ANALYSIS 1
(ENGLISH)
LECTURE 21

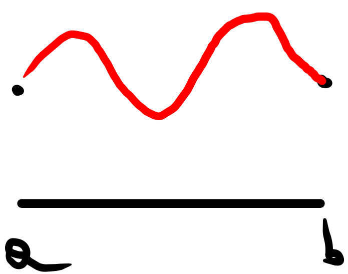
THM

$f : [a,b] \to \mathbb{R}$ C^0 & DIFFER. ON $]a,b[$

IF $f(a) = f(b) \Rightarrow \exists\, c \in\,]a,b[$ S.T. $f'(c) = 0$.

QUICK WORDS :

$f(a) = f(b) \implies$ f' has a 0 somewhere btw a & b

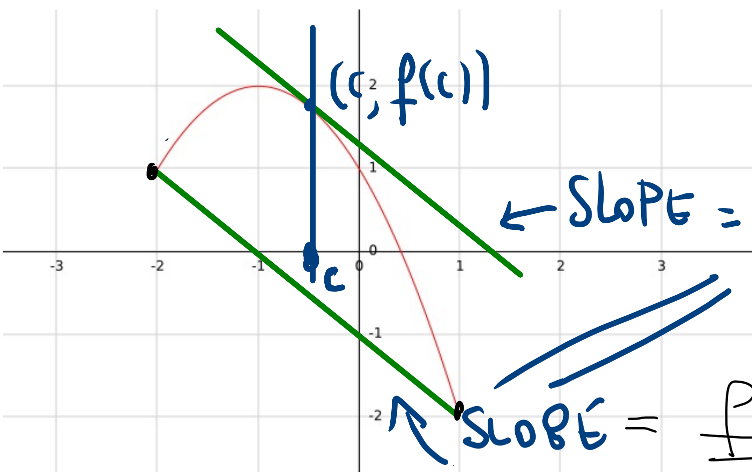


THM $f : [a,b] \to \mathbb{R}$, C^0 ON $[a,b]$

DIFF. ON $]a,b[$

$\Rightarrow \exists\, c \in\,]a,b[$ S.T.

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

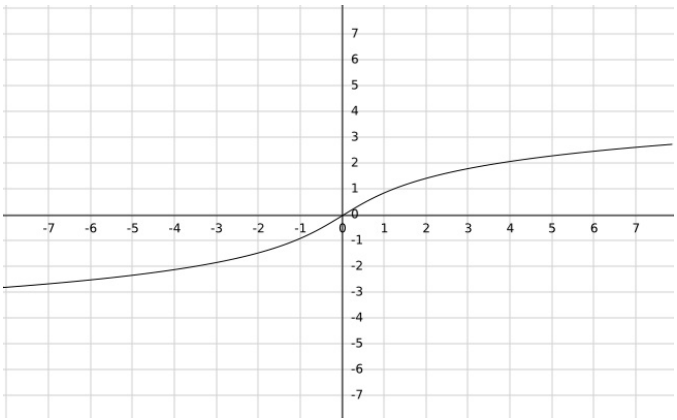


$\leftarrow$ SLOPE $= f'(c)$

SLOPE $= \dfrac{f(b)-f(a)}{b-a}$

DERIVATIVE & MONOTONE FCTS

$f: E \to \mathbb{R}$ MONOTONE INCREASING $x_0 \in E$



$x \geq x_0$

$\Downarrow$

$f(x) \geq f(x_0)$

$\Downarrow$

$$\frac{f(x)-f(x_0)}{x-x_0} \geq 0$$

$x_0 \geq x$

$\Downarrow$

$f(x_0) \geq f(x)$

$\Downarrow$

$$\frac{f(x)-f(x_0)}{x-x_0} \geq 0$$

$$\lim_{x \to x_0^+} \frac{f(x)-f(x_0)}{x-x_0} \geq 0 \qquad \lim_{x \to x_0^-} \frac{f(x)-f(x_0)}{x-x_0} \geq 0$$

IF f DIFF. AT $x_0 \Rightarrow f'(x_0) = \lim\limits_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$

$$\overset{\overset{0}{\wedge\!\shortparallel}}{\lim_{x\to x_0^+} \frac{f(x)-f(x_0)}{x-x_0}} = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \overset{\overset{0}{\wedge\!\shortparallel}}{\lim_{x\to x_0^-} \frac{f(x)-f(x_0)}{x-x_0}}$$

