ANALYSIS 1

(ENGLISH)

LECTURE 20

HIGHER DERIVATIVES

$f : E \longrightarrow \mathbb{R}$ $\qquad$ $f' : D(f') \longrightarrow \mathbb{R}$

$$D(f') = \Big\{ x \in \underset{\substack{\| \\ D(f)}}{E} \;\Big|\; f \text{ IS DIFFERENTIABLE AT } x \Big\}$$

WE CAN TRY TO COMPUTE $(f')'$

2nd DERIVATIVE of f

HIGHER DERIVATIVES

$f : E \longrightarrow \mathbb{R}$ $\qquad$ $f' : \mathrm{Dom}(f') \longrightarrow \mathbb{R}$

$$D(f') = \{ x \in E \mid f \text{ is DIFFERENTIABLE AT } x \}$$

WE CAN TRY TO COMPUTE $(f')'$

$$f'' := (f')' : D(f'') \longrightarrow \mathbb{R}$$

$$D(f'') = \{ y \in D(f') \mid f' \text{ IS DIFF. AT } y \}$$

2nd DERIVATIVE of f

[DERIV. OF f']

f' IS DIFF AT $y \in D(f')$

$\Updownarrow$ DEF

$$\lim_{h \to 0} \frac{f'(y+h) - f'(y)}{h} \text{ exists}$$

$= f''(y)$

FOR THIS LIMIT TO EXIST

$\Longrightarrow$ $D(f')$ NEEDS TO CONTAIN A NBD OF y

IN GENERAL: $f: E \to \mathbb{R}$

$$\rightsquigarrow f', \; f'', \; f'''$$

$$f''' \doteq (f'')' : \; D(f''') \to \mathbb{R}$$

$$\{x \in D(f'') \mid f'' \text{ IS DIFF. AT } x\}$$

$$f^{(n)} \doteq \left(f^{(n-1)}\right)' : \; D(f^{(n)}) \longrightarrow \mathbb{R}$$

n-th DERIVATIVE

DERIVATIVE OF THE $(n-1)$-st DERIVATIVE

EXAMPLE

$$f(x) = x^n$$

$n = 3$

$f'(x) = 3x^2$ $\quad f' : \mathbb{R} \longrightarrow \mathbb{R}$

DIFF OVER $\mathbb{R}$

$f''(x) = 3 \cdot 2\, x = 6x$ $\quad f'' : \mathbb{R} \longrightarrow \mathbb{R}$

DIFF. OVER $\mathbb{R}$

$f'''(x) = 6 \quad : \mathbb{R} \to \mathbb{R}$

$f^{(4)}(x) = 0 \quad \forall x \in \mathbb{R}$

$\forall k \geq 4 \quad f^{(k)}(x) = 0$

GENERAL n :

$f'(x) = n \cdot x^{n-1}$

if $n = 1$

$\Downarrow$

$f' = n \Rightarrow f'' = 0$

$f''(x) = n(n-1)\, x^{n-2}$

$n = 2$

$\Downarrow$

$f''(x) = n \cdot (n-1)$

if $n = 1$

GENERAL n :

$f'(x) = n \cdot x^{n-1}$ $\quad f' = n \Rightarrow f'' = 0$

$n = 2$

$f''(x) = n(n-1)\,x^{n-2}$

$\Downarrow$

$f''(x) = n \cdot (n-1)$

$$f^{(K)}(x) = \begin{cases} n(n-1)(n-2)\cdots(n-(K-1))\,\underline{\underline{x^{n-K}}} & \text{if } K \leq n \\ 0 & \text{if } K > n \end{cases}$$

$f = x^n$

EXAMPLE

$$f(x) = \cos(x) : \mathbb{R} \longrightarrow \mathbb{R}$$

DIFF / $\mathbb{R}$

$$f'(x) = -\sin(x) ; \mathbb{R} \longrightarrow \mathbb{R} \quad \text{DIFF}/\mathbb{R}$$

$$f''(x) = -\cos(x)$$

$$f'''(x) = \sin(x)$$

$$f^{(4)}(x) = \cos(x)$$

$$f^{(5)}(x) = -\sin(x)$$

$$n = 4k + i \quad , \quad i = 0, 1, 2, 3$$

$$f^{(n)}(x) = f^{(4k+i)}(x) = f^{(i)}(x)$$

EXAMPLE

$$f(x) = \arcsin(x^2)$$

$$(\arcsin t)' = \frac{1}{\sqrt{1-t}}$$

$$h(t) = \arcsin t$$

$$(h \circ g)(x)$$

$$g(x) = x^2$$

$$f'(x) = h'(g(x))\, g'(x)$$

$$= \frac{2x}{\sqrt{1-x^2}}$$

$$f''(x) = \left(\frac{2x}{\sqrt{1-x^2}}\right)' = \frac{N' \cdot D - D' \cdot N}{D^2}$$

