ANALYSIS 1

(ENGLISH)

LECTURE 1

INTRO: WHY ANALYSIS?

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UNDERSTANDING PROPERTIES OF

- REAL / COMPLEX NUMBERS
- FUNCTIONS OF REAL #'s WITH VALUES IN THE REAL #'s

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[CONTINUITY, DIFFERENTIABILITY, INTEGRALS, ...]

INTRO: WHY ANALYSIS?

UNDERSTANDING PROPERTIES OF

- REAL NUMBERS / COMPLEX NUMBERS
- FUNCTIONS OF REAL #'s WITH VALUES IN THE REAL #'s

[CONTINUITY, DIFFERENTIABILITY, INTEGRALS, ...]

LATER: ALSO FUNCTIONS OF COMPLEX #'s

WHY SHOULD YOU CARE ?

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- PHYSICS : $v(t) = \frac{d}{dt} p(t)$

velocity at time t

derivative

position of a particle at time t

WHY SHOULD YOU CARE?

- PHYSICS: $v(t) = \frac{d}{dt} p(t)$

  velocity

  derivative

  position

- ENGINEERING: $W(t) = \int_0^t \langle \vec{F}(t), \vec{p}(t) \rangle \, dt$

  work

  force on the particle

  position

WHY SHOULD YOU CARE?

- PHYSICS: $v(t) = \frac{d}{dt} p(t)$
- ENGINEERING: $W(t) = \int_0^t \langle \vec{F}(t), \vec{p}(t) \rangle dt$

  work   force   position
- BIOLOGY: $\begin{cases} R(t) = \text{\# rats at time } t \\ H(t) = \text{\# hawks} \text{ — '' —} \end{cases}$

WHY SHOULD YOU CARE?

- PHYSICS : $v(t) = \frac{d}{dt} p(t)$
- ENGINEERING : $W(t) = \int^{t} \langle \vec{F}(t), \vec{p}(t) \rangle \, dt$
- BIOLOGY :
$$\begin{cases} \frac{d}{dt} R(t) = \alpha R(t) - \beta R(t) H(t) \\ \frac{d}{dt} H(t) = -\gamma H(t) + \delta R(t) H(t) \end{cases}$$

MATHEMATICAL VIEWPOINT:

WE WOULD LIKE TO UNDERSTAND WHAT HAPPENS WHEN WE WORK WITH ∞

infinitely small

infinitely big.

$$v(t) = \frac{d}{dt} p(t) \qquad \frac{p(t+h) - p(t)}{(t+h) - t}$$

What happens when $h \ll 1$? LIMITS

PROOFS :

PROOFS: WHAT IS A PROOF?

START : ASSUMPTION / CONCEPT
(AXIOMS)
(HYPOTHESES OF A THM.)

METHOD : LOGICAL DEDUCTIONS
FROM OUR STARTING
ASSUMPTIONS TO
FIND NEW PROPERTIES.

PROOFS: SHOWING THAT EACH OBSERVATION FOLLOWS FROM THE PREVIOUS ONES IN A LOGICAL WAY

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CONSTRUCTIVE VS BY CONTRADICTION

CONSTRUCTIVE	BY CONTRADICTION
CONSTRUCT SOME OBJECT OR WE EXPLICITLY SHOW THAT PROPERTIES HOLD	WE NEGLECT THE CONCLUSION (THE PROPERTY WE WISH TO PROVE) AND OBTAIN A CONTRADIOTION.

PROOFS: SHOWING THAT EACH OBSERVATION FOLLOWS FROM THE PREVIOUS ONES IN A LOGICAL WAY

EXAMPLE

PROPOSITION: $\sqrt{3}$ IS A REAL NUMBER BUT IT IS NOT RATIONAL.

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PROPOSITION: $\sqrt{3}$ IS A REAL NUMBER BUT IT IS NOT RATIONAL.

