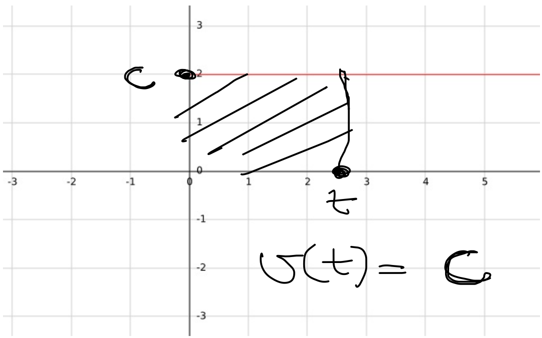
$\Downarrow$ SQUEEZE

$$\sum f\left(\frac{1}{n}\right) = +\infty$$

INTEGRATION

$v(t) =$ velocity of a particle at time t

$p(t) = ?$ $\quad$ $p(0) = 0$



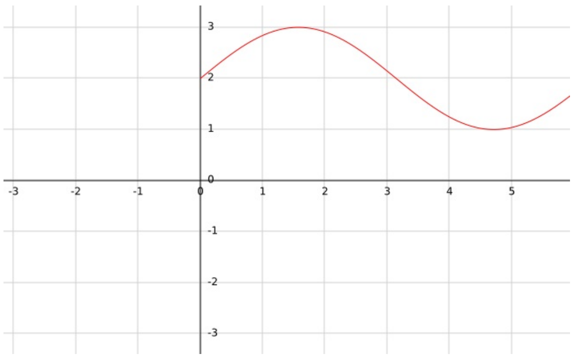
$v(t) = c$

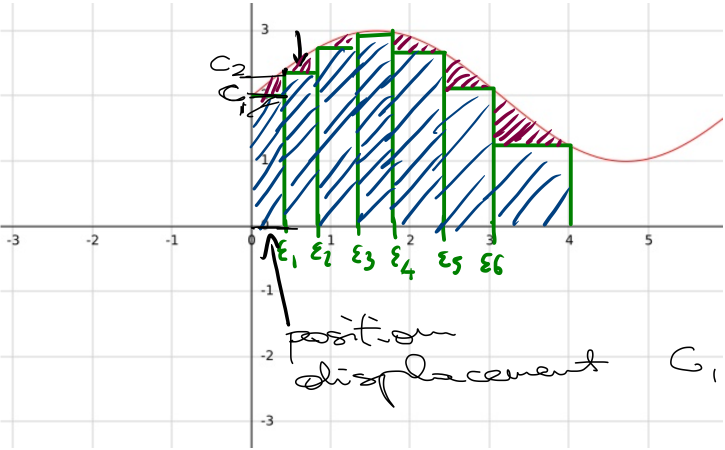
$p(t) = c \cdot t$

INTEGRATION

$v(t) =$ velocity of a particle at time t

$p(t) = ?$ $\quad\quad p(0) = 0$



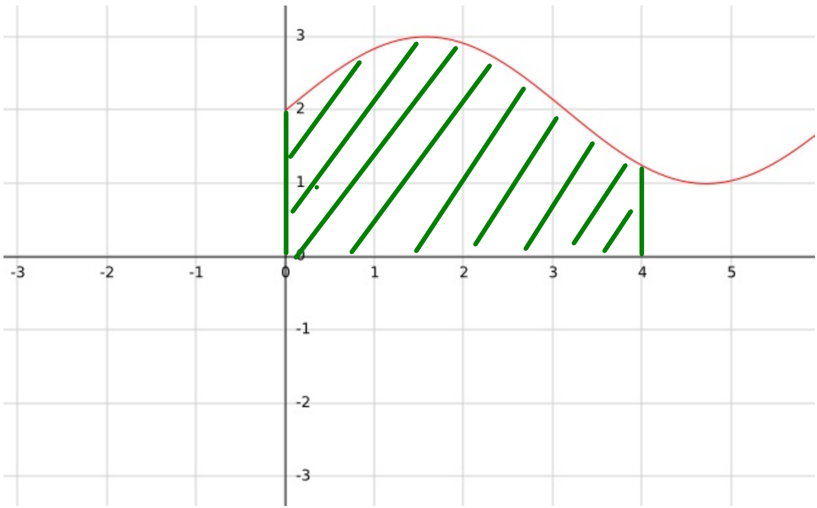


$$C_1 \cdot \varepsilon_1 + C_2(\varepsilon_2 - \varepsilon_1) +$$

$$+ C_{i+1}(\varepsilon_{i+1} - \varepsilon_i)$$

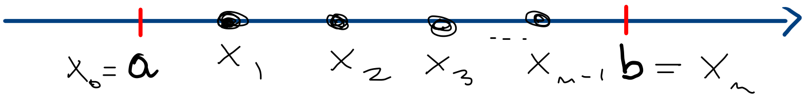
Q: ① HOW TO GENERALIZE THIS

② WHAT ARE WE DOING?



$$[a,b] \subseteq \mathbb{R}$$

σ = PARTITION OF $[a,b]$ =



$$\sigma = \{\underset{\substack{\parallel \\ a}}{x_0} < x_1 < x_2 < \dots < x_{n-1} < x_n = b\}$$

