ANALYSIS 1

(ENGLISH)

LECTURE 19

CONTINUITY VS DIFFERENTIABILITY

$$f : E \to \mathbb{R}, \quad x_0 \in E$$

- f IS CONTINUOUS AT x_0 IF

$$\lim_{x \to x_0} f(x) = f(x_0)$$

- f IS DIFFERENTIABLE AT x_0

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{(x - x_0)} \text{ EXISTS}$$

$$\lim_{x \to x_0} \frac{r(x)}{(x - x_0)} = 0 \;\Downarrow\; \lim_{x \to x_0} r(x) = 0 = \lim_{x\to x_0}\frac{r(x)}{(x-x_0)}\,(x-x_0) \quad (\to 0,\ \to 0)$$

$$\Updownarrow$$

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} f(x_0) + \cancel{f'(x_0)} \cdot \cancel{(x - x_0)} + \cancel{r(x)}, \quad (x-x_0) \to 0, \quad r(x) \to 0$$

PROP $f: E \longrightarrow \mathbb{R}$, $x_0 \in E$

IF f IS DIFFERENTIABLE AT x_0

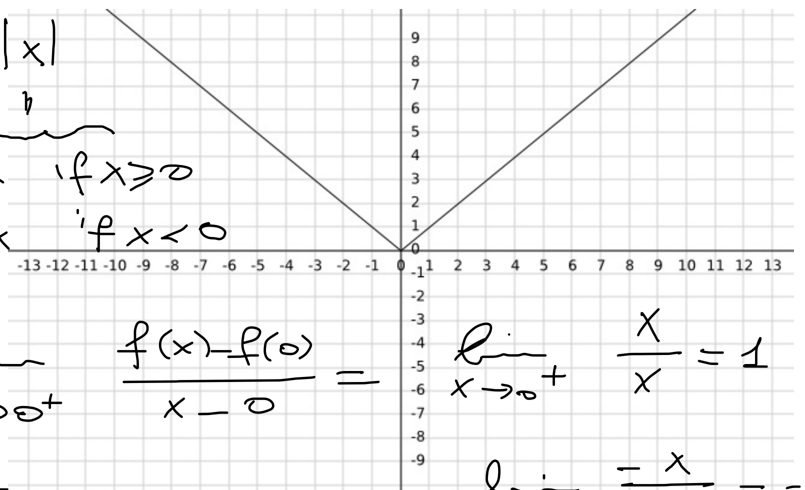
THEN f IS C^0 AT x_0

DOES THE VICEVERSA HOLD?

$f: E \to \mathbb{R}$, $x_0 \in E$

f C^0 AT x_0 $\overset{?}{\Longrightarrow}$ f DIFFERENTIABLE AT x_0

$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$



$x_0 = 0$

$|x|$ is C^0 AT 0

$\lim_{x \to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0^+} \frac{x}{x} = 1$

$\lim_{x \to 0^-} \cdots = \lim_{x \to 0^-} \frac{-x}{x} = -1$

$\lim_{x \to 0} \frac{f(x)-f(0)}{x-0}$ DOES NOT EXIST

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \quad 1 \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \quad 0 \end{cases}$$

$$0 \le \underbrace{|f(x)|}_{x\to 0 \,\to\, 0} \le \underbrace{|x|}_{x \to 0 \,\to\, 0}$$

$$\Longrightarrow \lim_{x \to 0} f(x) = 0$$

f IS C^0 AT 0 & THIS IS THE ONLY PT OF C^0

$\Longrightarrow f$ IS NOT DIFF. AT ANY PT $\in \mathbb{R}^*$

$\left[\lim_{x \to x_0} f(x) \text{ DOES NOT EXIST} \right]$ x_0

$$\lim_{x \to 0} \frac{f(x)}{x}$$ CANNOT EXIST

$\Longrightarrow$ f is NOT DIFF. AT $x=0$

$$g(x) = \frac{f(x)}{x} = \begin{cases} 1 & x \in \mathbb{Q}^* \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

$$x_n = \frac{1}{n}$$

$g(x_n) = 1 \quad \forall n$

$g(x_n) \to 1$

$$y_n = \frac{1}{n \cdot \pi}$$

$g(y_n) = 0 \quad \forall n$

$g(y_n) \to 0$

EXAMPLE

$f(x) = e^x$

FORMAL DEFINITION:

$$e^x := \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

CONV. ABS $\forall x \in \mathbb{R}$

WELL DEFINED: $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ CONVERGES

[D'ALEMBERT'S CRITERION]