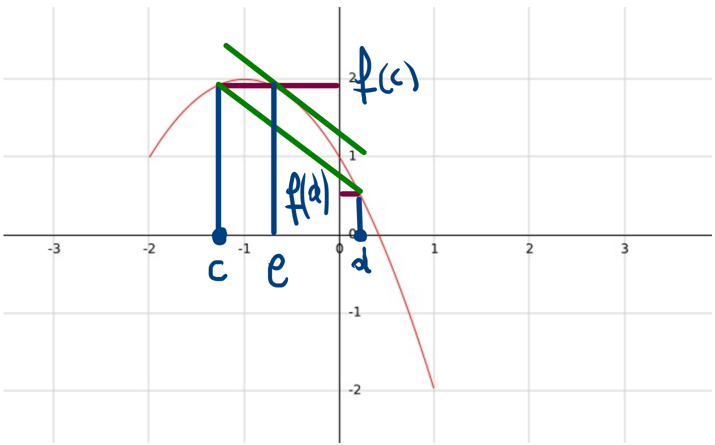
Q: DOES THE VICEVERSA HOLD?

IF $f : E \to \mathbb{R}$, $f'(x) \geq 0 \ \forall x \in E$

$\Longrightarrow$ f IS INCREASING?

SUPPOSE f IS NOT INCREASING

$\Longrightarrow \exists \ d > c \in E$ S.T. $f(d) < f(c)$



$$f'(e) = \frac{f(d) - f(c)}{d - c} < 0$$

VI 0

MEAN VALUE THEOREM

$e \in \,]c, d[$

THM $f : [a, b] \to \mathbb{R}$ C^0 & DIFFERENT. ON $f|_{]a,b[}$

THEN

1. f INCREASING $\iff f' \geq 0$

2. f DECREASING $\iff f' \leq 0$

3. $f' > 0 \implies f$ STR. INCREASING

4. $f' < 0 \implies f$ STR. DECREASING

5. $f' \gneq 0 \implies f$ increasing e.g. $f(x) = \begin{cases} 0 & x > 0 \\ -x^2 & x \leq 0 \end{cases}$

6. $f' \lneq 0 \implies f$ decreasing

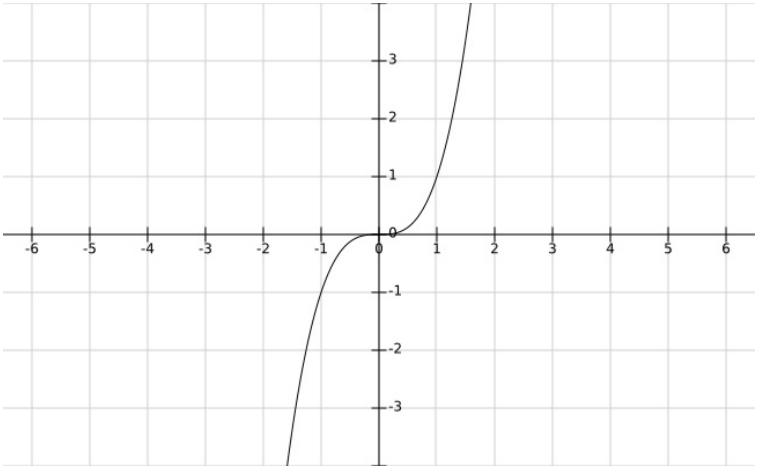
THM $f:[a,b]\to\mathbb{R}$ C^0 & DIFFERENT. ON $f|_{]a,b[}$

THEN

1. f INCREASING $\iff f'\geq 0$
2. f DECREASING $\iff f'\leq 0$
3. $f'>0$ $\boxed{\Longrightarrow}$ f STR. INCREASING
4. $f'<0$ $\boxed{\Longrightarrow}$ f STR. DECREASING

NOT $\iff$

RMK f STR. INCR. $\Rightarrow f'(x) > 0$?



