ANALYSIS 1

(ENGLISH)

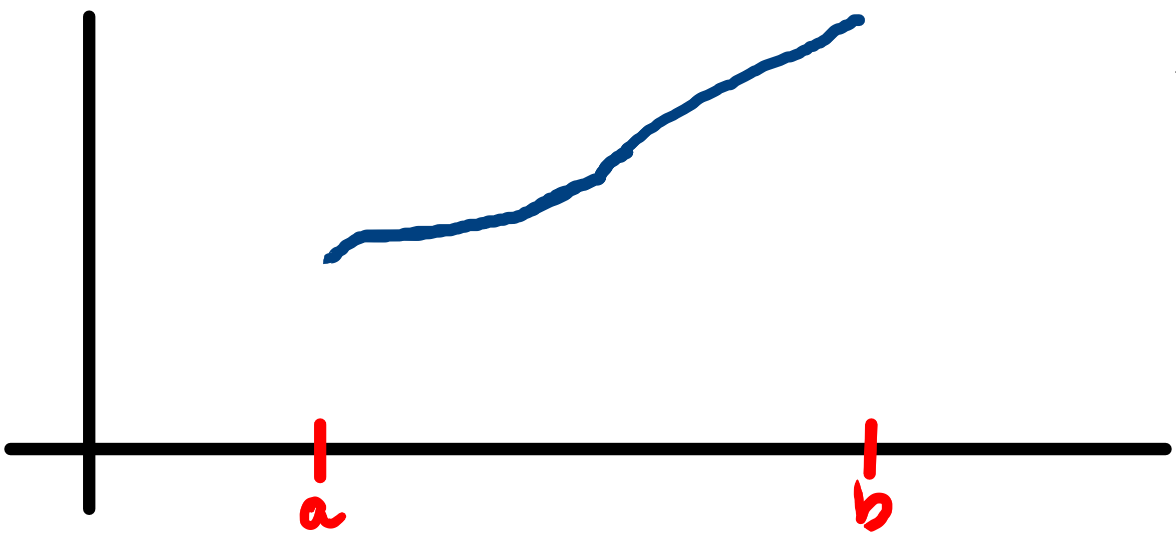
LECTURE 18

THM $f :]a, b[\longrightarrow \mathbb{R}$

CONTINUOUS &

STRICTLY MONOTONE

$\Longrightarrow R(f)$ IS AN OPEN INTERVAL



INT. VALUE THM:

$f : [a, b] \longrightarrow \mathbb{R}$

is $C^0 \Rightarrow R(f) = [m, M]$

global max/min

PROOF

$a_n \doteq a + \frac{1}{n}$ $\qquad$ $b_n \doteq b - \frac{1}{n}$

$a_n \to a$, $b_n \to b$

ASSUME f IS STR. INCR :

$f(a_n)$ STRICTLY DECREASING

$f(b_n)$ STRICTLY INCREASING

$$\Longrightarrow \lim_{n\to\infty} f(a_n) = \begin{cases} c \\ -\infty \end{cases}$$

$c < d$

$$\wedge$$

$$\lim_{n\to\infty} f(b_n) = \begin{cases} d \\ +\infty \end{cases}$$

PROOF

$a_n \doteq a + \frac{1}{n}$ $\qquad b_n \doteq b - \frac{1}{n}$

$a_n \to a$, $b_n \longrightarrow b$

$f(a_n)$ STRICTLY DECREASING

$f(b_n)$ STRICTLY INCREASING

$$\Rightarrow \lim_{n\to\infty} f(a_n) = \begin{cases} \boxed{c} \\ -\infty \end{cases}$$

$$\lim_{n\to\infty} f(b_n) = \begin{cases} \boxed{d} \\ +\infty \end{cases}$$

WE WANT TO SHOW THAT

$R(f) =]c, d[$

$$[f(a_n), f(b_n)]$$

$$\cap\!\!|$$ ⟵ STRICT MONOT. $\{f(a_n)\}$ $\{f(b_n)\}$

$$* \quad [f(a_{n+1}), f(b_{n+1})]$$

$$||$$ ⟵ f IS STR. INCR., f IS C^0

$$\underline{\underline{R\left(f\big|_{[a_{n+1}, b_{n+1}]}\right)}}$$

$c \quad d$

$\uparrow \quad \uparrow$

$[f(a_n), f(b_n)]$

$$R(f) = \bigcup_{n\in\mathbb{N}^*} R\left(f\big|_{[a_n, b_n]}\right) = (c, d)$$

$$\begin{matrix} a_n \downarrow a \\ b_n \uparrow b \end{matrix} \implies \forall c \in (a,b), \exists n_c \in \mathbb{N}^* \text{ S.T. } c \in [a_{n_c}, b_{n_c}]$$