DEF $f : E \to \mathbb{R}$ IS C^{n} IF

"IS OF CLASS C^n"

$f^{(n)}$ EXISTS AT ANY POINT OF E

AND $f^{(n)}$ IS CONTINUOUS

$n \in \mathbb{N}$, $n = 0$ $f^{(n)} \overset{\text{DEF}}{=} f$

REMARK AS $D(f^{(n)}) = E$

$\Longrightarrow D(f^{(n-1)}) = E$ + $f^{(n-1)}$ IS DIFF. ALL OVER E

$\Longrightarrow f^{(n-1)}$ IS CONTINUOUS ON E

IN PARTICULAR

IF $f \in C^{k}(E)$

$$\Downarrow$$

$f \in C^{k-1}(E)$

$$\Downarrow$$

$f \in C^{k-2}(E)$

IN GENERAL IF $f \in C^{k}(E)$

$$\Downarrow$$

$f \in C^{j}(E) \quad \forall\, 0 \leq j \leq k$

$$f(x) = \cos(x)$$

$$f^{(n)}(x) \text{ exists } \forall n \in \mathbb{N}$$

$$\|$$

$\cos x$ or

$-\sin x$ or

$-\cos x$ or

$\sin x$

$$\Longrightarrow f \in C^n(\mathbb{R}) \quad \forall n \in \mathbb{N}$$

EXAMPLE

$$f(x) = \operatorname{sgn}(x) \cdot x^2$$

$$\overbrace{\begin{array}{ccc} 1 & \emptyset & -1 \\ x > 0 & & x < 0 \\ & x = 0 & \end{array}}^{[\ ]}$$

EXAMPLE

$$f(x) = \operatorname{sgn}(x) \cdot x^2$$

$$f(x) = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases}$$

$! \quad \mathbb{R} \longrightarrow \mathbb{R}$

$$\lim_{x \to 0^-} \frac{-x^{\cancel{2}}}{\cancel{x}} = 0$$

$$\lim_{x \to 0^+} \frac{x^2}{x} = 0$$

$$f'(x) = \begin{cases} 2x & x > 0 \\ -2x & x < 0 \\ \underline{\underline{0}} & x = 0 \end{cases}$$

$$\lim_{x \to 0^-} \frac{-2\cancel{x}}{\cancel{x}} = -2$$

$$\lim_{x \to 0^+} \frac{2\cancel{x}}{\cancel{x}} = 2$$

$$f''(x) = \begin{cases} 2 & x > 0 \\ -2 & x < 0 \\ & x = 0 \end{cases}$$

$$\mathbb{R}^* \longrightarrow \mathbb{R}$$

EXAMPLE

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} : \mathbb{R} \to \mathbb{R}$$

NOWHERE CONTINUOUS

$$g(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} : \mathbb{R} \longrightarrow \mathbb{R}$$

IS DIFF. ONLY AT $x = 0$

& CONTINUOUS ONLY IN 0

$$h(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

$$h(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

h IS DIFF. AT $x = 0$

$$\lim_{x \to 0} \frac{h(x) - h(0)}{x} = \lim_{x \to 0} \frac{\begin{cases} x^2 - 0 & (1) \\ 0 - 0 & (2) \end{cases}}{x} = 0$$

$$(2) = 0$$

$$(1) = \frac{x^2}{x} = x \xrightarrow{x \to 0} 0$$

CAN WE COMPUTE $h''(x)$ AT $x = 0$? NO

h IS NOT C^0 WHEN $x \neq 0 \Longrightarrow h$ IS NOT DIFF AT $x \neq 0$

CAN WE COMPUTE $h''(x)$ AT $x=0$? NO

h IS NOT C^0 WHEN $x \neq 0$

$x \in \mathbb{Q}^*$

$$h\left(x+\frac{1}{n}\right) = \left(x+\frac{1}{n}\right)^2 \xrightarrow{n \to \infty} x^2$$