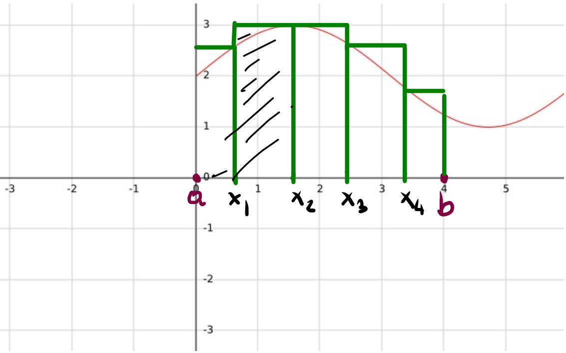
MESH(σ) = $\max |x_{i+1} - x_i|$

$f : [a,b] \longrightarrow \mathbb{R}$,

FIX σ = PARTITION OF $[a,b]$ =

$\{x_0 = a < x_1 < x_2 < x_3 < \dots < x_n = b\}$

UPPER DARBOUX SUM of σ:



OVER $[x_i, x_{i+1}]$ WE TAKE THE CONST. APPROX. GIVEN BY

$$\sup_{[x_i, x_{i+1}]} f \geq f(x) \quad \forall x$$

$f : [a,b] \longrightarrow \mathbb{R}$,

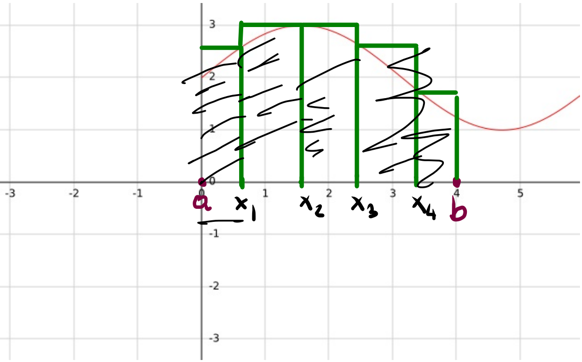
FIX σ = PARTITION OF $[a,b]$ =

$\{x_0 = a < x_1 < x_2 < x_3 < \ldots < x_n = b\}$

UPPER DARBOUX SUM OF σ :

$$\overline{S}_\sigma = \sum_{i=0}^{n-1} (x_{i+1} - x_i) \cdot f_i$$

$$\sup_{[x_i, x_{i+1}]} f$$

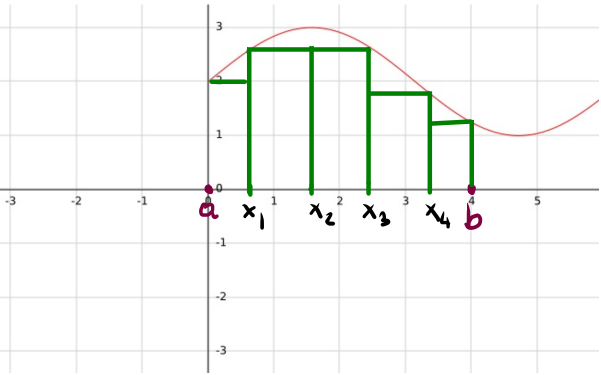


$f : [a,b] \longrightarrow \mathbb{R}$,

FIX σ = PARTITION OF $[a,b]$ =

$\{x_0 = a < x_1 < x_2 < x_3 < \ldots < x_n = b\}$

LOWER DARBOUX SUM of σ:



$\inf\limits_{[x_i, x_{i+1}]} f =: f_i($

$f : [a,b] \longrightarrow \mathbb{R}$,

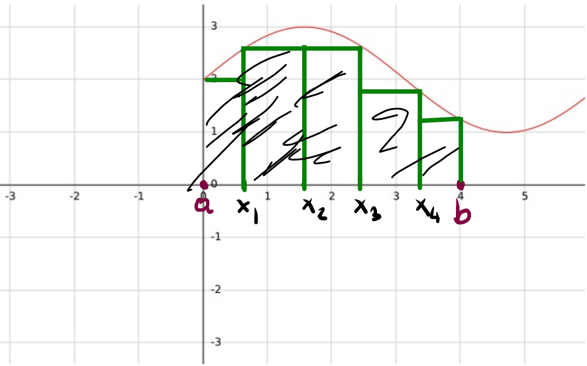
FIX σ = PARTITION OF $[a,b]$ =

$\{ x_0 = a < x_1 < x_2 < x_3 < \ldots < x_n = b \}$

LOWER DARBOUX SUM of σ :

$$\underline{S}_\sigma = \sum_{i=0}^{n-1} (x_{i+1} - x_i)\, \underline{f}_i$$

$$\underline{f}_i = \inf_{[x_i,\, x_{i+1}]} f$$



$f : [a, b] \to \mathbb{R}$

$$\sigma \begin{cases} \underline{S}_\sigma \\ \overline{S}_\sigma \end{cases}$$

$\underline{S}_\sigma$ ← LOWER DARBOUX SUM OF f WRT σ

$\overline{S}_\sigma$ ← UPPER DARBOUX SUM.

$$\underline{S}_\sigma \leq \overline{S}_\sigma$$

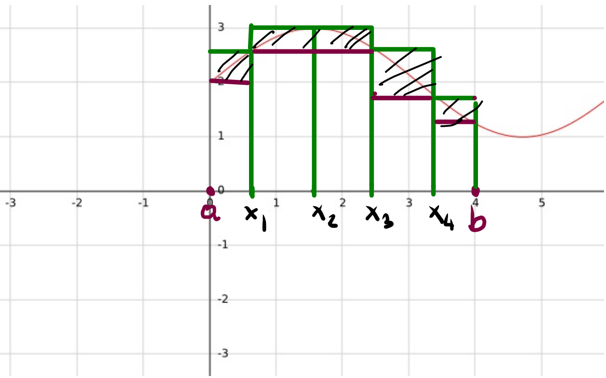
$$\Uparrow$$

$$\sup_{[x_i, x_{i+1}]} f \geq \inf_{[x_i, x_{i+1}]} f$$

$f : [a, b] \to \mathbb{R}$, σ PARTITION OF $[a, b]$

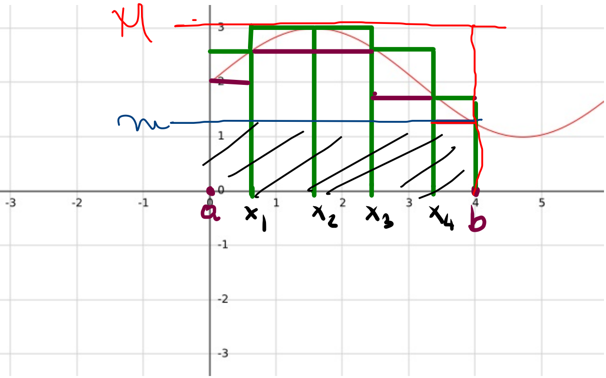
$$\underline{S}_\sigma \leq \overline{S}_\sigma$$