$\lim a_n = c$ $\qquad$ f STR. INCR

$\lim b_n = d$ $\qquad$ f IS C^0

$f(a_n) = c_n \longrightarrow c$ $\qquad$ $a_n \downarrow a$

$f(b_n) = d_n \longrightarrow d$ $\qquad$ $b_n \uparrow b$

$$R\left(f|_{[a_n, b_n]}\right) = [c_n, d_n]$$

$$R(f) = \bigcup_{n \in \mathbb{N}^*} [c_n, d_n]$$

$$\overset{?}{=} \bigcup_{n \in \mathbb{N}^*} R\left(f|_{[a_n, b_n]}\right) = R\left(f|_{\cup [a_n, b_n]}\right)$$

$$R(f) = \bigcup_{n \in \mathbb{N}^*} [c_n, d_n] \qquad R(f|_{(a,b)})$$

$$\overset{?}{=} \bigcup_{n \in \mathbb{N}^*} R(f|_{[a_n, b_n]}) = R(f|_{\cup [a_n, b_n]})$$

$$\bigcup [a_n, b_n] = (a, b)$$

$$a < c < b$$

$$\exists n \quad a < a + \frac{1}{n} < c < b - \frac{1}{n} < b$$

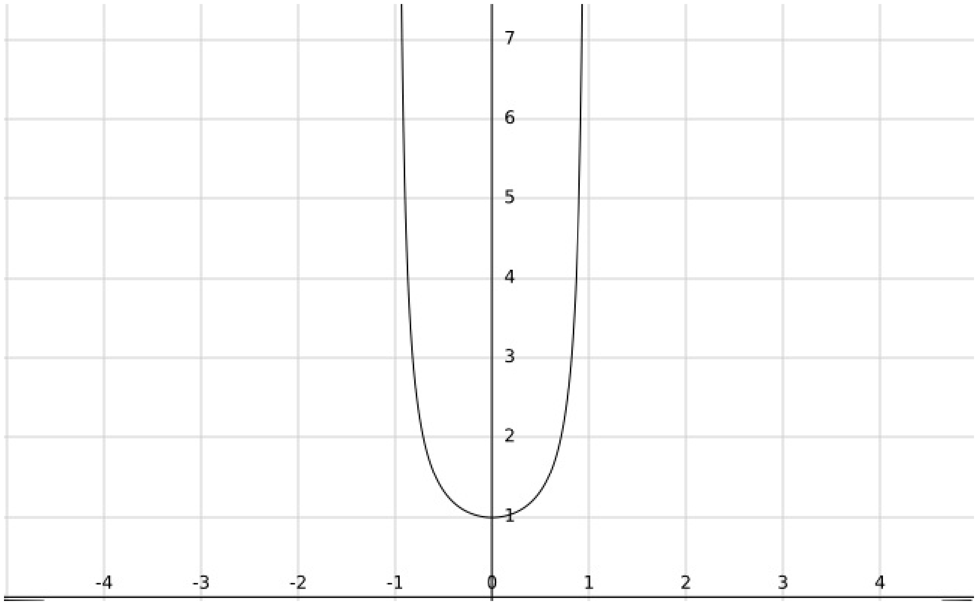
$$[a_n, b_n] \quad \exists n$$

$c_n \to c$

$d_n \to c$

$$\bigcup [c_n, d_n] = (c, d)$$

EXAMPLE



$$f(x) = \frac{1}{1-x^2}$$

$$f : (-1, 1) \to \mathbb{R}$$

f IS NOT MONOTONE

$$f(0) = 1 \qquad f\left(-\frac{1}{2}\right) = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$$