$$h\left(x+\frac{\pi}{n}\right) = 0 \longrightarrow 0$$

$\lim_{y \to x} h(y)$ DOES NOT EXIST

LOCAL EXTREMA

DEF $f : E \to \mathbb{R}$, $x_0 \in E$

f HAS A LOCAL MAXIMUM AT x_0

IF $\exists\, \delta > 0$ SUCH THAT $\forall x \in E$

IF $|x - x_0| < \delta \implies f(x_0) \geq f(x)$

LOCAL EXTREMA

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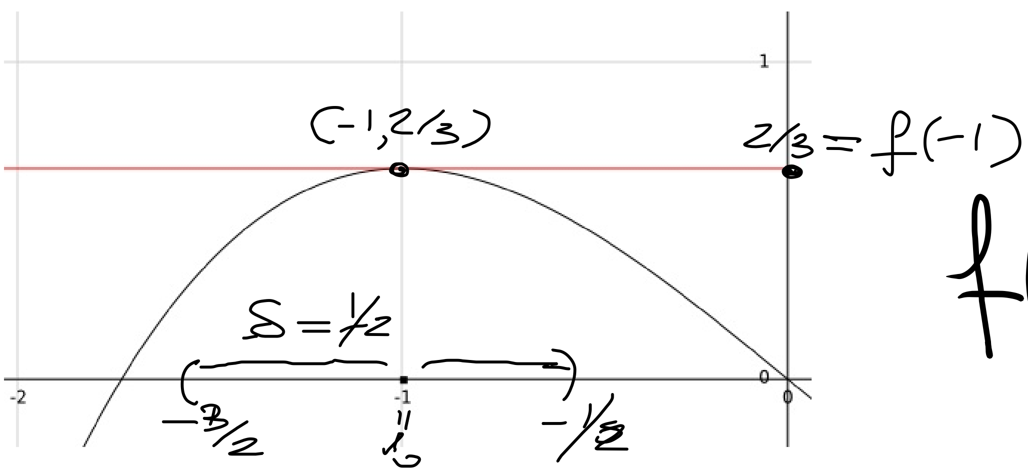