$$f(x) = x^3$$

EXAMPLE

$$f(x) = e^x$$

$$f'(x) = e^x > 0 \quad \forall x \in \mathbb{R}$$

hence, $f(x)$ is strictly increasing

EXAMPLE

$$f(x) = a^x, \quad a \in \mathbb{R}^*_+$$
$$a \neq 1$$

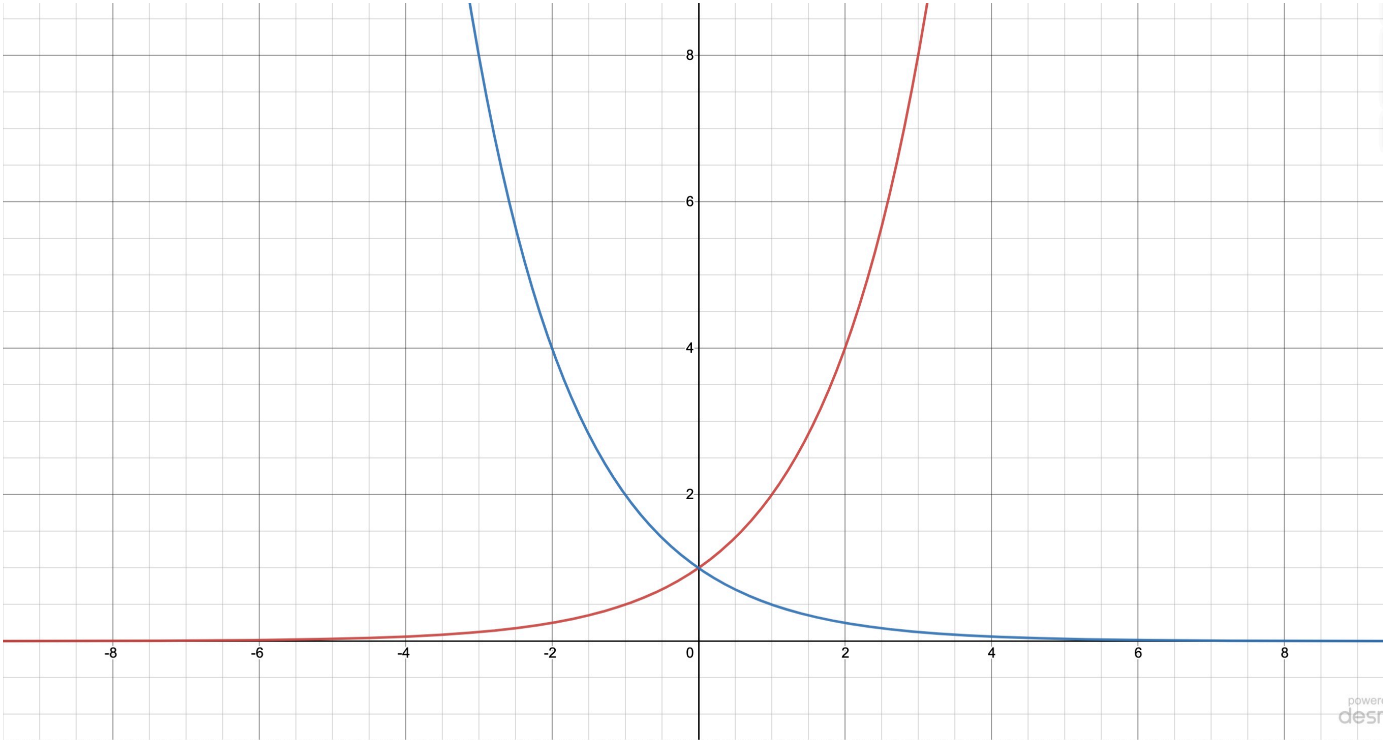
$$f(x) = e^{\log a \cdot x}$$

$$f'(x) = \log a \;\; a^x$$

$$\Rightarrow \text{ IF } \underset{a>1}{\log a > 0} \Rightarrow f'(x) > 0$$

$$\text{IF } \underset{0<a<1}{\log a < 0} \Rightarrow f'(x) < 0$$

$a > 1$, f strictly increasing
 ,,
$0 < a < 1$, f str. decreasing



8
6
4
2
0
-8
-6
-4
-2
2
4
6
8

L'HÔPITAL RULE

EXAMPLE $\lim_{x\to\infty} \frac{e^x}{x^2}$ $\left(\frac{+\infty}{+\infty}\right)$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