$$\shortparallel$$

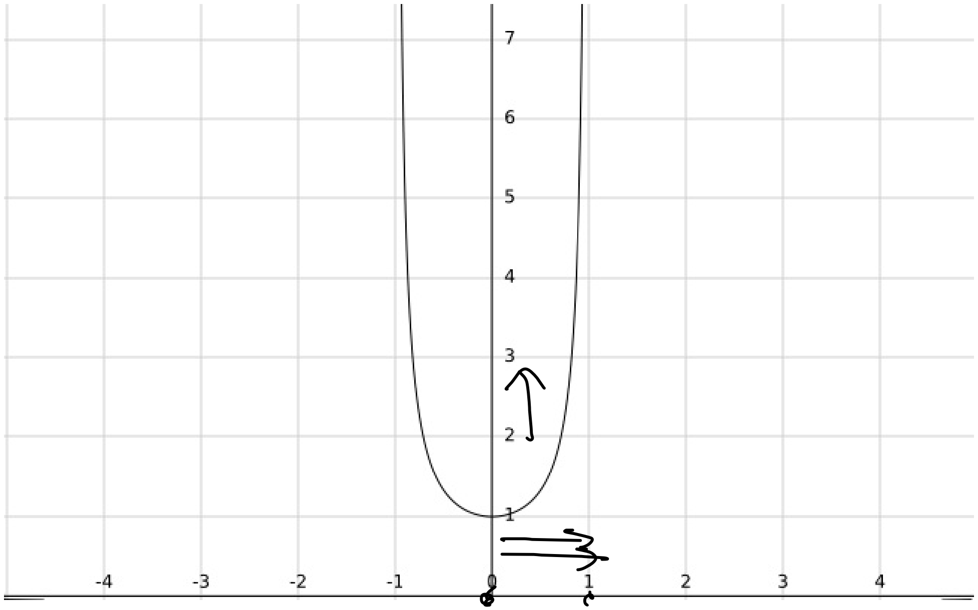
$$f\left(\frac{1}{2}\right)$$

$$-\frac{1}{2} < 0 < \frac{1}{2} \qquad \frac{4}{3} > 0 < \frac{4}{3}$$

EXAMPLE

EVEN FCT.

$$f(x) = \frac{1}{1-x^2}$$



$f : (-1, 1) \longrightarrow \mathbb{R}$

f IS NOT MONOTONE

$f(0) = 1$

$\lim_{x \to 1^-} f(x) = +\infty$.

$R(f) = [1, +\infty)$

$f|_{[0,1)} : [0, 1) \longrightarrow \mathbb{R}$ STR. INCR.

$$f(x) = \frac{1}{1-x^2} \qquad f : (-1, 1) \longrightarrow \mathbb{R}$$

$$f\big|_{[0,1)} : [0,1) \longrightarrow \mathbb{R} \quad \text{STR. INCR.}$$

$$R\left(f\big|_{[0,1)}\right) = [1, +\infty) \qquad \lim_{x \to 1^-} f(x) = +\infty$$

$$\Downarrow$$

$$R\left(f\big|_{[0, y_{n+1}]}\right) = [1, f(y_{n+1})]$$

$$\cup$$

$$R\left(f\big|_{[0, y_n]}\right) = \boxed{1}, f(y_n)]^* \qquad y_n = 1 - \frac{1}{n}, \text{ THEN}$$

$$f(y_n) \longrightarrow +\infty$$

$$[1, +\infty)$$

$$\parallel$$

$$[0,1) = \bigcup_{n>0} [0, y_n] \qquad f(y_n) = \frac{1}{1-(1-\frac{1}{n})^2}$$

$$\Rightarrow R\left(f\big|_{[0,1)}\right) = \bigcup_{n>0} R\left(f\big|_{[0, y_n]}\right)$$