EXAMPLE $f(x) = e^x$

FORMAL DEFINITION:

$$e^x := \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$e^x : \mathbb{R} \longrightarrow \mathbb{R}$

$e^x \in C^0(\mathbb{R})$

WELL DEFINED: $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ CONVERGES

PROPERTIES:

- $e^0 = 1 = 1 + \frac{0}{1!} + \frac{0}{2!} + \frac{0}{3!} + \dots$
- $e^x \cdot e^y = e^{x+y}$
- $e^{-x} = \frac{1}{e^x} \Longleftarrow e^x \cdot e^{-x} = e^0 = 1$

IS e^x DIFFERENTIABLE?

FOR $x_0 \in \mathbb{R}$

DOES $\displaystyle\lim_{h \to 0} \frac{e^{x_0+h} - e^{x_0}}{h}$ EXIST?

$$\lim_{h \to 0} \frac{e^{x_0+h} - e^{x_0}}{h} = \lim_{x \to x_0} \frac{e^{x} - e^{x_0}}{x - x_0} \qquad x = x_0 + h$$

$$\lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} \frac{e^x \cdot e^h - e^x}{h} =$$

$$\lim_{h \to 0} \frac{e^x \left(e^h - 1\right)}{h} = e^x \qquad \to 1$$

$$(e^x)' = e^x$$

TO SHOW WE PROVE THAT

$$S(h) = \frac{e^h - 1}{h} = \frac{\sum_{k=1}^{\infty} \frac{h^k}{k!}}{h} = \frac{h \cdot \sum_{k=1}^{\infty} \frac{h^{k-1}}{k!}}{\cancel{h}}$$

$$\lim_{h \to 0} S(h) = S(0) = 1$$

$$* \quad \sum_{k=1}^{\infty} \frac{h^{k-1}}{k!} = s(h)$$

$s(0) = 1$

s is C^0 AT 0

CONVERGES ABSOLUTELY

$t(h) \xrightarrow{h \to 0} 1$

$$\frac{\frac{h^{\cancel{k}}}{(k+1)\cancel{!}}}{\cancel{\frac{h^{k-1}}{k!}}} \xrightarrow{k\to\infty} 0$$

$$t(h) = \sum_{k=1}^{\infty} \frac{|h|^{k-1}}{k!}$$

geometric series

$$1 \le \cancel{|s(h)|} \le \sum_{k=1}^{\infty} |h|^{k-1} = \frac{1}{1-h} \xrightarrow{h\to 0} 1$$

PROP $f, g : E \to \mathbb{R}$, $x_0 \in E$

f, g DIFFERENTIABLE AT x_0

① THEN $f \cdot g$ DIFFERENTIABLE AT x_0

LEIBNIZ RULE

$$(f \cdot g)'(x_0) = f'(x_0) \cdot g(x_0) + g'(x_0) \cdot f(x_0)$$

② $g(x_0) \neq 0 \Rightarrow \frac{f}{g}$ DIFFERENTIABLE AT x_0

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)\, g(x_0) - f(x_0)\, g'(x_0)}{g(x_0)^2}$$

3 $f+g$ is DIFF. AT x_0

& $(f+g)'(x_0) = f'(x_0)+g'(x_0)$

EXAMPLE

$$h(x) = \underbrace{x^2}_{f} \cdot \underbrace{e^x}_{g}$$

$$h'(x) = f'(x) \cdot g(x) + g'(x) \cdot f(x)$$

$$= 2 \cdot x^{2-1} \cdot e^x + e^x \cdot x^2$$

$$= e^x \cdot (2x + x^2)$$

EXAMPLE

$$h(x) = \tan(x) = \frac{\overset{=f}{\sin x}}{\underset{=g}{\cos x}}$$

$$D(h) = \mathbb{R} \setminus \left\{ \frac{\pi}{2} + k\pi, \; k \in \mathbb{Z} \right\}$$