$\forall\ x>0$

$$1 + x + \frac{x^2}{2} + \frac{x^3}{6} \Rightarrow \lim_{x\to\infty} \frac{1 + x + x^2/2 + x^3/6}{x^2}$$

$$= \underbrace{\lim_{x\to\infty} \frac{1}{x^2}}_{0} + \underbrace{\lim_{x\to+\infty} \frac{1}{x}}_{0} + \underbrace{\lim_{x\to\infty} \frac{1}{2}}_{1/2} + \underbrace{\lim_{x\to\infty} \frac{x}{6}}_{+\infty} = +\infty$$

As $\frac{e^x}{x^2} > \frac{1+x+\frac{x^2}{2}+\frac{x^3}{6}}{x^2}$, $\frac{e^x}{x^2} \xrightarrow[x\to\infty]{} +\infty$

THM $f, g :]a,b[\setminus \{x_0\} \to \mathbb{R}$ $\quad a, b \in \overline{\mathbb{R}}$

DIFFERENTIABLE

ASSUME $g(x) \neq 0 \neq g'(x)$ FOR $x \approx x_0$

& $\lim\limits_{x \to x_0} \frac{f(x)}{g(x)} = \frac{0}{0}, \frac{\infty}{\infty}$

THEN

IF $\lim\limits_{x \to x_0} \frac{f'(x)}{g'(x)} = \underset{\overline{\mathbb{R}}}{\underset{\Uparrow}{\mu}} \Longrightarrow \lim\limits_{x \to x_0} \frac{f(x)}{g(x)} = \underset{\overline{\mathbb{R}}}{\underset{\Uparrow}{\mu}}$

THM $f, g :]a,b[\to \mathbb{R}$ $a, b \in \overline{\mathbb{R}}$

DIFFERENTIABLE

ASSUME $g(x) \neq 0 \neq g'(x)$ FOR $x \approx b$

& $\lim_{x \to b^-} \frac{f(x)}{g(x)} = \frac{0}{0}, \frac{\infty}{\infty}$

if $\lim_{x \to b^-} \frac{f'(x)}{g'(x)} = \mu \in \overline{\mathbb{R}}$, then $\lim_{x \to b^-} \frac{f(x)}{g(x)} = \mu \in \overline{\mathbb{R}}$

EXAMPLE

$$\lim_{x\to+\infty} \frac{e^x}{x^n} \qquad \frac{+\infty}{+\infty}$$

$(n \in \mathbb{N}_{>0})$

$$\lim_{x\to+\infty} \frac{e^x}{x^n} \overset{H}{=} \lim_{x\to+\infty} \frac{e^x}{nx^{n-1}} \overset{H}{=} \lim_{x\to+\infty} \frac{e^x}{n(n-1)x^{n-2}}$$

$$\underbrace{\overset{H}{=} \cdots \overset{H}{=}}_{\substack{\text{inductively} \\ \text{(do it } n \text{ times)}}} \lim_{x\to+\infty} \frac{e^x}{n(n-1)\cdot \ldots \cdot 2\cdot 1\, x^0} =$$

$$= \lim_{x\to+\infty} \frac{e^x}{n!} = +\infty$$

EXAMPLE

$$\lim_{x \to +\infty} \frac{\log x}{x^n} \qquad \frac{+\infty}{+\infty}$$

$$\lim_{x \to +\infty} \frac{\log x}{x^n} \overset{H}{=} \lim_{x \to +\infty} \frac{1/x}{n x^{n-1}} =$$

$$= \lim_{x \to +\infty} \left(\frac{1}{n} \cdot \frac{1}{x^n} \right) = 0$$

EXAMPLE

$$\lim_{x \to 0} \frac{\arcsin(x)}{\sin(x)} \qquad \frac{0}{0}$$

$$\overset{H}{=} \lim_{x \to 0} \frac{\frac{1}{\sqrt{1-x^2}}}{\cos(x)} = \frac{1}{1} = 1$$

plug in

comment:

both $\sin(x)$ (our g) and $\cos(x)$ (our g') are 0 somewhere, but not in a small punctured interval around 0