$$f(x) = \frac{x^3}{3} - x$$

$$f(-1) = 2/3 \qquad f'(1) = 0$$

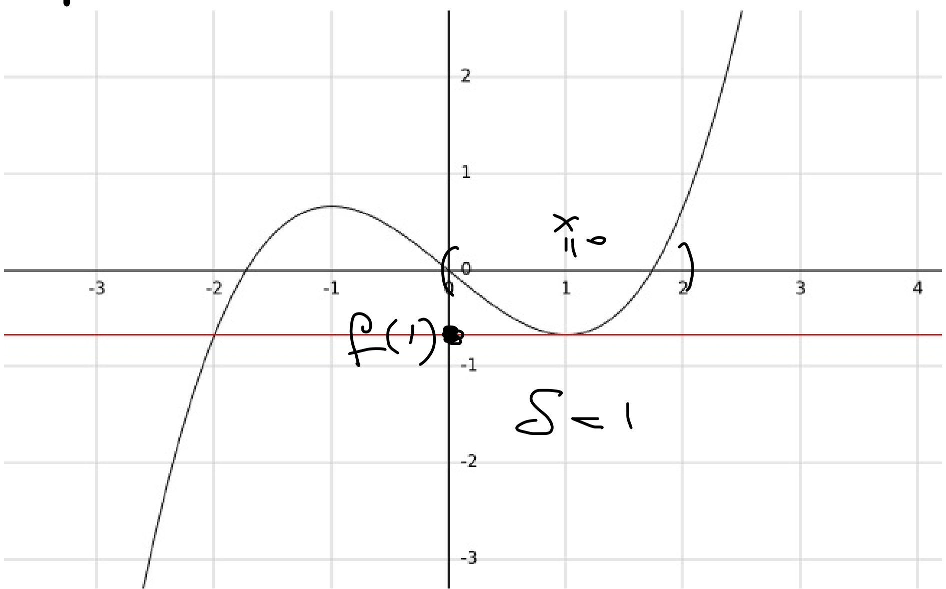
LOCAL EXTREMA

DEF $f : E \to \mathbb{R}$, $x_0 \in E$

f HAS A LOCAL MINIMUM AT x_0

IF $\exists\, \delta > 0$ SUCH THAT $\forall x \in E$

IF $|x - x_0| < \delta \implies f(x_0) \leq f(x)$



$f : E \to \mathbb{R}$, $x_0 \in E$

GLOBAL MAXIMUM: $f(x_0) \geq f(x)$

$\forall x \in E$

$\Updownarrow$

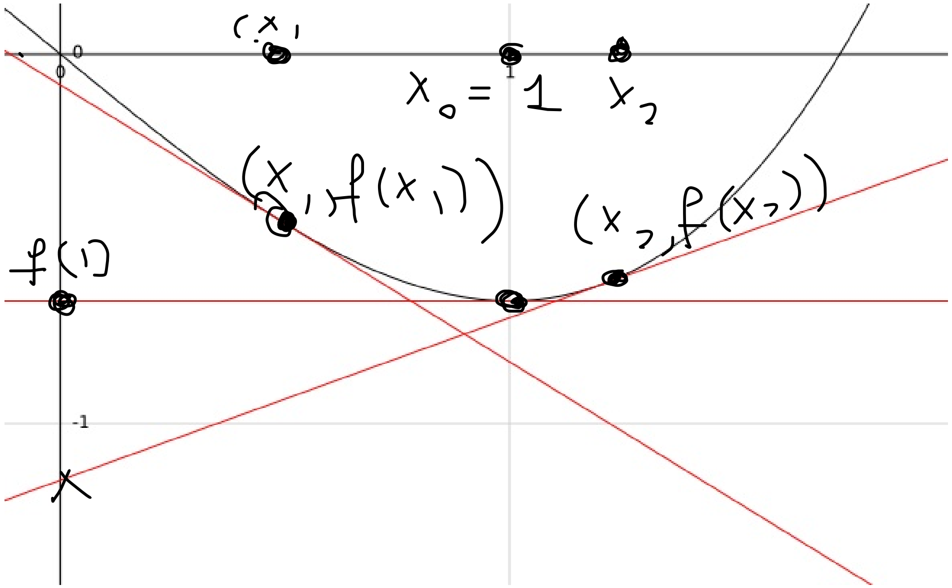
$f(x_0) = \max_E f$

GLOBAL MINIMUM: $f(x_0) \leq f(x)$

$\forall x \in E$

$\Updownarrow$

$f(x_0) = \min_E f$



PROP $f : E \to \mathbb{R}$, $x_0 \in E$ AT WHICH f HAS LOCAL MAX / MIN

ASSUME f IS DIFFERENTIABLE AT x_0.

THEN $f'(x_0) = 0$

PROP $f : E \to \mathbb{R}$, $x_0 \in E$ LOCAL MAX | MIN

ASSUME f IS DIFFERENTIABLE AT x_0

THEN $f'(x_0) = 0$

PROOF LET'S ASSUME $x_0 \in E$ IS LOCAL MAX

$\Rightarrow \exists \delta > 0$ S.T. $\forall x \in E$ w/ $|x - x_0| < \delta$, $f(x_0) \geq f(x)$

f DIFF AT x_0: $f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ ≤ 0

PROP $f: E \to \mathbb{R}$, $x_0 \in E$ LOCAL MAX | MIN

ASSUME f IS DIFFERENTIABLE AT x_0

THEN $f'(x_0) = 0$

$f(x) - f(x_0) \leq 0$

PROOF LET'S ASSUME $x_0 \in E$ IS LOCAL MAX

$\Rightarrow \exists \delta > 0$ S.T. $\forall x \in \mathbb{R}$ w/ $|x - x_0| < \delta$, $f(x_0) \geq f(x)$

f DIFF AT x_0: $f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ ≤ 0

IF $x < x_0$ $\Rightarrow \frac{f(x) - f(x_0)}{x - x_0} \geq 0 \Rightarrow \lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0} \geq 0$

IF $x > x_0$ $\Rightarrow \frac{f(x) - f(x_0)}{x - x_0} \leq 0 \Rightarrow \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} \leq 0$

THEN $f'(x_0) = \lim_{x \to x_0^+} \underbrace{\frac{f(x) - f(x_0)}{x - x_0}}_{\geq 0} = \lim_{x \to x_0^-} \underbrace{\frac{f(x) - f(x_0)}{x - x_0}}_{\leq 0}$

$\Rightarrow \quad f'(x_0) = 0$ ∎

DEF $f : E \to \mathbb{R}$, $x_0 \in E$.

x_0 IS A STATIONARY POINT FOR f
IF $f'(x_0) = 0$.

THEN $f'(x_0) = \lim_{x \to x_0^+} \underbrace{\frac{f(x)-f(x_0)}{x-x_0}}_{\geq 0} = \lim_{x \to x_0^-} \underbrace{\frac{f(x)-f(x_0)}{x-x_0}}_{\leq 0}$

$\Rightarrow \quad f'(x_0) = 0$ ∎

DEF $f : E \to \mathbb{R}$, $x_0 \in E$.

x_0 IS A STATIONARY POINT FOR f

IF $f'(x_0) = 0$.

JUST PROVED:

f HAS LOCAL MAX/MIN AT x_0 & f DIFF AT x_0 $\Longrightarrow$ x_0 STATIONARY PT. FOR f

DOES THE VICEVERSA HOLD?

THAT, $f'(x_0)=0 \Longrightarrow x_0$ LOCAL MAX/MIN?

[POLL]

~~TRUE~~ FALSE

DOES THE VICEVERSA HOLD ?

THAT, $f'(x_0)=0 \implies x_0$ LOCAL MAX/MIN ?