$$h'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2} =$$

$$= \frac{\cos(x)\cos(x) - (-\sin(x))\sin(x)}{(\cos(x))^2}$$

$$= \frac{\cos(x)^2 + \sin(x)^2}{\cos(x)^2} = \frac{1}{\cos(x)^2}$$

COMPOSITION & DERIVATIVES

$f : E \to \mathbb{R}$, $g : F \to \mathbb{R}$

IF $R(f) \subseteq D(g)$, THEN

$g \circ f : E \to \mathbb{R}$

$(g \circ f)(x) = g(f(x))$

COMPOSITION & DERIVATIVES

$$f : E \longrightarrow \mathbb{R}\ ,\quad g : F \longrightarrow \mathbb{R}$$

IF $R(f) \subseteq D(g) = S$, THEN

$$g \circ f : E \longrightarrow \mathbb{R}$$

PROP LET $x_0 \in E$. IF f IS DIFFERENT. AT x_0 & g IS DIFFERENTIABLE AT $f(x_0)$ THEN $g \circ f$ IS DIFFERENTIABLE AT x_0 AND

$$(g \circ f)'(x_0) = g'(\underline{f(x_0)}) \cdot f'(x_0)$$

EXAMPLE

$$f(x) = \cos(x^2) + \frac{1}{x^2+1}$$

$$f(x) = (g \circ h)(x)$$

$$g(t) \doteq \cos(t) + \frac{1}{t+1} : \mathbb{R}\setminus\{-1\} \to \mathbb{R}$$

C^0 & DIFF OVER THEIR DOMAIN

$$h(x) = x^2 : \mathbb{R} \to \mathbb{R}$$

$$R(h) \subseteq D(g)$$

$$\mathbb{R}_+ \subseteq \mathbb{R}\setminus\{-1\}$$

$g \circ h = f : \mathbb{R} \to \mathbb{R}$

DIFF. OVER $\mathbb{R}$

$$f'(x) = (g \circ h)'(x) = g'(h(x)) \cdot h'(x)$$

$$g'(t) = -\sin(t) + \frac{-1}{(t+1)^2} \qquad h'(x) = 2x$$

$$f'(x) = (g \circ h)'(x) = g'(h(x)) \cdot h'(x)$$

$$g'(t) = -\sin(t) + \frac{-1}{(t+1)^2} \qquad h'(x) = 2x$$

$$f'(t) = \left(-\sin(x^2) - \frac{1}{(x^2+1)^2} \right)(2x)$$

PROOF WE NEED TO SHOW THAT

$$\lim_{x \to x_0} \frac{(g \circ f)(x) - (g \circ f)(x_0)}{x - x_0} \quad \text{EXISTS.}$$

$$\|$$

$$\lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0}$$

PROOF WE NEED TO SHOW THAT

$$\lim_{x \to x_0} \frac{(g \circ f)(x) - (g \circ f)(x_0)}{x - x_0} \text{ EXISTS.}$$

$$\| $$

$$\lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0}$$

$$\|$$

$$\lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \cdot \boxed{\frac{f(x) - f(x_0)}{x - x_0}} \xrightarrow{x \to x_0} f'(x_0)$$

PROOF WE NEED TO SHOW THAT

$$\lim_{x \to x_0} \frac{(g \circ f)(x) - (g \circ f)(x_0)}{x - x_0} = g'(f(x_0)) f'(x_0)$$

$$\Uparrow$$

$$= \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0}$$

$$g'(f(x_0)) \xleftarrow{x \to x_0} \qquad \xrightarrow{x \to x_0} f'(x_0)$$

$$= \lim_{x \to x_0} \boxed{\frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)}} \cdot \boxed{\frac{f(x) - f(x_0)}{x - x_0}}$$

$A \subseteq \mathbb{R}$, $A \neq \emptyset$

If $\inf A$, $\sup A$ $\in A \implies A$ is a closed interval.

FALSE: $\{-1, 1\} = A$

INTERVAL: a, b

$\inf A = -1$ $\sup A = 1$

$f : \mathbb{N} \longrightarrow \mathbb{R}$ fct. s.t.

$f(n) < \frac{1}{n} \implies \sum f(n)$ CONVERGES.