RATIONAL NUMBERS: $\frac{a}{b}$, $a, b \in \mathbb{Z}$, $b \neq 0$

WITH THE CONDITION $\frac{a \cdot c}{b \cdot c} = \frac{a}{b}$, $\forall c \in \mathbb{Z}^*$

PROP. $\sqrt{3}$ IS IRRATIONAL

PROOF ASSUME $\sqrt{3} = \frac{a}{b}$, a, b are integers
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WE MAY ASSUME:

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$b \neq 0$

WE MAY ASSUME: a, b relatively prime

$\gcd(a, b) = 1$

$a = a' \cdot \gcd$

$b = b' \cdot \gcd$

$$\sqrt{3} = \frac{a}{b} \iff b\sqrt{3} = a$$

$$\implies b^2 \cdot 3 = a^2$$

$$3b^2 = a^2 \implies 3 \mid 3b^2 \text{ and } 3 \mid a^2$$

$3b^2 = a^2 \xRightarrow{} 3 \mid 3b^2$ and $3 \mid a^2$ $= a \cdot a$

LOGIC IMPLICATION

$\xRightarrow[\text{PRIME}]{3\text{ IS}} 3 \mid a \Longrightarrow a = 3 \cdot r$ integer

p prime

if $p \mid c \cdot d \Rightarrow p \mid c$ OR $p \mid d$

$$3b^2 = a^2 \xRightarrow[\text{LOGIC IMPLICATION}]{} 3 \mid 3b^2 \text{ and } 3 \mid a^2$$

$$\xRightarrow[\text{PRIME}]{3 \text{ IS}} 3 \mid a \implies a = 3 \cdot r$$

$$\implies 3b^2 = a^2 = 9r^2$$

$$\Downarrow$$

$$\cancel{3}b^2 = \cancel{9}r^2$$

$$b^2 = 3r^2 \implies 3 \mid b^2 = b \cdot b$$

$$\implies 3 \mid b$$

$$3 \mid \gcd(a,b) \neq 1$$

PROP. $\sqrt{3}$ IS IRRATIONAL

PROOF ASSUME $\sqrt{3} = \dfrac{a}{b}$, a, b are integers

$b \neq 0$

$\boxed{\text{WE MAY ASSUME: } a, b \text{ relatively prime, that is, no common divisors for } a, b}$

$$3b^2 = a^2 \implies 3 \mid 3b^2 \text{ and } 3 \mid a^2$$

LOGIC IMPLICATION

$$\overset{3 \text{ IS}}{\underset{\text{PRIME}}{\implies}} \quad 3 \mid a \implies a = 3 \cdot r$$

$$\implies 3b^2 = a^2 = (3r)^2 = 9r^2$$

$$\implies b^2 = 3r^2$$

$$\implies 3 \mid b^2 \text{ so that } 3 \mid b$$

$$\implies 3 \mid a \wedge 3 \mid b \longrightarrow \text{contradiction}$$

$$\blacksquare$$

PROOFS: SHOWING THAT EACH OBSERVATION FOLLOWS FROM THE PREVIOUS ONES IN A LOGICAL WAY

ATTENTION: NO WRONG ASSUMPTIONS

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"**PROP.**" 1 IS THE LARGEST INTEGER

PROOF. LET ℓ BE THE LARGEST INTEGER

$$\Longrightarrow \quad \ell \geq \ell^2$$

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"**PROP.**" 1 IS THE LARGEST INTEGER

PROOF. LET ℓ BE THE LARGEST INTEGER

$$\Longrightarrow \quad \ell \geqslant \ell^2 \quad \Longleftrightarrow \quad \ell - \ell^2 \geqslant 0$$

PROOFS: SHOWING THAT EACH OBSERVATION FOLLOWS FROM THE PREVIOUS ONES IN A LOGICAL WAY

ATTENTION: NO WRONG ASSUMPTIONS

"**PROP.**" 1 IS THE LARGEST INTEGER

PROOF. LET ℓ BE THE LARGEST INTEGER

$$\Longrightarrow \quad \ell \geq \ell^2 \iff \underset{\ell(1-\ell)}{\ell - \ell^2} \geq 0$$

$$\ell(1-\ell) \geq 0$$

$\ell(1-\ell) \geq 0$ [ℓ IS AN INTEGER]

EITHER $\ell \geq 0 \wedge (1-\ell) \geq 0$

~~OR~~ $\cancel{\ell \leq 0 \wedge (1-\ell) \leq 0}$

$1 \in$ INTEGERS $\Longrightarrow \ell = 1$

$\geq$

0

$2 \in \mathbb{Z}$ $\quad 2 > 1$

IS THIS PROOF CORRECT?

(VOTE THE POLL)

IS THIS PROOF CORRECT?

NO !!!

IS THIS PROOF CORRECT?

NO !!!