[POLL]

TRUE $\boxed{\text{FALSE}}$

EXAMPLE

$f(x) = -x^3$

$f'(x) = -3x^2$

$f'(0) = 0$

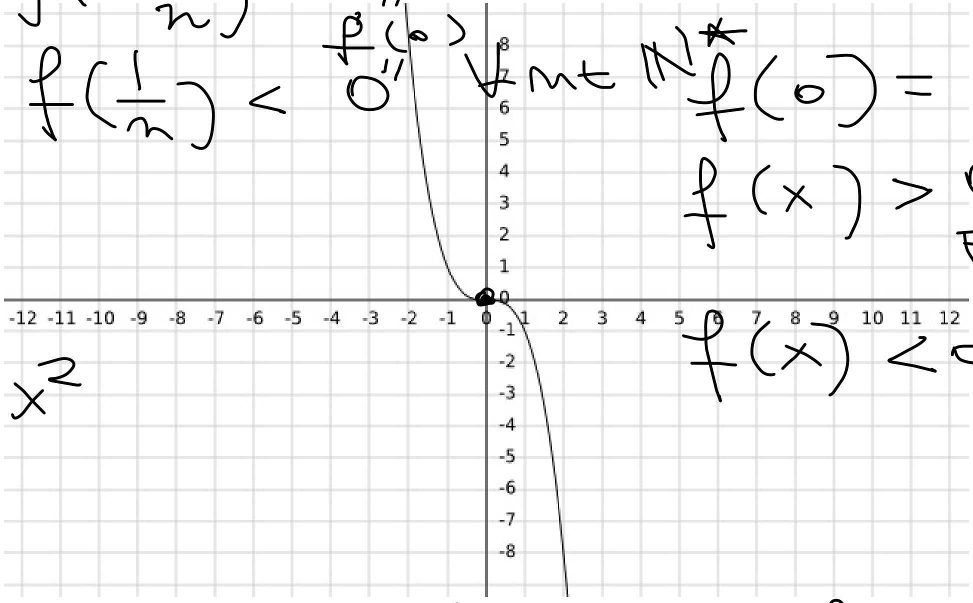
$f(-\frac{1}{n}) > 0 \quad \forall n \in \mathbb{N}^*$

$f(\frac{1}{n}) < 0 \quad \forall n \in \mathbb{N}^*$

$f(0) = 0$

$f(x) > 0$ FOR $x<0$

$f(x) < 0$ FOR $x>0$



$x=0$ IS A STAT. PT for f. BUT f HAS NO LOCAL MAX/MIN AT $x=0$?

$f : [a,b] \to \mathbb{R}$ ASSUME $f|_{]a,b[}$ IS DIFFERENTIABLE &

RESTRICTION OF f TO $]a,b[$

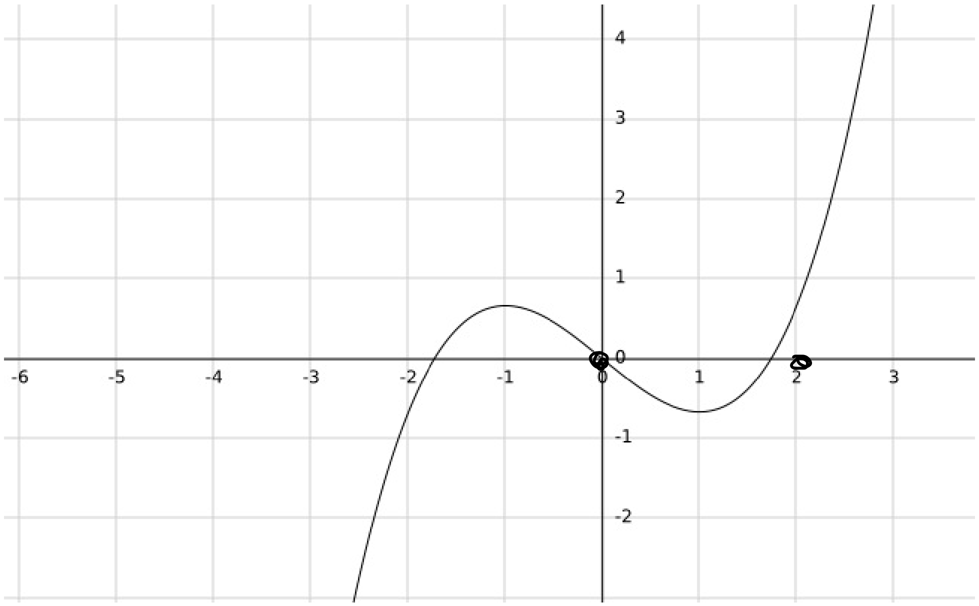
C^0 IN a, b

$\overset{\text{EXTR}}{\underset{\text{VAL THM}}{\Longrightarrow}} \quad R(f) = [m, M]$

HOW TO FIND $M = \max\limits_{[a,b]} f$ & $m = \min\limits_{[a,b]} f$?

