# ANALYSIS 1
## (ENGLISH)
## LECTURE 22

# THM [TAYLOR'S FORMULA]

$f : ]a, b[ \to \mathbb{R}$ , $x_0 \in ]a, b[$

ASSUME THAT $f$ IS DIFF $(n+1)$ TIMES

THEN THE DEVELOPM. OF ORDER $n$ AROUND $x_0$ EXISTS

$\lim_{x \to x_0} \varepsilon_n(x) = 0$

$$f(x) = \sum_{i=0}^{n} a_i (x - x_0)^i + \varepsilon_n(x) \cdot (x - x_0)^n$$

$$a_0 = f(x_0), \quad a_1 = f'(x_0), \quad a_i = \frac{f^{(i)}(x_0)}{i!}$$

**EXAMPLE** $f(x) = \dfrac{1}{\sqrt{1-x}}$ $\quad x = 0$

$f$ is differentiable around $x = 0$

$C^\infty$ (when $1-x \to (0, \mathbb{R})$)

$C^\infty((0,\mathbb{R}))$ polynomial

$f(x) = g \circ h(x)$

$h(x) = \sqrt{1-x} = \sqrt{\phantom{x}} \circ (1-x)$

$C^\infty((-1,1))$

$g(t) = \dfrac{1}{t} \quad C^\infty(\mathbb{R}^*)$

$$\left(\sqrt{x}\right)^{(n)} = \left(x^{\frac{1}{2}}\right)^{(n)} = C \cdot x^{\frac{1}{2}-n}$$

$$C = \frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-1-1\right) \cdots \left(\frac{1}{2}-(n-1)\right)$$

this is OKAY when $x \approx 0$ $(x \in (-1,1))$

$$f(x) = (1-x)^{-\frac{1}{2}} \quad \text{for } x_0 = 0$$

$$f(0) = 1$$

$$f'(0) = \left[+\frac{1}{2}\,\underline{\underline{(1-x)}}^{-\frac{3}{2}}\right](0) = +\frac{1}{2}$$

$$f''(0) = \left[\frac{1}{2}\left(+\frac{3}{2}\right)(1-x)^{-\frac{5}{2}}\right](0) = \frac{1}{2}\cdot\frac{3}{2}$$

$$\vdots$$

$$f^{(n)}(0) = \left[\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdots\frac{(2n-1)}{2}(1-x)^{-\frac{1}{2}-n}\right](0)$$

$n^{th}$-order expansion of $f$

$$f(x) = 1 + \frac{1}{2}x + \frac{\frac{1}{2}\cdot\frac{3}{2}}{2}x^2 + \dots + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{(2i-1)}{2}}{i!}x^i + \dots$$

# EXPANSIONS & COMPOSITION OF FUNCTIONS

$f : ]a, b[ \to \mathbb{R}$, $x_0 \in ]a,b[$, $g : ]c, d[ \to \mathbb{R}$, $y_0 = f(x_0) \in ]c,d[$

ASSUME $g \circ f$ IS $C^n(]a,b[)$

$$\lim_{x \to x_0} \varepsilon_n(x) = 0$$

$$g \circ f(x) = \sum_{i=0}^{n} \frac{(g \circ f)^{(i)}}{i!} (x - x_0)^i + \underline{\underline{\varepsilon_n(x)(x-x_0)^n}}$$

$$= g \circ f(x_0) + g'(f(x_0)) \cdot f'(x_0)(x - x_0) + ?\ \ldots$$

CAN WE COMPUTE THE EXPANSION OF $g \circ f$ USING THOSE OF $g, f$ ?