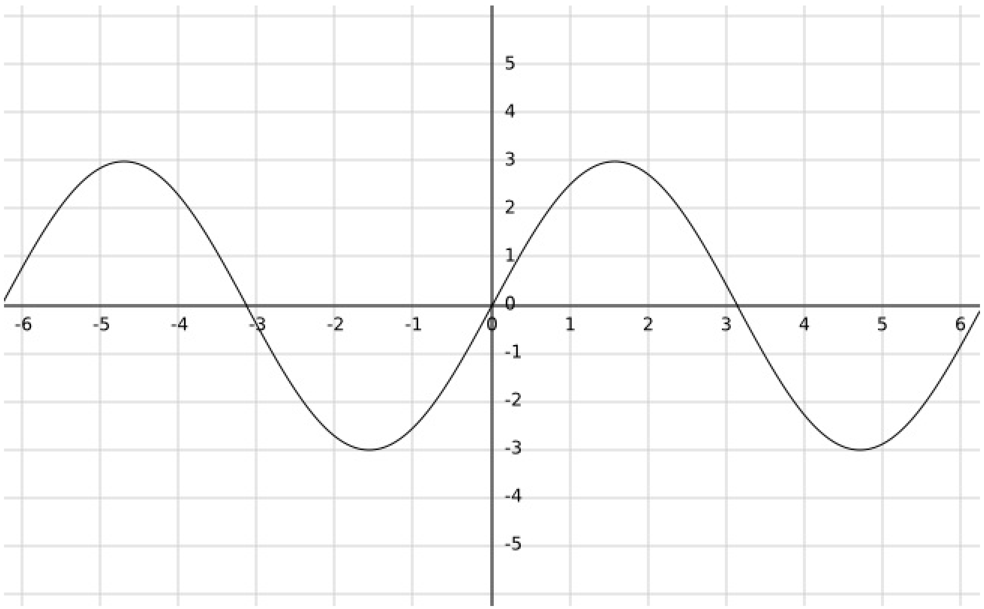
EXAMPLE $f(x) = 3\sin(x)$

$f : \mathbb{R} \longrightarrow \mathbb{R}$

NOT MON.

C^∞

$R(f) = [-3, 3]$



THM

$f : E \to \mathbb{R}$ CONTINUOUS

E ← INTERVAL

$$f \text{ IS STRICTLY MONOTONE} \iff f \text{ IS INJECTIVE}$$

[f STR. MON. $\Longrightarrow$ f INJ.

[$c < d \in E \overset{\text{STR,}}{\underset{\text{INCR.}}{\Longrightarrow}} f(c) < f(d)$]

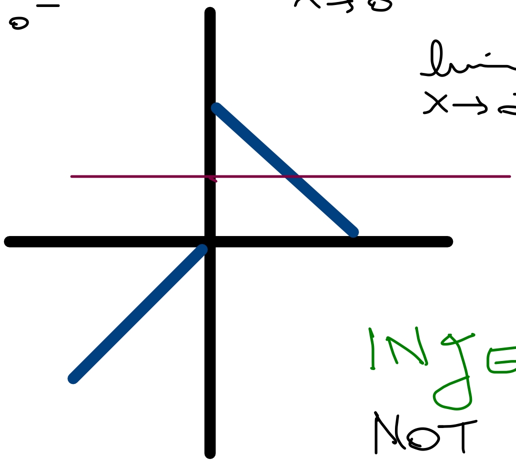
f INJ $\Longrightarrow$ f STRICTLY MON]

STRICTLY MON $\Rightarrow$ INJECTIVE — ALWAYS TRUE

INJECTIVE $\Rightarrow$ STRICTLY MON — NEEDS C^0

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x = 0 \qquad \nRightarrow$$

$$\lim_{x\to 0^+} 2-x = 2$$



$$f(c) = f(d)$$

$$c > d$$

INJECTIVE

NOT STR. MON.

$$\underset{-2}{\underset{\parallel}{f(-2)}} < \underset{0}{\underset{\parallel}{f(0)}}$$

$$f : [-2, 2[\to \mathbb{R}$$

$$f(x) = \begin{cases} x & \text{IF } -2 \leq x \leq 0 \\ 2-x & \text{IF } 0 < x < 2 \end{cases}$$

$$\underset{3/2}{\underset{\parallel}{f\left(\tfrac{1}{2}\right)}} > \underset{1}{\underset{\parallel}{f(1)}}$$

PROOF INJ. $\Rightarrow$ STR. MONOT.

SUPPOSE f NOT STR. MONOT., BUT NOT CONST.

$c < d < e \in D(f)$

$f(c) \leq f(d) \geq f(e)$ OR

$f(c) \geq f(d) \leq f(e)$

PROOF INJ. $\Rightarrow$ STR. MONOT.