EXAMPLE

$$f(x) = \frac{1}{3}x^3 - x \text{ ON } [0,2]$$



STATIONARY PTS:

$f'(x) = 0$ in $]0,2[$

$\parallel$

$x^2 - 1 = 0$

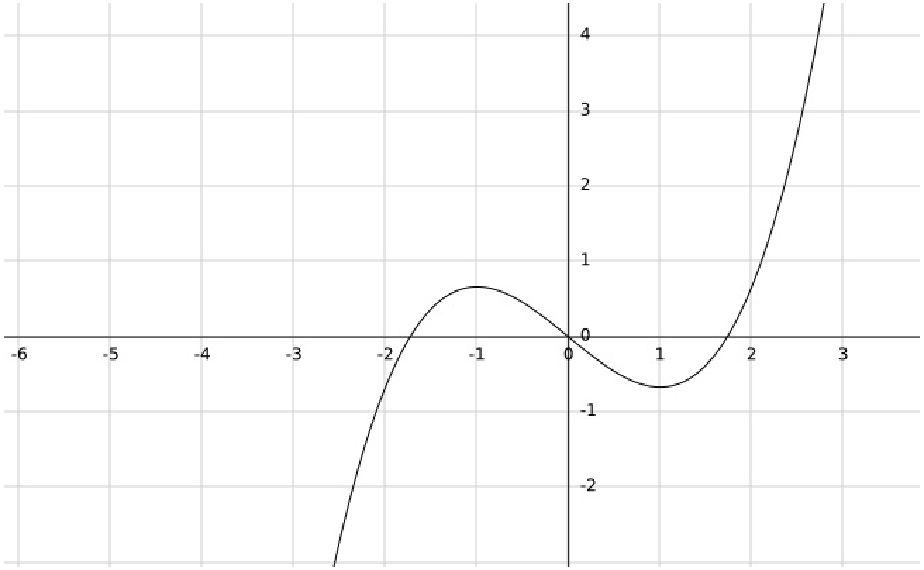
SOL'S : $x = 1$ ~~-1~~

f HAS A LOCAL MIN AT $x = 1$

OF VALUE $f(1) = -\frac{2}{3}$

EXAMPLE

$$f(x) = \frac{1}{3}x^3 - x \quad \text{ON } [0,2]$$



STATIONARY PTS:

$$f'(x) = 0 \quad \text{in }]0,2[$$

$$\| \quad$$

$$x^2 - 1 = 0$$

SOL'S: $x = 1, \cancel{-1}$

$$M \doteq \max_{[a,b]} f = \max\{f(0), f(2)\} = \max\{0, \underbrace{\tfrac{8}{3} - 2}_{2/3}\} = \frac{2}{3}$$

$$m \doteq \min_{[a,b]} f = f(1) = -\frac{2}{3}$$

$f : [a,b] \to \mathbb{R}$ ASSUME $f|_{]a,b[}$ IS DIFFERENTIABLE &

RESTRICTION OF f TO $]a,b[$

C^0 IN a,b

$\xRightarrow[\text{VAL THM}]{\text{EXTR.}} R(f) = [m, M]$

HOW TO FIND $M = \max_{[a,b]} f$?

- COMPUTE STATIONARY PTS:

$f : [a,b] \to \mathbb{R}$ ASSUME $f|_{]a,b[}$ IS DIFFERENTIABLE &

RESTRICTION OF f TO $]a,b[$

C^0 IN a, b

$\xRightarrow[\text{VAL THM}]{\text{EXTR.}}$ $R(f) = [m, M]$

HOW TO FIND $M = \max_{[a,b]} f$?

- COMPUTE STATIONARY PTS:

SOLVE EQN $f'(x) = 0$ in $]a,b[\rightsquigarrow \{x_i\}$ ← SET OF SOL'S

- $M = \max\{f(a), f(b), f(x_1), f(x_2), \dots, f(x_i), \dots\}$

$m = \min\{f(a), f(b), f(x_1), f(x_2), \dots, f(x_i), \dots\}$

ROLLE'S THM

$f : [a, b] \to \mathbb{R}$ C^0 & DIFF ON $]a, b[$

ASSUME $f(a) = f(b)$

WHAT CAN WE SAY ABOUT STATIONARY PTS OF f ?