$$(g\circ f)''(x_0) = \left(g'(f(x_0))\,f'(x_0)\right)'(x_0)$$

$$= g''(f(x))\cdot f'(x)\cdot f'(x) + g'(f(x))\cdot f''(x)$$

$$= g''(f(x))\cdot\left(f'(x)\right)^2 + g'(f(x))\cdot f''(x)$$

# EXPANSIONS & COMPOSITION OF FUNCTIONS

$f : ]a, b[ \to \mathbb{R}$ , $g : ]c, d[ \to \mathbb{R}$

$a \ni x_0$ $\qquad a_i = \dfrac{f^{(i)}(x_0)}{i!}$ $\qquad c \ni y_0 = f(x_0)$

$$f(x) = \cancel{y_0}\, a_0 + a_1(x - x_0) + a_2(x-x_0)^2 + \dots + a_n(x-x_0)^n + (x-x_0)^n \varepsilon_n(x)$$

$$g(y) = g(y_0) + b_1(y - \underline{y_0}) + b_2(y-y_0)^2 + \dots + b_n(y-y_0)^n + (y-y_0)^n \eta_n(y)$$

$$(g \circ f)(x) = g(f(x_0)) + b_1\big(a_1(x-x_0) + a_2(x-x_0)^2 + \dots + \dots + a_n(x-x_0)^n + \text{err.}\big) + b_2\big[(a_1(x-x_0) + \dots + \dots + a_n(x-x_0)^n + \text{err}\big]^2 + \dots + b_n(\dots)^n$$

# EXAMPLE

$$h(x) = e^{\sin(x)} = g \circ f(x)$$

$$g(y) = e^{y}, \quad f(x) = \sin(x), \quad x_0 = 0$$

$$y_0 = 0$$

$$f(x_0) = f(0) = 0 = y_0$$

$$f(x) = 0 + 1(x - x_0) + 0(x - x_0)^2 + \frac{(-1)}{6}(x - x_0)^3 + ERR$$

$$= \underbrace{x - \frac{1}{6}x^3 + ERR}$$

$$g(y) = 1 + (y - \cancel{y_0}) + \frac{(y - \cancel{y_0})^2}{2} + \frac{(y - \cancel{y_0})^3}{6} + \frac{(y - \cancel{y_0})^4}{24} + ERR$$

$$(g \circ f)(x) = 1 + \left(x - \frac{1}{6}x^3 + ERR_4\right) + \frac{1}{2}\left(x - \frac{1}{6}x^3 + ERR_4\right)^2 +$$

$$+ \frac{1}{6}\left(x - \frac{1}{6}x^3 + ERR_4\right)^3 + \frac{1}{24}\left(x - \frac{1}{6}x^3 + ERR_4\right)^4 + ERR_4$$

$$(g\circ f)(x) = 1 + \left(x - \frac{1}{6}x^3 + ERR_4\right) + \frac{1}{2}\left(x - \frac{1}{6}x^3 + ERR_4\right)^2 +$$

$$+ \frac{1}{6}\left(x - \frac{1}{6}x^3 + ERR_4\right)^3 + \frac{1}{24}\underline{\underline{\left(x - \frac{1}{6}x^3 + ERR_4\right)^4}} + ERR_4$$

$$1 + \underline{\underline{x}} - \underline{\underline{\frac{1}{6}}}\cancel{x^3} + ERR_4 + \underline{\underline{\frac{1}{2}}}\Big(x^2 + \left[\frac{1}{36}x^6\right] + ERR_8 +$$

$$- \frac{1}{3}x^4 + ERR_5 + ERR_7\Big) + \cancel{\frac{1}{6}}\Big(\cancel{x^3} \left[-\frac{1}{6^3}x^9\right] +$$

$$ERR_{12} + \left[3x^2\left(-\frac{1}{6}x^3\right)\right] + ERR_{>4}\Big) +$$

$$\frac{1}{24}\left(x^4 + ERR_{>4}\right) =$$

$$\mu(x)\cdot x^4, \quad \mu(x) \xrightarrow{x\to 0} 0$$

$$=$$

$$1 + x + \frac{1}{2}x^2 + 0\cdot x^3 + \left(-\frac{1}{6} + \frac{1}{24}\right)x^4 + ERR_4$$