INTEGERS = $\{0, 1, -1, 2, -2, \ldots\}$

PROOFS: SHOWING THAT EACH OBSERVATION FOLLOWS FROM THE PREVIOUS ONES IN A LOGICAL WAY

ATTENTION: NO WRONG ASSUMPTIONS

"**PROP.**" 1 IS THE LARGEST INTEGER

PROOF. LET ℓ BE THE LARGEST INTEGER

WRONG!! WHY? $\ell \in \mathbb{Z} \Longrightarrow \underbrace{\ell+1}_{\in \mathbb{Z}} > \ell$

BASIC NOTIONS

SETS A SET IS A COLLECTION OF OBJECTS ("ELEMENTS")

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EXAMPLE NATURAL NUMBERS

$$\text{NATURAL NUMBERS} = \{\underbrace{0}_{\text{ELEMENTS}}, 1, 2, 3, \ldots, n, n+1, \ldots\}$$

ELEMENTS

BASIC NOTIONS

SETS

A SET IS A COLLECTION OF OBJECTS ("ELEMENTS")

EXAMPLE NATURAL NUMBERS

$$\text{NATURAL NUMBERS} = \{0, 1, 2, 3, \ldots, n, n+1, \ldots\}$$

NOTATION: $a \in S$, $a \notin S$

$$p \mid d$$

SUBSET: A SET T IS A SUBSET OF SET S, IF ALL ELEMENTS OF T ARE ELEMENTS OF S

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NOTATION: $T \subseteq S$ $\quad$ $T \subset S$

TO SAY THAT T IS A SUBSET OF S, $T \neq S$

$$T \subsetneq S$$

$$T \not\subseteq S \quad \text{(NOT A SUBSET)}$$

SUBSET: A SET T IS A SUBSET OF SET S, IF ALL ELEMENTS OF T ARE ELEMENTS OF S.

NOTATION: $T \subseteq S$

EXAMPLE

EVEN NATURAL #'s = $\{0, 2, 4, 6, \ldots, 2n, \ldots\}$

$2\mathbb{N}$

$\cap$

$\mathbb{N}$

2 WAYS TO DESCRIBE SETS

2 WAYS TO DESCRIBE SETS

- LISTING ALL ELEMENTS

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- LISTING ALL ELEMENTS

NATURAL #'S $= \{0, 1, 2, 3, 4, \dots, n, n+1, \dots\}$

2 WAYS TO DESCRIBE SETS

- LISTING ALL ELEMENTS

NATURAL #'s $= \{0, 1, 2, 3, 4, \ldots, n, n+1, \ldots\}$

- IMPOSING CONDITIONS

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- LISTING ALL ELEMENTS

NATURAL #'s $= \{0, 1, 2, 3, 4, \ldots, n, n+1, \ldots\}$

- IMPOSING CONDITIONS

$\{ x \mid x$ SATISFIES SOME CONDITION $\}$

EVEN NATURAL #'s : $\{ x \mid x \in \mathbb{N} \wedge 2 \mid x \}$

OPERATIONS ON SETS

S, T SETS

OPERATIONS ON SETS

S, T SETS

- $S \cup T = \{x \mid x \in S \text{ OR } x \in T\}$

union

- $S \cap T = \{x \mid x \in S \text{ AND } x \in T\}$

intersection

- $S \setminus T = \{x \mid x \in S \text{ AND } x \notin T\}$

complement of T inside S

IMPORTANT SETS

- $\emptyset$ = EMPTY SET

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- $\mathbb{N}$ = NATURAL NUMBERS $= \{0, 1, 2, 3, \ldots\}$

  ORDERING: $p, q \in \mathbb{N}$

  $p > q$

  $p < q$

  $p = q$

IMPORTANT SETS

- $\emptyset$ = EMPTY SET

  ALWAYS : $\emptyset \subseteq S \leftarrow$

- $\mathbb{N}$ = NATURAL NUMBERS $= \{0, 1, 2, 3, \dots\}$

- $\mathbb{Z}$ = INTEGERS

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  ALWAYS : $\emptyset \subseteq S$

- $\mathbb{N}$ = NATURAL NUMBERS $= \{0, 1, 2, 3, \ldots\}$
- $\mathbb{Z}$ = INTEGERS $= \{0, 1, -1, 2, -2, \ldots\}$