ROLLE'S THM

$f : [a, b] \to \mathbb{R}$ C^0 & DIFF ON $]a, b[$

ASSUME $f(a) = f(b)$

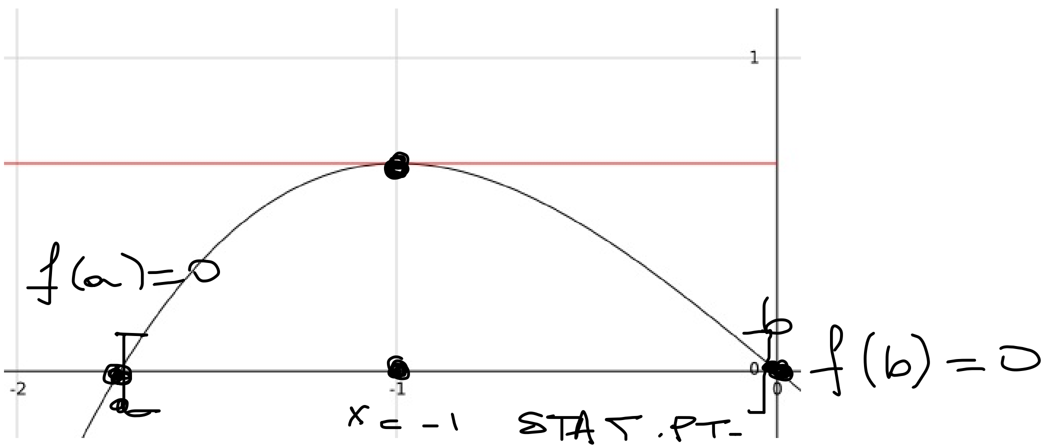
WHAT CAN WE SAY ABOUT STATIONARY PTS OF f?

CASE 1 : f IS CONSTANT $\Longrightarrow f'(x) = 0 \quad \forall x \in]a,b[$

CASE 2 :

f NOT CONSTANT

$\Longrightarrow M > m$



THM

$f : [a,b] \to \mathbb{R}$ C^0 & DIFFER. ON $]a,b[$

IF $f(a) = f(b) \Rightarrow \exists\, c \in]a,b[$ S.T. $f'(c) = 0$.

QUICK WORDS !

$f(a) = f(b) \implies$ f' has a 0 somewhere btw a & b

THM $f : [a,b] \to \mathbb{R}$ C^0 & DIFFER. ON $]a,b[$

IF $f(a) = f(b) \Rightarrow \exists\, c \in]a,b[$ S.T. $f'(c) = 0$.