Non **EXAMPLE**

$$\lim_{x \to 2} \frac{3x}{x^2} = \frac{3 \cdot 2}{2^2} = \frac{6}{4}$$

$\downarrow$

C^0

at 2

$$(3x)' = 3$$

$$(x^2)' = 2x$$

$$\lim_{x \to 2} \frac{(3x)'}{(x^2)'} = \frac{3}{4} \neq \frac{6}{4}$$

NON **EXAMPLE**

$$\lim_{x \to +\infty} \frac{x + \cos x}{x} = \lim_{x \to +\infty} \left(1 + \frac{\cos x}{x} \right) = 1 \text{, if we used H:}$$

$$\lim_{x \to +\infty} \frac{1 - \sin(x)}{1} = \lim_{x \to +\infty} \left(1 - \sin(x) \right) \quad \text{DNE}$$

$$\lim_{x \to +\infty} \frac{x + x \cdot \cos x}{x} = \lim_{x \to +\infty} \left(1 + \cos(x) \right) \quad \text{DNE}$$

H not applicable here, as

$$\lim_{x \to +\infty} \left(x + x\cos x \right) \quad \text{DNE}$$

EXAMPLE

$$\lim_{x \to +\infty} \frac{p(x)}{q(x)}$$

$p(x)$ deg $= p$ POLYNOMIAL IN x

$q(x)$ deg $= q$ POLYNOMIAL IN x

With L'Hôpital we can retrieve:

$$\lim \begin{cases} \frac{a_p}{a_q} & \text{if } p = q \ (a_p, a_q \text{ leading coeff}) \\ 0 & \text{if } q > p \\ \infty & \text{if } p > q \ (\text{sign depends on sign of } a_p/a_q) \end{cases}$$

TAYLOR APPROXIMATION

$\sin(x)$ $\qquad$ $\sin(0) = 0$

$|\sin(x)| \leqslant |x|$

Q: HOW DO WE EXPLICITLY COMPUTE $\sin(x)$, $x \approx 0$?

WE KNOW $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$

n	$\frac{1}{n}$	$\sin\left(\frac{1}{n}\right)$	$\frac{1}{n} - \sin\left(\frac{1}{n}\right)$
1	1	0.8414709848078965	0.1585290151921035
2	0.5	0.479425538604203	0.020574461395796995
3	0.3333333333333333	0.3271946967961522	0.006138636537181108
4	0.25	0.24740395925452294	0.002596040745477063
5	0.2	0.19866933079506122	0.0013306692049387947
6	0.16666666666666666	0.16589613269341502	0.0007705339732516359
7	0.14285714285714285	0.14237172979226365	0.0004854130648792032
8	0.125	0.12467473338522769	0.0003252666147723071
9	0.1111111111111111	0.11088262850995298	0.00022848260115812535
10	0.1	0.09983341664682815	0.0001665833531718508
11	0.09090909090909091	0.09078392350887036	0.00012516740022054662
12	0.08333333333333333	0.08323691620031025	$9.641713302308008e-05$
13	0.07692307692307693	0.0768472383413981	$7.583858167882485e-05$
14	0.07142857142857142	0.07136784834007859	$6.0723088492836697e-05$
15	0.06666666666666667	0.066617294923393	$4.937174327367122e-05$
16	0.0625	0.0624593178423802	$4.0682157619799375e-05$
17	0.058823529411764705	0.05878961167637315	$3.391773539155457e-05$
18	0.05555555555555555	0.05552698200473387	$2.857355082168389e-05$
19	0.05263157894736842	0.05260728333807213	$2.42956092962876e-05$
20	0.05	0.04997916927067833	$2.083072932167196e-05$

HOW DO WE CONTROL THE ERROR IN AN APPROXIMATION?

$\sin(x)$, $x \approx 0$

error

$$\sin(x) = \sin(0) + E(x)$$

HOW DO WE CONTROL THE ERROR IN AN APPROXIMATION?