SUPPOSE f NOT STR. MONOT., NOT CONS

$$c < d < e$$

$$f(c) < f(d) > f(e) \qquad \text{OR}$$

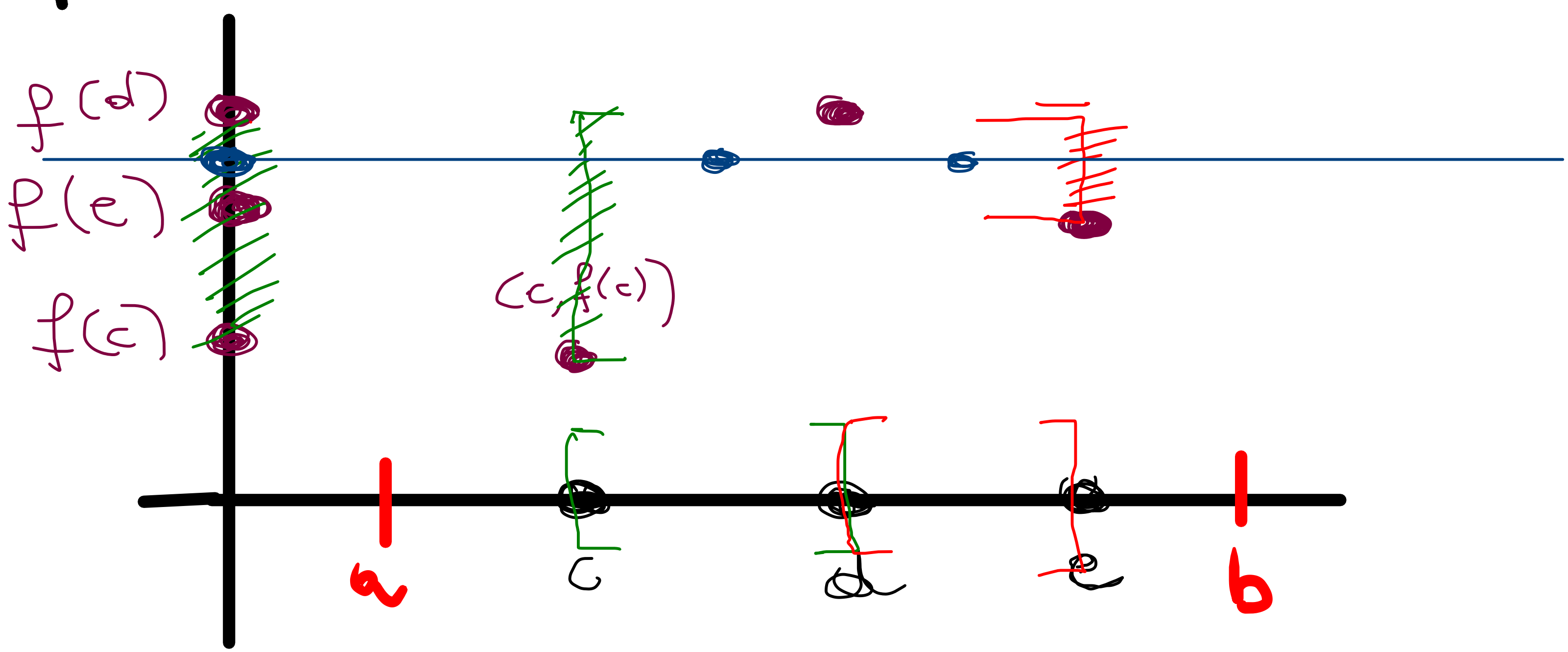
$$f(c) > f(d) < f(e)$$

PROOF INJ. $\Rightarrow$ STR. MONOT.

SUPPOSE NOT STR. MONOT.

$$c < d < e$$

$$f(c) < f(d) > f(e)$$



$f : E \to \mathbb{R}$ IS C^0 & STR. MONOT.

$\uparrow$ INTERVAL

$$F \doteq R(f)$$

$\Rightarrow f : E \to F$ IS BIJECTIVE

$\Rightarrow f^{-1} : F \to E$ EXISTS

<u>Q</u>: IS f^{-1} C^0 ?

THM $f: E \to \mathbb{R}$ IS C^0 & STR. MONOT.

E: INTERVAL

$F \doteq R(f)$

$f^{-1}: F \to E$ IS C^0 AT ANY POINT.

INJ + SURJ.