FALSE: $f(n) = \frac{1}{n+1} < \frac{1}{n}$

HARM. SERIES $\implies \sum_{n \geq 0} \frac{1}{n+1}$ DOES NOT CONVERGE

SQUEEZE THM.

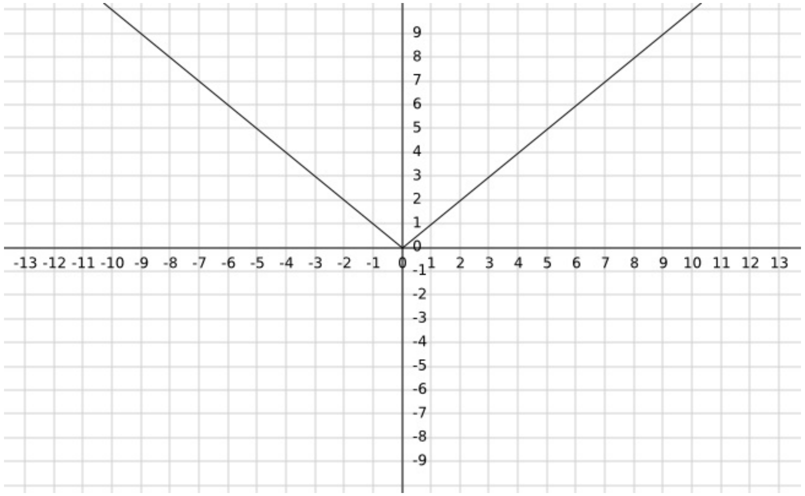
$$0 \le x_n \le y_n$$

IF $\sum^{\infty} y_n$ CONV. $\Longrightarrow \sum^{\infty} x_n$ CONV.

IF $\sum^{\infty} x_n = +\infty \Longrightarrow \sum^{\infty} y_n = +\infty$

$$f(n) < \frac{1}{n}$$

LEFT & RIGHT DERIVATIVE



$f(x) = |x|$

f IS NOT DIFF AT $x = 0$

BUT $\lim\limits_{x \to 0^+} \dfrac{f(x) - f(0)}{x - 0}$ EXISTS

$\lim\limits_{x \to 0^-}$ ——————"——————

$f: E \to \mathbb{R}$, $x_0 \in \mathbb{R}$

ASSUME THAT f IS DEFINED TO THE RIGHT OF x_0 $\quad]x_0, x_0+\delta[\subseteq E$, $\exists \delta > 0$

$f : E \to \mathbb{R}$, $x_0 \in E$

ASSUME THAT f IS DEFINED TO THE RIGHT OF x_0 $[x_0, x_0+\delta[\subseteq E$, $\exists \delta > 0$

IF $\lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0}$ EXISTS

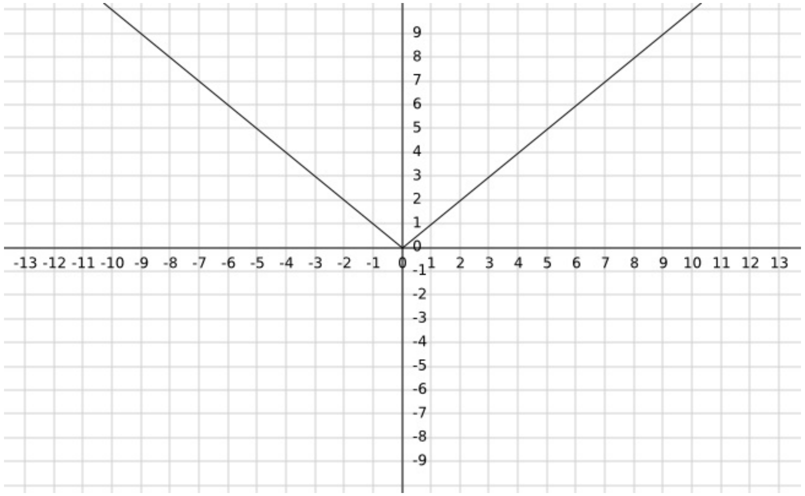
THEN IT IS CALLED THE RIGHT DERIVATIVE OF f AT x_0

$f : E \to \mathbb{R}$, $x_0 \in E$

ASSUME THAT f IS DEFINED TO THE LEFT OF x_0 $]x_0 - \delta, x_0] \subseteq E$, $\exists \delta > 0$