$\sin(x)$, $x \approx 0$

$$\sin(x) = \sin(0) + \underbrace{E(x)}_{\hookrightarrow \sin(x)} \qquad |E(x)| \leq |x|$$

MEAN VALUE THM:
$$\frac{\sin(x) - \sin(0)}{x - 0} = \cos(c)$$

FOR SOME $c \in]0, x[$

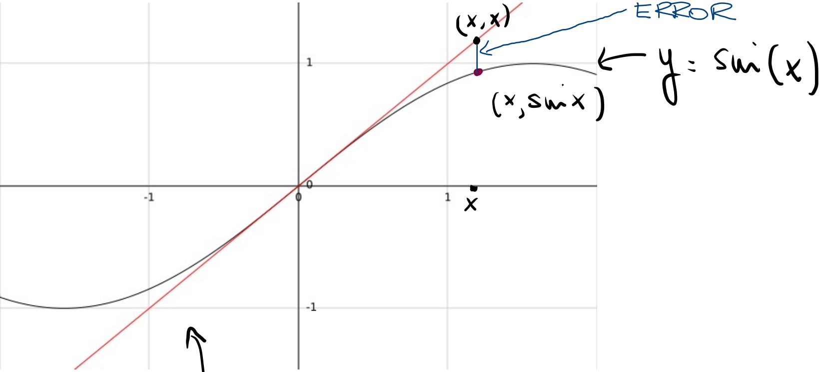
so, $E(x) = x \cdot \cos(c)$ for some $0 < c < x$

WE CAN DO EVEN BETTER:

$$\sin'(0) = \lim_{x \to 0} \frac{\sin(x) - \sin(0)}{x - 0}$$

$$\Updownarrow$$

$$0 = \lim_{x \to 0} \frac{\sin(x) - \sin(0) - \sin'(0)(x-0)}{x - 0}$$



$$y = \sin(0) + \sin'(0)(x - 0)$$

WE CAN DO EVEN BETTER:

$$\sin'(0) = \lim_{x \to 0} \frac{\sin(x) - \sin(0)}{x - 0}$$

$$\Updownarrow$$

$$0 = \lim_{x \to 0} \frac{\sin(x) - \sin(0) - \sin'(0)(x-0)}{x - 0}$$

$$\sin(x) = \sin(0) + \sin'(0)(x-0) + E_1(x)$$

ERROR

$$\lim_{x \to 0} \frac{E_1(x)}{(x-0)} = 0$$

IN GENERAL : WE WOULD LIKE TO APPROXIMATE $f(x)$, $x \approx x_0$

CHOSEN POINT

BY MEANS OF SIMPLE FUNCTIONS

& BE ABLE TO CONTROL THE ERROR

SIMPLE FUNCTIONS :
- CONSTANT
- LINEAR
- POLYNOMIALS

EXAMPLE

$f(x) = e^x \qquad x_0 = 0$

$$f(x) = \underbrace{a_0 + a_1 x + a_2 x^2}_{\text{deg 2 POLY}} + E_2(x)$$

← ERROR FCT

WE WANT THIS TO GROW SLOWER THAN $(x-x_0)^2$

$$\lim_{x \to x_0} \frac{E_2(x)}{(x-x_0)^2} = 0$$

EXAMPLE

$f(x) = e^x \qquad x_0 = 0$

$$f(x) = \underbrace{a_0 + a_1 x + a_2 x^2}_{\text{deg 2 POLY}} + E_2(x)$$

ERROR FCT

WE WANT THIS TO GROW SLOWER THAN $(x-x_0)^2$

$$\lim_{x \to x_0} \frac{E_2(x)}{(x-x_0)^2} = 0$$

$$f(0) = a_0 + \underbrace{E_2(0)}_{0}$$

$$\Rightarrow a_0 = 1$$

$$f'(0) = a_1 + 2a_2 \cdot 0 + E_2'(0)$$

$$E_2'(0) = \lim_{x \to 0} \frac{E_2(x) - E_2(0)}{(x-0)} = \lim_{x \to 0} \frac{E_2}{(x-0)^2}(x-0) = 0$$