# EXAMPLE

$$h(x) = e^{\cos(x)} = g \circ f(x)$$

$$g(y) = e^{y}, \quad f(x) = \cos(x), \quad x_0 = 0$$

$\cos(x_0) = 1$

$\parallel$

$y_0$

$$f(x) = \underbrace{1 + \frac{x^2}{2} + x^3 \varepsilon_3(x)}$$

$$g(y) = g(1) + g'(1)(y-1) + \frac{g''(1)}{2} \cdot (y-1)^2 + ERR_2$$

TAYLOR EXPANSION of $g$ around $y_0$

$$g \circ f(x) = e^{\cos(x)} = e^1 + e^1 \cdot \left(\frac{x^2}{2} + ERR_3\right) + \frac{e^1}{2}\left(\frac{x^2}{2} + ERR_3\right)^2$$

$$= e^1 + e^1 \cdot \frac{x^2}{2} + ERR_3$$

$$g(y) = e^1 + e^1(y-1) + e^1(y-1)^2 + ERR_2$$

# TAYLOR EXPANSION & LIMITS

## EXAMPLE

$$\lim_{x\to 0} \frac{\left(e^x-1\right)^2+\sin\left(x^2\right)}{\cos(x)-1} = \left[\frac{0}{0}\right]$$

$x \to x_0 \to 0$

$$ERR_n = \varepsilon_n(x)\cdot(x-x_0)^n$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots + \frac{x^i}{i!} + \dots + ERR_n$$

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} + \dots$$

$$= x^2 + ERR_5$$

$$\sin(y) = y - \frac{y^3}{3!} + \frac{y^5}{5!} + \dots$$

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} \longrightarrow \cos(x) - 1 = \boxed{-\frac{x^2}{2} + ERR_3}$$

$$\left(e^x-1\right)^2 = \left(x + \frac{x^2}{2} + \frac{x^3}{3!} + ERR_3\right)^2 =$$

$$= x^2 + \cancel{2}\cdot x\cdot\frac{x^2}{\cancel{2}} + ERR_3 = \boxed{x^2 + x^3 + ERR_3}$$

$$\lim_{x\to 0} \frac{(e^x-1)^2+\sin(x^2)}{\cos(x)-1} = \quad \cancel{x}\to 0$$

$$x\,(\,= \gamma_3(x)\cdot x^3$$

$$\lim_{x\to 0} \frac{x^2+x^3+\eta_3(x)\cdot x^3+x^2+\lambda_5(x)\cdot x^5}{-\frac{x^2}{2}+\varepsilon_3(x)\cdot x^3} =$$

$$\lim_{x\to 0} \frac{2x^2+x^3+\gamma_3(x)\cdot x^3}{-\frac{x^2}{2}+\varepsilon_3(x)\cdot x^3} =$$

$$\lim_{x\to 0} \frac{\cancel{x^2}\left(2+\underbrace{x}_{\to 0}+\underbrace{\gamma_3(x)\cdot x}_{\to 0}\right)}{\cancel{x^2}\left(-\frac{1}{2}+\varepsilon_3(x)\cdot x\right)} = -4$$

# TAYLOR EXPANSION & LIMITS

## EXAMPLE

$$\lim_{t \to +\infty} \sin\left(\frac{1}{t^5}\right) \cdot t^4$$

$\sin\left(\frac{1}{t^5}\right) \to 0$, $\quad t^4 \to +\infty$

$$\frac{1}{t} = x$$

$$\lim_{x \to 0^+} \sin x^5 \cdot \frac{1}{x^4} =$$

$$\lim_{x \to 0^+} \frac{\sin x^5}{x^4} = \lim_{x \to 0^+} \frac{\cancel{x^5} + \varepsilon_{15}(x)\, x^{15}}{\cancel{x^4}} = 0$$