PROOF $m = \min_{[a,b]} f$ & $M = \max_{[a,b]} f$ EXIST

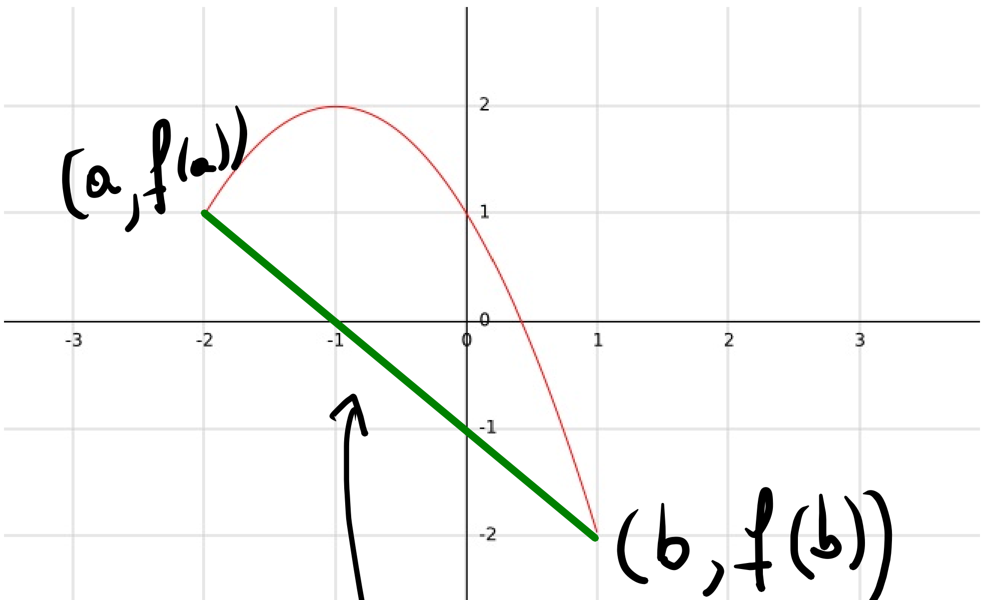
AT LEAST 1 OF THEM $\neq f(a), f(b)$

$\Rightarrow$ A STATIONARY PTS EXIST FOR f IN $]a,b[$

MEAN VALUE THM

SIMILAR SETUP AS BEFORE

$f : [a, b] \to \mathbb{R}$, C^0 & DIFF. in $]a, b[$



$f(x) = -s^2 - 2s + 1$

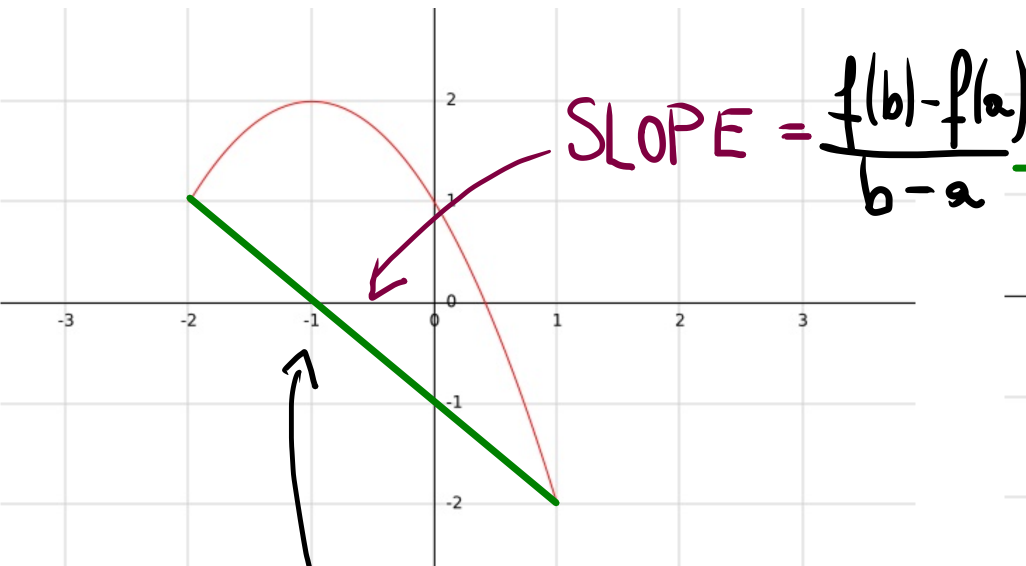
on $[\underset{a}{\underset{\shortparallel}{-2}}, \underset{b}{\underset{\shortparallel}{1}}]$

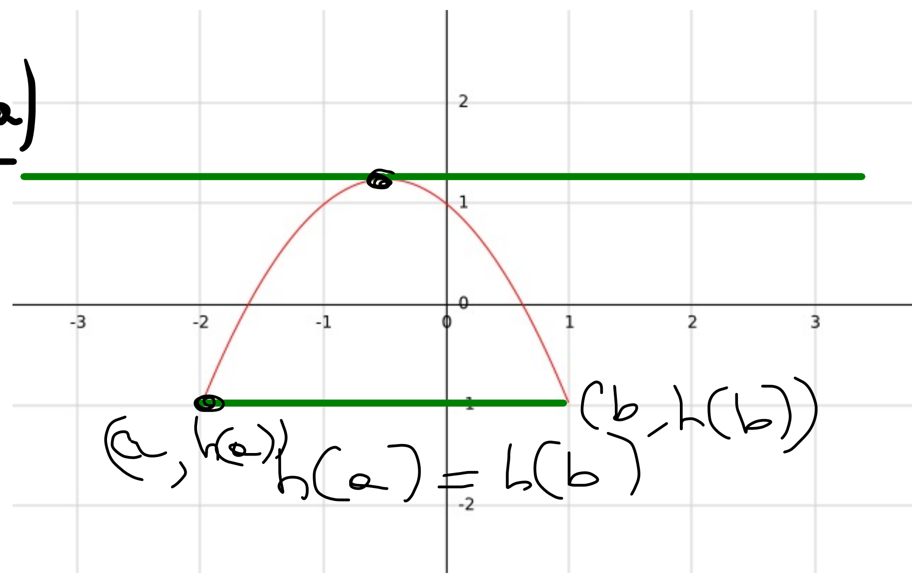
line connecting THESE 2 PTS

MEAN VALUE THM

SIMILAR SETUP AS BEFORE

$f : [a,b] \to \mathbb{R}$, C^0 & DIFF. in $]a,b[$





IF THIS LINE WAS HORIZONTAL

$\Rightarrow \exists$ STATIONARY PT.

THM $f : [a,b] \to \mathbb{R}$, C^0 ON $[a,b]$,
DIFF. ON $]a,b[$

$\Rightarrow \exists\ c \in\ ]a,b[$ S.T.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$