IF $$\lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0}$$ EXISTS

THEN IT IS CALLED THE LEFT DERIVATIVE OF f AT x_0



$f(x) = |x|$

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{x}{x} = 1$$

$$\lim_{x\to 0^-} \text{——— '' ———} = \lim_{x\to 0^-} \frac{-x}{x} = -1$$

PROP $f: E \to \mathbb{R}$, $x_0 \in E$

f IS DIFFERENTIABLE AT x_0

$\Updownarrow$

THE LEFT & RIGHT DERIVATIVE OF f AT x_0 EXIST & COINCIDE

$$\lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} = \ell = f'(x_0)$$

$$\ell = \lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0}$$

DERIVATIVE & INVERSE FUNCTIONS

Q: HOW TO COMPUTE $(\operatorname{Arcsin}(x))'$?

$(\sqrt{x})' = ?$

INVERSE OF x^2

$(\operatorname{Log}(x))' = ?$

INVERSE OF e^x

INVERSE OF $\sin(x)$

DERIVATIVE & INVERSE FUNCTIONS

Q: HOW TO COMPUTE $(\text{Arcsin}(x))'$?

$(\sqrt{x})' = ?$

INVERSE OF x^2

$(\log(x))' = ?$

INVERSE OF $\sin(x)$

Q: IF f IS INVERTIBLE & DIFFERENTIABLE, $(f^{-1})' = ?$

DERIVATIVE OF f^{-1}

THM $f : E \to \mathbb{R}$ CONTINUOUS

INTERVAL — WE ALLOW THE POSS. OF $\pm\infty$

f IS STRICTLY MONOTONE

$\Updownarrow$

f IS INJECTIVE $\Longrightarrow$ f^{-1} EXISTS [D(f) INTERVAL] & f^{-1} IS C^0

IF E IS OPEN INTERVAL

$\Downarrow$

$f(E)$ IS OPEN INTERVAL

Q : IF f IS INVERTIBLE & DIFFERENTIABLE, $(f^{-1})' = ?$

TAKE $y_0 \in \mathrm{Dom}(f^{-1})$ S.T. f^{-1} IS DEFINED ON A NBD OF y_0

$]y_0 - \delta, y_0 + \delta[$ $\delta > 0$

$$\lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} =$$

Q: IF f IS INVERTIBLE & DIFFERENTIABLE, $(f^{-1})' = ?$

$R(f)$ //

$$\lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} =$$

TAKE $y, y_0 \in D(f^{-1})$ S.T. f^{-1} IS DEFINED ON A NBD OF y_0

$]y_0 - \delta, y_0 + \delta[$

$\delta > 0$

$y = f(x)$

$$\lim_{x \to x_0} \frac{f^{-1}(f(x)) - f^{-1}(f(x_0))}{f(x) - f(x_0)} =$$

$f^{-1} \circ f = id_{Dom(f)}$

$f(x_0) = y_0$

$y \to y_0$

$$= \lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{f'(x_0)}$$

$x \to x_0 \overset{f \ C^0}{\underset{\text{NIV}}{\Longleftrightarrow}} f(x) \to f(x_0)$

THM $f : I \to \mathbb{R}$ f STRICTLY MONOTONE & C^0

OPEN INTERVAL

ASSUME f DIFFERENTIABLE AT $x_0 \in I$ & $f'(x_0) \neq 0$.

THEN $f^{-1} : R(f) \to \mathbb{R}$ IS DIFF.

OPEN INTERVAL

AT $f(x_0) = y_0$

& $$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$$

THM $f : I \to \mathbb{R}$ f STRICTLY MONOTONE

OPEN INTERVAL

ASSUME f DIFFERENTIABLE AT $x_0 \in I$

& $f'(x_0) \neq 0$.

THEN f^{-1} IS DIFFERENTIABLE AT

$y_0 = f(x_0)$

EXAMPLE

$$f(x) = \sin(x) : \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\to \mathbb{R}$$

INVERTIBLE

OPEN INT.