IMPORTANT SETS

- $\emptyset$ = EMPTY SET

  ALWAYS : $\emptyset \subseteq S \longleftarrow$

- $\mathbb{N}$ = NATURAL NUMBERS = $\{0, 1, 2, 3, \ldots\}$

- $\mathbb{Z}$ = INTEGERS = $\{0, 1, -1, 2, -2, \ldots\}$

- $\mathbb{Q}$ = RATIONAL NUMBERS

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$$= \left\{ x \;\middle|\; x = \frac{a}{b} \;,\; a, b \in \mathbb{Z}, b \neq 0 \right.$$

IN ADDITION: $\dfrac{a \cdot c}{b \cdot c} = \dfrac{a}{b}$

$$\left. \text{for any } a, b, c \in \mathbb{Z} \;,\; b, c \neq 0 \right\}$$

ORDERING :

OPERATIONS : $(+, 0)$ $(\cdot\,, 1)$

NEW: $\forall b \in \mathbb{Q}^* \quad \exists\, b^{-1} \in \mathbb{Q}$ s.t. $b^{-1} \cdot b = 1$

ALSO $\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q}$

ALSO $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q}$

$$\underset{\mathbb{Z}}{x} = \frac{x}{1} \in \mathbb{Q}$$

- $\mathbb{R}$ = REAL NUMBERS

ALSO $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q}$

$$\underset{\substack{\uparrow \\ \mathbb{Z}}}{x} = \frac{x}{1} \in \mathbb{Q}$$

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HOW DO WE DEFINE REAL #'s ?

ALSO $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q}$

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HOW DO WE DEFINE REAL #'s ?

INTUITIVELY: $\underbrace{1347\ldots6}_{\text{FINITELY MANY DIGITS}}$.

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HOW DO WE DEFINE REAL #'S ?

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THE SITUATION IS MORE COMPLICATED

PROP. $0.\overline{9} = 1$

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PROOF.

THE SITUATION IS MORE COMPLICATED

PROP. $0.\overline{9} = 1$

PROOF. $10 \cdot 0.\overline{9} - 0.\overline{9} = 9$

$9.\overline{9}$ $=$

$\parallel$

$9 \cdot (0.\overline{9})$

THE SITUATION IS MORE COMPLICATED

PROP. $0.\overline{9} = 1$

PROOF.

$$\underbrace{10 \cdot 0.\overline{9} - 0.\overline{9}}_{= \, 9 \cdot 0.\overline{9}} = \underbrace{9.\overline{9} - 0.\overline{9}}_{= \, 9}$$

$$\Downarrow$$

IF $x = 0.\overline{9}$ THEN $9x = 9$

$$\Longrightarrow x = 1$$

ALTERNATIVE PROOF. (MORE "ANALYTIC")

$$0.\overline{9} \overset{?}{=} 9/10 + \frac{9}{100} + \frac{9}{1000} +$$

$$9/1000 + \ldots + \frac{9}{10^n}$$

$$\text{"="} \quad \sum_{n=1}^{\infty} \frac{9}{10^n}$$

ALTERNATIVE PROOF. (MORE "ANALYTIC")

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IS THIS WELL DEFINED?

ALTERNATIVE PROOF. (MORE "ANALYTIC")

$$0.\overline{9} = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots$$

$$= \sum_{i=1}^{\infty} \frac{9}{10^i} = 9 \cdot \sum_{i=1}^{\infty} \frac{1}{10^i}$$

$$9 \cdot \frac{1}{10^i}$$

ALTERNATIVE PROOF. (MORE "ANALYTIC")

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$$= \sum_{i=1}^{\infty} \frac{9}{10^i} = 9 \cdot \sum_{i=1}^{\infty} \frac{1}{10^i}.$$

CLAIM Take $a > 0$, $a \neq 1$.

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CLAIM

Take $a > 0$, $a \neq 1$.

$$a + a^2 + a^3 + \ldots + a^n =$$

ALTERNATIVE PROOF. (MORE "ANALYTIC")

$$0.\overline{9} = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots$$

$$= \sum_{i=1}^{\infty} \frac{9}{10^i} = 9 \cdot \sum_{i=1}^{\infty} \frac{1}{10^i}.$$

CLAIM Take $a > 0$, $a \neq 1$.

$$a + a^2 + a^3 + \cdots + a^n = \frac{a - a^{n+1}}{1 - a}$$

ALTERNATIVE PROOF. (MORE "ANALYTIC")

$$0.\overline{9} = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots$$

$$= \sum_{i=1}^{\infty} \frac{9}{10^i} = 9 \cdot \sum_{i=1}^{\infty} \frac{1}{10^i}$$