$\xrightarrow{x \to 0} 0$

$$\sin y = y - \frac{y^3}{3!} + \dots$$

$$\boxed{\sin x^5 = x^5 - \frac{x^{15}}{3!} + \dots}$$

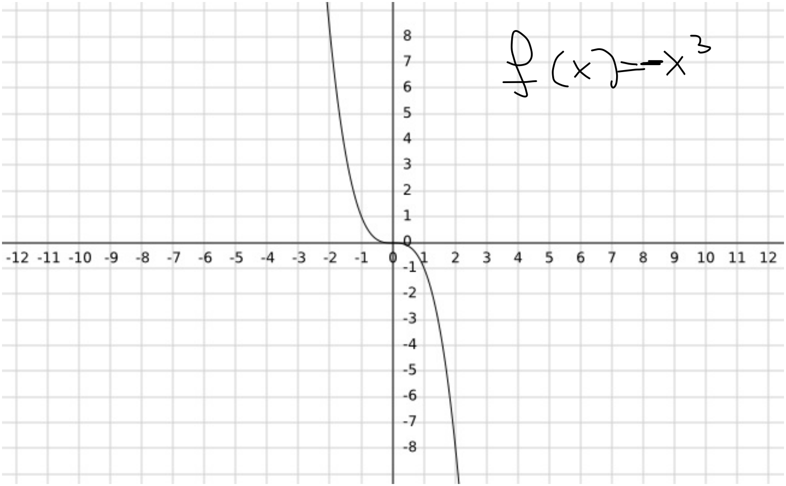
$$\lim_{x \to 0^+} \frac{\sin(x)}{x^4} = \frac{x - \frac{x^3}{6} + \frac{x^5}{5!} + \overset{\to 0}{\varepsilon_5(x)}\, x^5}{x^4}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} -$$

$$= \lim_{x \to 0^+} \frac{x - \frac{x^3}{6}}{x^4} + \overset{\to 0}{f(x)}$$

$$= \lim_{x \to 0^+} \frac{x - \frac{x^3}{6}}{x^4} = +\infty$$

# TAYLOR EXPANSION & LOCAL EXTREMA

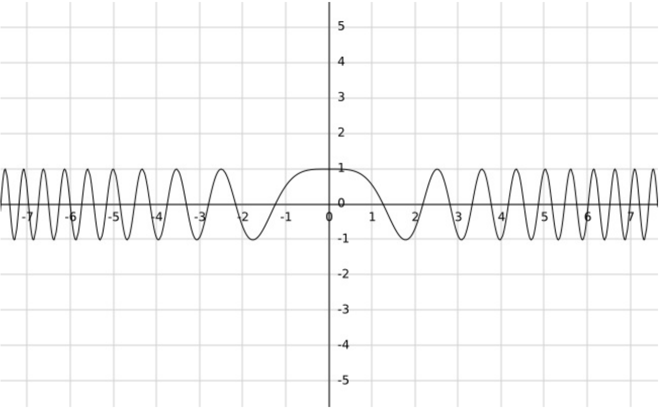



$f(x) = -x^3$

VICEVERSA NOT TRUE

- If $f$ has a local max/min at a pt $x_0$

$\Longrightarrow f'(x_0) = 0$




$f(x) = \cos(x^2)$

0 is a stat. pt

$f'(x) = -\sin(x^2)(2x)$

$f'(0) = 0$

INTUITION: 0 is a pt at which $f$ has a local max. ← TRUE around $x_0 = 0$

$\cos(y) = 1 - \frac{y^2}{2} + \frac{y^4}{4!} - \dots \geq 0$

$0 \leq |x^8| \ll |x^4|$

↑ FOR $x \ll 1$

$\cos(x^2) = 1 - \underbrace{\frac{x^4}{2}}_{\geq 0} + \frac{x^8}{4!} - \dots$

$f : \,]a, b[\, \to \mathbb{R}$, $x_0 \in \,]a, b[$

ASSUME THAT $f', f^{(2)}, \dots, f^{(n-1)}, f^{(n)}$ EXIST AROUND $x_0$ & $f'(x_0) = 0 = f^{(2)}(x_0) = f^{(3)}(x_0) = \dots = f^{(n-1)}(x_0) = 0$

$f^{(n)}(x_0) \neq 0$

$$f(x) = f(x_0) + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \varepsilon_n(x)(x - x_0)^n \qquad \varepsilon_n(x) \xrightarrow{x \to x_0} 0$$

$$f(x) \approx f(x_0) + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + (x-x_0)^n \varepsilon_n(x)$$

$$\varepsilon_n(x) \xrightarrow{x \to x_0} 0$$

① $n$ EVEN around $x_0$:

$f^{(n)}(x_0) > 0 \implies f(x) \geq f(x_0)$ around $x_0$

$f$ has a local minimum at $x_0$

[for all $x$ in some interval of the form $(x_0-\delta, x_0+\delta)$

$f^{(n)}(x_0) < 0 \implies f(x) \leq f(x_0)$ around $x_0$

$f$ has a local maximum at $x_0$

② $n$ ODD around $x_0$:

$$(x-x_0)^n \begin{cases} > 0 & \text{when } x > x_0 \\ < 0 & \text{when } x < x_0 \end{cases}$$

if $f^{(n)}(x_0) > 0$

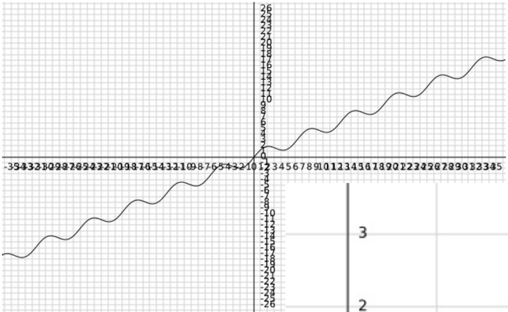
$f(x) \geq f(x_0)$ $\quad x > x_0$

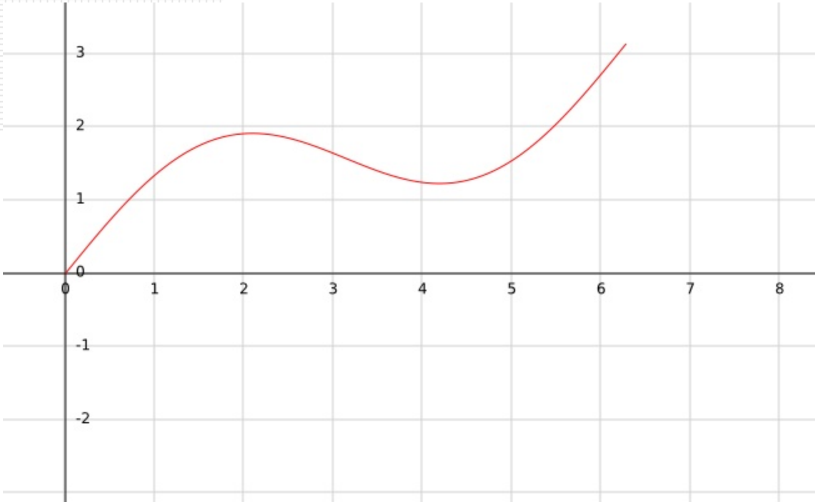
$f(x) \leq f(x_0)$ $\quad x < x_0$

# EXAMPLE

$$f(x) = \sin(x) + \frac{1}{2}x$$

$$x \in [0, 2\pi]$$





# EXAMPLE

$$f(x) = \sin(x) + \frac{1}{2}x$$
$$x \in [0, 2\pi]$$

$$M = \max_{[0,2\pi]} f \qquad m = \min_{[0,2\pi]} f$$

$$f'(x) = \cos(x) + \frac{1}{2}$$

STAT. PTS: $f'(x) = 0 \iff \cos(x) = -\frac{1}{2}$

$$\Updownarrow$$

$$x = \frac{2\pi}{3} + 2k\pi \qquad k \in \mathbb{Z}$$
$$\frac{-2\pi}{3} + 2k\pi$$

STAT. PTS in $[0, 2\pi]$ ARE $x = \frac{2\pi}{3}, \frac{4\pi}{3}$

$f''(2\pi/3) < 0$ LOCAL MAX

$f''(4\pi/3) > 0$ LOCAL MIN

$$\max_{[0,2\pi]} f = \max\left\{ f(0),\ f(2\pi),\ f(2\pi/3) \right\} = \max\left\{ 0,\ \pi,\ \frac{\sqrt{3}}{2} + \pi/3 \right\}$$

$$\min_{[0,2\pi]} f = \min\left\{ f(0),\ f(2\pi),\ f(4\pi/3) \right\} = \min\left\{ 0,\ \pi,\ 2\pi/3 - \frac{\sqrt{3}}{2} \right\}$$

**DEF** $f : ]a,b[ \to \mathbb{R}$, $x_0 \in D(f)$

$x_0$ IS AN INFLECTION PT. IF

$\exists \alpha > 0$

FOR $x \in ]x_0, x_0+\alpha[ \Rightarrow (x, f(x))$ IS ABOVE THE TANGENT LINE TO $(x_0, f(x_0))$