$$a_1 = f'(0) = 1$$

EXAMPLE

$f(x) = e^x \qquad x_0 = 0$

$$f(x) = \underbrace{a_0 + a_1 x + a_2 x^2}_{\text{deg 2 POLY}} + E_2(x)$$

ERROR FCT

WE WANT THIS TO GROW SLOWER THAN $(x-x_0)^2$

$$\lim_{x \to x_0} \frac{E_2(x)}{(x-x_0)^2} = 0$$

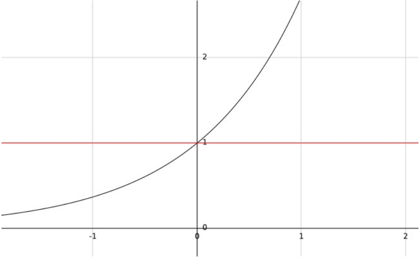
$f(0) = a_0 + E_2(0)$

$\Rightarrow a_0 = 1$

$f'(0) = a_1 + \cancel{2a_2 \cdot 0} + \cancel{E_2'(0)}$

$\Rightarrow a_1 = 1$

$f''(0) = 2a_2 + \underbrace{E_2''(0)}_{=0} \Longrightarrow a_2 = \frac{1}{2} f''(0) = \frac{1}{2}$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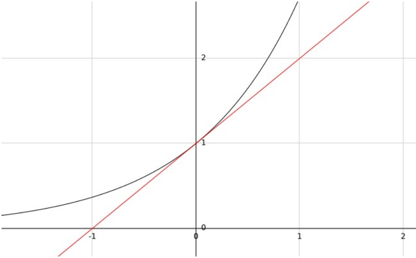
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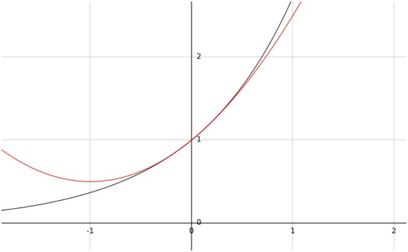
2
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0
-1
0
1
2

DEF $f :]a, b[\to \mathbb{R}$, $x_0 \in]a,b[$

WE SAY THAT f HAS A DEVELOPMENT TO THE n^{th} ORDER AROUND x_0, $n \in \mathbb{N}$

IF $\exists\ a_0, a_1, \dots, a_n \in \mathbb{R}$ S.T

$$f(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots + a_n(x-x_0)^n + \underbrace{\varepsilon_n(x)(x-x_0)^n}_{E_n(x)}$$

& $\lim_{x \to x_0} \varepsilon_n(x) = 0$

$f :]a, b[\to \mathbb{R}$, $x_0 \in]a, b[$

- IF A DEVELOPMENT OF ORDER n EXISTS IT IS UNIQUE !!

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \varepsilon_n(x)(x - x_0)^n$$

$$f(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)^2 + \dots + b_n(x - x_0)^n + \eta_n(x)(x - x_0)^n$$

THM [TAYLOR'S FORMULA]

$f:]a,b[\to \mathbb{R}$, $x_0 \in]a,b[$

ASSUME THAT f IS DIFF $(n+1)$ TIMES

THEN THE DEVELOPM. OF ORDER n AROUND x_0 EXISTS

$$f(x) = \sum_{i=0}^{n} a_i (x-x_0)^i + E_n(x)$$

$$a_0 = f(x_0), \quad a_1 = f'(x_0), \quad a_i = \frac{f^{(i)}(x_0)}{i!}$$

$$f(x) = \sum_{i=0}^{n} a_i (x-x_0)^i + E_n(x)$$

$$a_0 = f(x_0),\ a_1 = f'(x_0),\ a_i = \frac{f^{(i)}(x_0)}{i!}$$

WE CAN EVEN COMPUTE $E_n(x)$:

(by using MVT in smart way and the fact we have n+1 derivatives)

$$E_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}, \quad \begin{array}{ll} x_0 < c < x & (\text{if } x > x_0) \\ x < c < x_0 & (\text{if } x < 0) \end{array}$$

COR [TAYLOR'S FORMULA, 2]

$f:]a,b[\to \mathbb{R}$, $x_0 \in]a,b[$

ASSUME THAT f IS C^n

THEN

$$f(x) = \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!}(x-x_0)^i + \varepsilon_n(x)\cdot(x-x_0)^n$$