$R(f) = (-1, 1)$

$$f^{-1}(y) = \arcsin(y) : (-1, 1) \longrightarrow \mathbb{R}$$

$$(\arcsin)'(y) =$$

$$y = \sin(x)$$

EXAMPLE

$$f(x) = \sin(x) : \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\to \mathbb{R}$$

INVERTIBLE

OPEN INT.

$$f^{-1}(y) = \arcsin(y)$$

$$(\arcsin)'(\underbrace{y}_{= \sin(x)}) = \frac{1}{(\sin)'(x)} = \frac{1}{\cos(x)}$$

EXAMPLE

$$f(x) = \underset{\text{INVERTIBLE}}{\sin(x)} : \underset{\text{OPEN INT.}}{\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[} \to \mathbb{R}$$

$$f^{-1}(y) = \arcsin(y)$$

$$(\arcsin)'(\underset{\sin(x)}{\underset{\parallel}{y}}) = \frac{1}{(\sin)'(x)} = \frac{1}{\underset{\geqslant 0 \text{ ON }]-\pi/2, \pi/2[}{\cos(x)}}$$

EXAMPLE

$$f(x) = \sin(x) : \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\to \mathbb{R}$$

INVERTIBLE   OPEN INT.

$$f^{-1}(y) = \arcsin(y)$$

$$(\arcsin)'(y) = \frac{1}{(\sin)'(x)} = \frac{1}{\cos(x)}$$

$y = \sin(x)$

$\cos(x) \geq 0$ ON $\left]-\pi/2, \pi/2\right[$

$$= \frac{1}{\sqrt{1 - \sin^2(x)}} = \frac{1}{\sqrt{1 - y^2}}$$

EXAMPLE

$$f(x) = \underset{\text{INVERTIBLE}}{\sin(x)} : \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\to \mathbb{R}$$

OPEN INT.

$$f^{-1}(y) = \arcsin(y)$$

$$(\arcsin)'(\underset{\overset{\parallel}{\sin(x)}}{y}) = \frac{1}{(\sin)'(x)} = \frac{1}{\cos(x)}$$

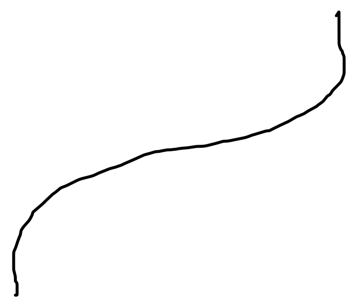
$\cos(x) \geq 0$ ON $\left]-\pi/2, \pi/2\right[$

$$= \frac{1}{\sqrt{1-\sin^2(x)}} = \frac{1}{\sqrt{1-\sin^2(\arcsin(y))}} = \frac{1}{\sqrt{1-y^2}}$$

$\sin x : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow \mathbb{R}$

$R = [-1, 1]$

$\arcsin x : [-1, 1] \longrightarrow \mathbb{R}$



EXAMPLE

$$f(x) = \sin(x) : \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\to \mathbb{R}$$

INVERTIBLE

OPEN INT.

$$f^{-1}(y) = \arcsin(y)$$

$$(\arcsin)'(y) = \frac{1}{(\sin)'(x)} = \frac{1}{\cos(x)}$$

$y = \sin(x)$

$\cos(x) \geq 0$ ON $\left]-\pi/2, \pi/2\right[$

$$= \frac{1}{\sqrt{1-\sin^2(x)}} = \frac{1}{\sqrt{1-\sin^2(\arcsin(y))}} = \frac{1}{\sqrt{1-y^2}}$$

EXAMPLE

$$f(x) = x^b, \quad b \in \mathbb{N}$$

EXAMPLE

$f(x) = x^b, \quad b \in \mathbb{N}$

$f^{-1}(y) = \sqrt[b]{y} \qquad \sqrt[b]{x^b} = x$

b EVEN : f IS EVEN

f NOT INV.

f IS STR. ~~MON~~ INCR. ON $[0,+\infty)$ $\quad f^{-1}(y) = \sqrt[b]{y}$

f IS INV. ON $[0,+\infty)$ $\quad y \in \mathbb{R}_+$

b ODD : f IS ODD.