CLAIM Take $a > 0$, $a \neq 1$.

$$a + a^2 + a^3 + \cdots + a^n = \frac{a - a^{n+1}}{1 - a}$$

PROOF $(1-a)(a + a^2 + \cdots + a^n) =$

$$a + \cancel{a^2} + \cancel{a^3} + \cdots + \cancel{a^n}$$
$$\cancel{- a^2} \quad \cancel{- a^3} \quad \cancel{- a^n} - a^{n+1} = a - a^{n+1}$$

$$9 \cdot \sum_{i=1}^{k} \frac{1}{10^i} = 9 \cdot \left(\frac{1}{10} + \frac{1}{10^2} + \cdots + \frac{1}{10^k} \right)$$

$$= 9 \cdot \frac{\left(\frac{1}{10} - \frac{1}{10^{k+1}} \right)}{1 - \frac{1}{10}}$$

going to 0

$$9 \cdot \sum_{i=1}^{\infty} \frac{1}{10^i} = 9 \cdot \frac{\left(\frac{1}{10} \right)}{1 - \frac{1}{10}} = 1$$

$$9 \cdot \sum_{i=1}^{n} \frac{1}{10^i} = 9 \cdot \left[\frac{\frac{1}{10} - \left(\frac{1}{10}\right)^{n+1}}{1 - \frac{1}{10}} \right]$$

$\Downarrow$ TAKING LIMIT FOR

$n \longrightarrow \infty$

HOW TO DEFINE $\mathbb{R}$?

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THERE IS AN ACTUAL SOLUTION
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RATHER THAN CONSTRUCTING $\mathbb{R}$ WE LIST ITS MOST IMPORTANT PROPERTIES:

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THERE IS AN ACTUAL SOLUTION TO THIS PROBLEM

RATHER THAN CONSTRUCTING $\mathbb{R}$ WE LIST ITS MOST IMPORTANT PROPERTIES :

1. $\mathbb{Q} \subseteq \mathbb{R}$ $\left(\subsetneq \quad \underset{\mathbb{R}}{\sqrt{3}} \notin \mathbb{Q} \right)$

HOW TO DEFINE $\mathbb{R}$?

THERE IS AN ACTUAL SOLUTION TO THIS PROBLEM.

RATHER THAN CONSTRUCTING $\mathbb{R}$ WE LIST ITS MOST IMPORTANT PROPERTIES:

1. $\mathbb{Q} \subseteq \mathbb{R}$
2. $\mathbb{R}$ IS AN ORDERED FIELD:

ORDERING: $(\leq \quad)$

FIELD: $(+, 0)$, $(\cdot, 1)$ $\quad \forall x \in \mathbb{R}^* \; \exists x^{-1}$

HOW TO DEFINE $\mathbb{R}$?

THERE IS AN ACTUAL SOLUTION TO THIS PROBLEM

RATHER THAN CONSTRUCTING $\mathbb{R}$ WE LIST ITS MOST IMPORTANT PROPERTIES :

1. $\mathbb{Q} \subseteq \mathbb{R}$
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INTUITIVELY :

HOW TO DEFINE $\mathbb{R}$?

THERE IS AN ACTUAL SOLUTION TO THIS PROBLEM.

RATHER THAN CONSTRUCTING $\mathbb{R}$ WE LIST ITS MOST IMPORTANT PROPERTIES:

1. $\mathbb{Q} \subseteq \mathbb{R}$
2. $\mathbb{R}$ IS AN ORDERED FIELD
3. $\mathbb{R}$ SATISFIES "THE INFIMUM AXIOM"

INFIMUM AXIOM

IF $S \subseteq \mathbb{R}^*_+$ THEN THERE EXISTS A LARGEST $\ell \in \mathbb{R}$ IN THE SET

$$\{ x \in \mathbb{R} \mid x \leq s, \forall s \in S \}$$

IMPORTANT SETS

- $\emptyset$ = EMPTY SET

$\cap|$

- $\mathbb{N}$ = NATURAL #'s

$\cap|$

- $\mathbb{Z}$ = INTEGERS

$\cap|$

- $\mathbb{Q}$ = RATIONAL #'s

$\cap|$

- $\mathbb{R}$ = REAL #'s

SETS: SOME MORE NOTATION

$\mathbb{N}^*$, $\mathbb{Z}^*$, $\mathbb{Q}^*$, $\mathbb{R}^*$

non-zero elt's

of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ respectively