& $\lim_{n\to 0} \varepsilon_n(x) = 0$

EXAMPLE

$f(x) = \cos(x) : \mathbb{R} \to \mathbb{R}$

DIFF/$\mathbb{R}$

$f'(x) = -\sin(x) : \mathbb{R} \longrightarrow \mathbb{R}$ DIFF/$\mathbb{R}$

$f''(x) = -\cos(x)$ $f'''(x) = \sin(x)$

$f^{(4)}(x) = \cos(x)$ $f^{(5)}(x) = -\sin(x)$

$n = 4k + i, \quad i = 0, 1, 2, 3$

$$f^{(n)}(x) = f^{(4k+i)}(x) = f^{(i)}(x)$$

The derivatives at 0 are:

$1, 0, -1, 0, 1, 0, -1, 0, 1, \ldots$

0^{th} f

EXAMPLE

$f(x) = \cos x \quad , \quad x_0 = 0$

$$\cos(x) \approx 1 - \frac{1}{2}(x-0)^2 + \frac{1}{4!}x^4 - \frac{x^6}{6!} + \dots$$

$f^{(4)}(0)$ $\qquad$ $(x-x_0)^4$

$$= \underbrace{\left(\sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!}\right)}_{2n \,\&\, 2n+1 \text{ order developement}} + \varepsilon_{2n+1}(x)\, x^{2n+1}$$

EXAMPLE

$$f(x) = \frac{1}{1-x}, \quad x_0 = 0$$

By induction, we can show

$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}, \text{ so } f^{(n)}(0) = n!$$

So,

$$f(x) = 1 + x + x^2 + \dots + x^n + x^n \varepsilon_n(x)$$

$$\frac{n!}{n!}(x-0)^n$$

EXAMPLE

$f(x) = \log x$, $\quad x_0 = 1$

recall, $f'(x) = \frac{1}{x} = x^{-1}$

recursively, one shows $f^{(n)}(x) = (-1)^{n-1} \frac{(n-1)!}{x^n}$

so, $f^{(n)}(1) = (-1)^{n-1}(n-1)!$

so, $\frac{f^{(n)}(1)}{n!}(x-1)^n = (-1)^{n-1}\frac{(n-1)!}{n!}(x-1)^n =$

$= \frac{(-1)^{n-1}}{n}(x-1)^n$, so $f(x) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}(x-1)^k + \varepsilon_n(x)(x-1)^n$

EXPANSIONS & COMPOSITION OF FUNCTIONS

$$f :]a, b[\to \mathbb{R} , \quad g :]c, d[\to \mathbb{R}$$

$x_0 \in]a,b[$, $y_0 \in]c,d[$, $y_0 = f(x_0)$

ASSUME $g \circ f$ IS $C^n(]a, b[)$

$$g \circ f(x) = \sum_{i=0}^{n} \frac{(g \circ f)^{(i)}}{i!} (x - x_0)^i + \varepsilon_n(x)(x - x_0)^n$$

$$= g \circ f(x_0) + g'(f(x_0)) \cdot f'(x_0)(x - x_0) + \ ? \ \ldots$$

CAN WE COMPUTE THE EXPANSION OF $g \circ f$ USING THOSE OF g, f ?

EXPANSIONS & COMPOSITION OF FUNCTIONS

$f :]a, b[\to \mathbb{R}$, $a < x_0 < b$, $g :]c, d[\to \mathbb{R}$, $c < y_0 < d$

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + (x - x_0)^n \varepsilon_n(x)$$

$$g(y) = \sum_{k=0}^{n} \frac{g^{(k)}(y_0)}{k!} (y - y_0)^k + (y - y_0)^n \eta_n(y)$$

EXAMPLE $h(x) = e^{\cos(x)} = g \circ f(x)$

$$g(t) = e^{t}, \quad f(x) = \cos(x), \quad x_0 = 0$$

$$f(x) = 1 + \frac{x^2}{2} + x^3 \varepsilon_3(x)$$

$$g(t) = 1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + t^3 \eta_3(t)$$

$$g \circ f(x) = e^{\cos(x)} = 1 + 1 + \frac{x^2}{2} + x^3 \varepsilon_3(x) + \frac{1}{2} \cdot \left(1 + \frac{x^2}{2} + x^3 \varepsilon_3(x)\right)^2 + \frac{\left(1 + \frac{x^2}{2} + x^3 \varepsilon_3(x)\right)^3}{6} + \left(1 + \frac{x^2}{2} + x^3 \varepsilon_3(x)\right)^3 \eta_3(\;)$$