$$\overline{S}_\sigma - \underline{S}_\sigma =$$



$f : [a, b] \to \mathbb{R}$, σ PARTITION OF $[a, b]$

$$m \cdot (b-a) \leq \underline{S}_\sigma \leq \overline{S}_\sigma \leq M(b-a)$$



$$M \doteq \sup_{[a,b]} f$$

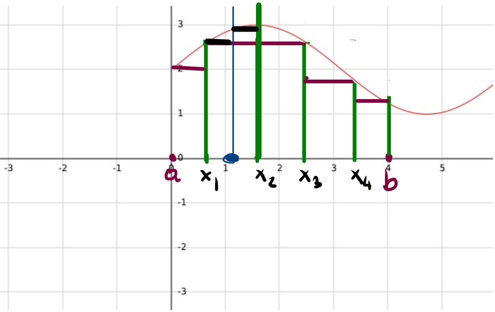
$$m \doteq \inf_{[a,b]} f$$

$$f : [a, b] \to \mathbb{R}$$

$$\begin{tikzcd} & \underline{s}_\sigma \\ \sigma \arrow[ur] \arrow[dr] & \\ & \overline{S}_\sigma \end{tikzcd}$$

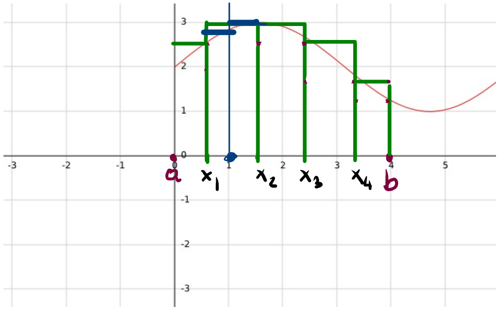
$$\underset{\inf\limits_{[a,b]} f}{\underset{\parallel}{m}}(b-a) \leq \underline{s}_\sigma \leq \overline{S}_\sigma \leq \underset{\sup\limits_{[a,b]} f}{\underset{\parallel}{M}}(b-a)$$

$$f : [a, b] \to \mathbb{R}$$



IF WE REFINE THE PARTITION σ W/ A NEW PARTITION σ'

$$\Longrightarrow \overline{S}_{\sigma'} \leq \overline{S}_{\sigma}$$



$$\underline{S}_{\sigma'} \geq \underline{S}_{\sigma}$$

DEF $f : [a,b] \to \mathbb{R}$

- LOWER DARBOUX INTEGRAL

$$\underline{S} = \sup \left\{ \underline{S}_\sigma \;\middle|\; \sigma \text{ is a partition of } [a,b] \right\}$$

- UPPER DARBOUX INTEGRAL

$$\overline{S} = \inf \left\{ \overline{S}_\sigma \;\middle|\; \sigma \text{ is a partition of } [a,b] \right\}$$

- f is INTEGRABLE if

$$\overline{S} = \underline{S}$$

DEF $f : [a, b] \to \mathbb{R}$

f IS **INTEGRABLE** IF
(RIEMANN INTEGR.)

$$\overline{S} = \underline{S}$$

DEF $f : [a,b] \to \mathbb{R}$

f IS INTEGRABLE (RIEMANN INTEGR.) IF

$$\overline{S} = \underline{S}$$

RMK $\forall \sigma$ PARTITION OF $[a,b]$

$$\underline{S}_\sigma \leq \underline{S} \boxed{\leq} \overline{S} \leq \overline{S}_\sigma$$

$\underline{S} := \sup \underline{S}_\sigma$ $\qquad$ $\overline{S} := \inf \overline{S}_\sigma$

DEF $f:[a,b]\to\mathbb{R}$

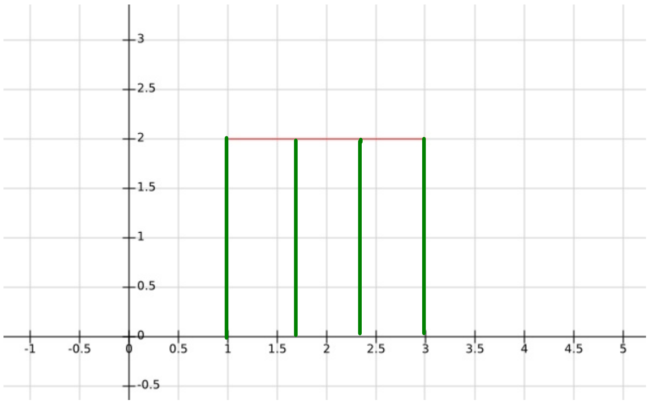
f IS **INTEGRABLE** (RIEMANN INTEGR.) IF