f IS STR. INCR. ON $\mathbb{R}$ $\quad \mathbb{R}$

f IS INV. ON $\mathbb{R}$ $\quad f^{-1}(\cancel{y}) = \sqrt[b]{y}$

EXAMPLE

$$f(x) = x^b, \quad b \in \mathbb{N}$$

$$f^{-1}(y) = \sqrt[b]{y} = y^{\frac{1}{b}}, \quad \sqrt[b]{x^b} = x$$

$$(f^{-1})'(y_0) = \frac{1}{b\, x_0^{b-1}} = \frac{x_0}{b\, x_0^b} =$$

$y_0 \neq 0$

$y_0 = x_0^b$

$y_0 \neq 0$

b even

$$= \frac{\sqrt[b]{y_0}}{b\, y_0} = \frac{1}{b} (y_0)^{\frac{1}{b}} (y_0)^{-1} =$$

$$= \frac{1}{b} (y_0)^{\frac{1}{b} - 1}$$

EXAMPLE

$$f(x) = x^b, \quad b \in \mathbb{N}$$

$$f^{-1}(y) = \sqrt[b]{y} \qquad \sqrt[b]{x^b} = x$$

$$(f^{-1})'(y_0) = \frac{1}{b\, x_0^{b-1}} = \frac{x_0}{b\, x_0^b} =$$

$$\overset{x_0^b}{\shortparallel}$$

$$= \frac{\sqrt[b]{y_0}}{b\, y_0} = \frac{1}{b}\,(y_0)^{\frac{1}{b}-1}$$

EXAMPLE $f(x) = e^x$

$e^x > 0 \quad \forall x \in \mathbb{R}$

$f^{-1}(y) = \mathrm{Log}(y)$ ← SAME AS $\log(y)$ LOGARITHM IN BASE e

$D(f^{-1}) = R(f) = (0, +\infty)$

$f'(x) = e^x$

$x \geq 0$

$e^x \geq 1 + x \implies R\left(e^x\big|_{[0,+\infty)}\right) = [1, +\infty]$

$(\log)'(y) = \dfrac{1}{e^x}$

$y = e^x$, $\dfrac{1}{e^x} = 1/y$

$e^{-x} = \dfrac{1}{e^x}$

$\Downarrow$

$R\left(e^x\big|_{(-\infty,0]}\right) = (0, 1]$

EXAMPLE $f(x) = e^x$

$f^{-1}(y) = \mathrm{Log}(y)$ ⟵ SAME AS $\log(y)$ LOGARITHM IN BASE e

$$(f^{-1})'(y_0) = \frac{1}{e^{x_0}}$$

$$y_0 = e^{x_0}$$

HIGHER DERIVATIVES

$f : E \longrightarrow \mathbb{R}$ $\rightsquigarrow$ $f' : \mathrm{Dom}(f') \longrightarrow \mathbb{R}$

$$D(f') = \left\{ x \in E \;\middle|\; f \text{ is DIFFERENTIABLE AT } x \right\}$$

WE CAN TRY TO COMPUTE $(f')'$

2nd DERIVATIVE of f

HIGHER DERIVATIVES

$f : E \longrightarrow \mathbb{R}$  $f' : \mathrm{Dom}(f') \longrightarrow \mathbb{R}$

$$D(f') = \{ x \in E \mid f \text{ IS DIFFERENTIABLE AT } x \}$$

WE CAN TRY TO COMPUTE $(f')'$

$$f'' := (f')' : \underset{\{ y \in D(f') \mid f' \text{ IS DIFF. AT } y \}}{\underset{\parallel}{D(f'')}} \longrightarrow \mathbb{R}$$

2nd DERIVATIVE of f

[DERIV. OF f']

EXAMPLE

$$f(x) = \cos(x)$$

EXAMPLE

$$f(x) = x^n$$

IN GENERAL: $f : E \to \mathbb{R}$

$\rightsquigarrow f', f'', f'''$

$f''' \doteq (f'')' : D(f''') \to \mathbb{R}$

$D(f''') \doteq \{x \in D(f'') \mid f'' \text{ IS DIFF. AT } x\}$

$f^{(n)} \doteq \left(f^{(n-1)}\right)' : D(f^{(n)}) \longrightarrow \mathbb{R}$

$f^{(n)}$: n-th DERIVATIVE

$\left(f^{(n-1)}\right)'$: DERIVATIVE OF THE $(n-1)$-st DERIVATIVE