AND $x \in ]x_0-\alpha, x_0[ \Rightarrow (x, f(x))$ IS BELOW THE TANGENT LINE TO $(x_0, f(x_0))$

[OR VICEVERSA]

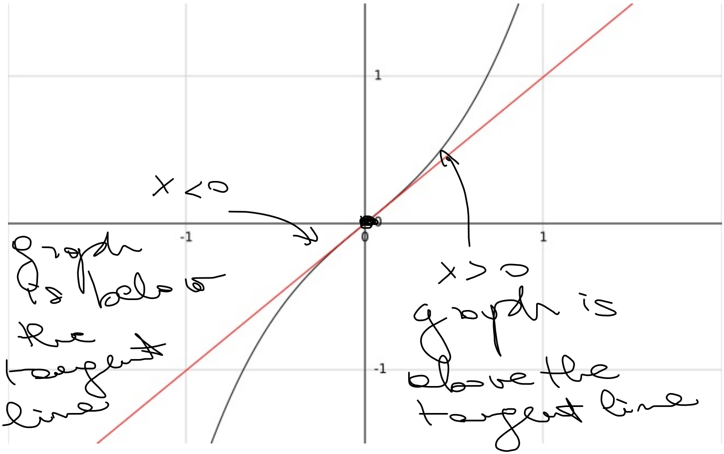
**DEF** $f : \,]a,b[\, \to \mathbb{R}$, $x_0 \in Dom(f)$

$x_0$ IS AN **INFLECTION PT.** IF $\exists\, \alpha > 0$

FOR $x \in \,]x_0, x_0+\alpha[\, \Rightarrow (x, f(x))$ IS ~~ABOVE~~ BELOW THE TANGENT LINE TO $(x, f(x))$

AND $x \in \,]x_0-\alpha, x_0[\, \Rightarrow (x, f(x))$ IS ~~BELOW~~ ABOVE THE TANGENT LINE TO $(x, f(x))$

[OR VICEVERSA]




$f(x) = x + x^3$

$x_0 = 0$

$f'(0) = 1$

$y - f(0) = 1 \cdot (x - 0)$

TANGENT LINE $y = x$

**PROP** $f : ]a,b[ \to \mathbb{R}$, $x_0 \in Dom(f)$

ASSUME $f$ HAS $n$ DERIVATIVES ON $]a,b[$

ASSUME $f^{(n)}$ IS $C^0$ AT $x_0$ &

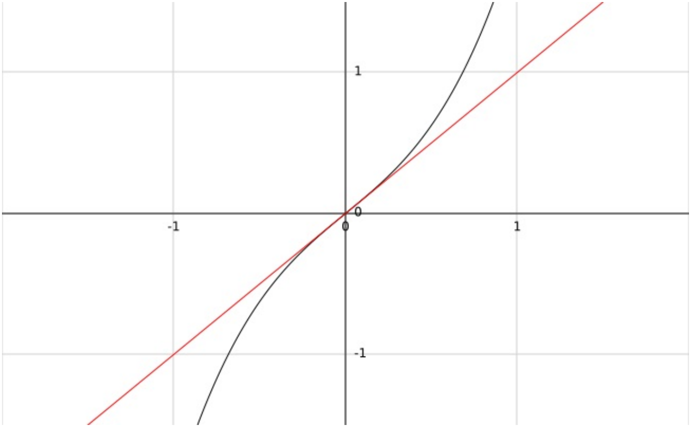
$$f''(x_0) = f'''(x_0) = \dots = f^{(n-1)}(x_0) = 0$$

IF $n$ IS ODD & $f^{(n)}(x_0) \neq 0$

$\Rightarrow$ $f$ has an inflection pt at $x_0$.

# PROOF

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + f^{(n)}(x_0)(x-x_0)^n + (x-x_0)^n \varepsilon_n(x)$$



STATIONARY PT $\left(f'(x_0)=0\right)$

$$n \doteq \min\left\{ k \in \mathbb{N}^* \mid f^{(k)}(x_0) \neq 0 \right\}$$

$n$ even :

$n$ : odd

# EXAMPLE

$$f(x) = \sin(x) - x$$

$$f'(x) = \cos(x) - 1 \Rightarrow f'(0) = 0$$