$$\overline{S} = \underline{S}$$

NOTATION: $\displaystyle\int_a^b f(x)\,dx = \underline{S} = \overline{S}$

EXAMPLE

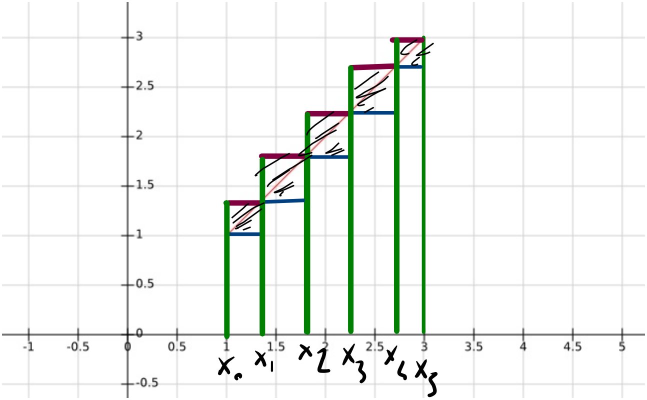
$f:[1,3]\to\mathbb{R} \quad f(x)=2$



$$\int_1^3 f(x)\,dx = 2\cdot(3-1)=4$$

EXAMPLE

$f(x) = x$, $6 = \{1 < x_1 < x_2 < \cdots < x_n = 3\}$



$\underline{S}_6 =$

$$\sum_{i=0}^{n-1} x_i (x_{i+1} - x_i)$$

$\overline{S}_6 =$

$$\sum_{i=0}^{n-1} x_{i+1} (x_{i+1} - x_i)$$

$$\overline{S}_6 - \underline{S}_6 = //// = \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2$$

TAKE $\sigma_n = \{ x_0 = 1 < x_1 = 1 + \frac{2}{n} <$

$< x_2 = 1 + 2 \cdot \frac{2}{n} <$

$\vdots$

$< x_i = 1 + i \cdot \frac{2}{n} <$

$\vdots$

$< x_n = 1 + n \cdot \frac{2}{n} = 3 \}$

$$\Longrightarrow \underline{S}_{\sigma_n} = \sum_{i=0}^{n-1} \left(1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) =$$

$$= \sum_{i=0}^{n-1} \frac{2}{n} + \sum_{i=0}^{n-1} \left(\frac{2}{n}\right)^2 \cdot i =$$

$$= 2 + \left(\frac{2}{n}\right)^{\cancel{2}} \left(\frac{\cancel{n}(n-1)}{\cancel{2}}\right) \xrightarrow{n \to \infty} 4$$

TAKE $\sigma_n = \{ x_0 = 1 < x_1 = 1 + \frac{2}{n} <$

$< x_2 = 1 + 2 \cdot \frac{2}{n} <$

$\vdots$

$< x_i = 1 + i \cdot \frac{2}{n} <$

$\vdots$

$< x_n = 1 + n \cdot \frac{2}{n} = 3 \}$

$$\Longrightarrow \overline{S}_{\sigma_n} = \sum_{i=0}^{n-1} \left(1 + \frac{2(i+1)}{n}\right)\left(\frac{2}{n}\right) =$$

$$= \sum_{i=0}^{n-1} \frac{2}{n} + \sum_{i=0}^{n-1} \left(\frac{2}{n}\right)^2 \cdot (i+1) =$$

$$= 2 + \left(\frac{2}{n}\right)^2 \left(\frac{n(n+1)}{2} - 1\right) \xrightarrow{n \to \infty} 4$$

$$\underline{S}_{\sigma_n} \leq \underline{S} \leq \overline{S} \leq \overline{S}_{\sigma_n}$$

$$\underline{S}_{\sigma_n} = 2 + \frac{2}{n}(n-1) \to 4$$

$$\overline{S}_{\sigma_n} = 2 + \left(\frac{2}{n}\right)^2 \left(\frac{n(n+1)}{2} - 1\right) \to 4$$

$$\Downarrow$$

$$\underline{S} = \overline{S} = 4$$