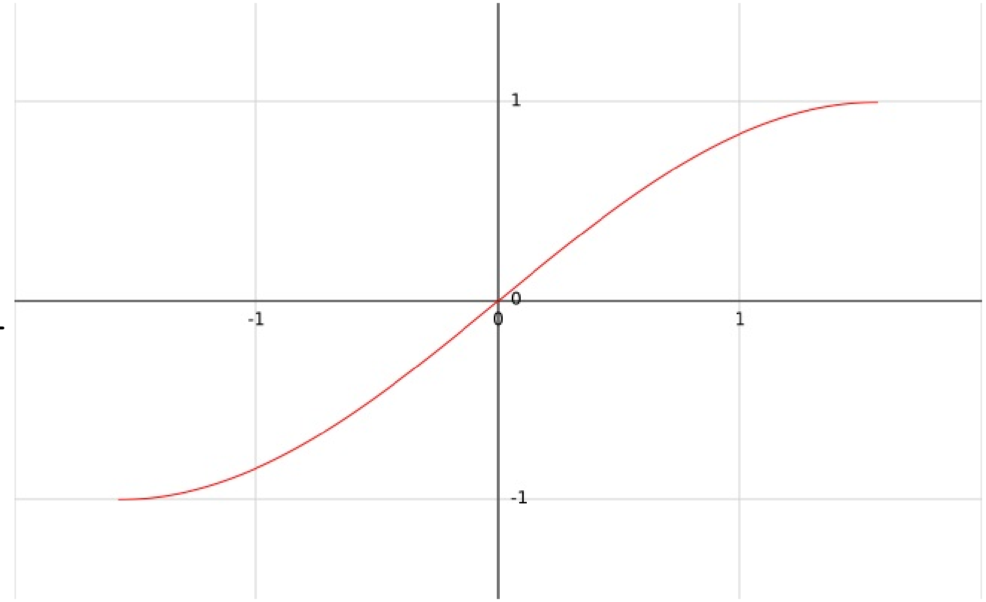
EXAMPLE

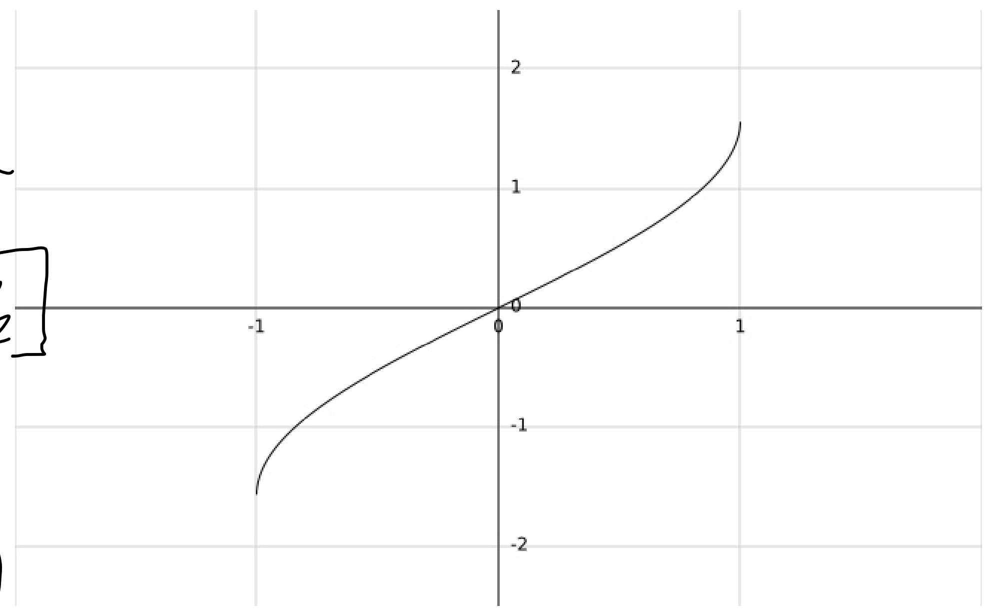
$f(x) = \sin x$

$f : [-\pi/2, \pi/2] \to \mathbb{R}$

$R(f) = [-1, 1]$

f IS STR. INCR.





$f^{-1} : [-1, 1] \to \mathbb{R}$

$R(f^{-1}) = [-\pi/2, \pi/2]$

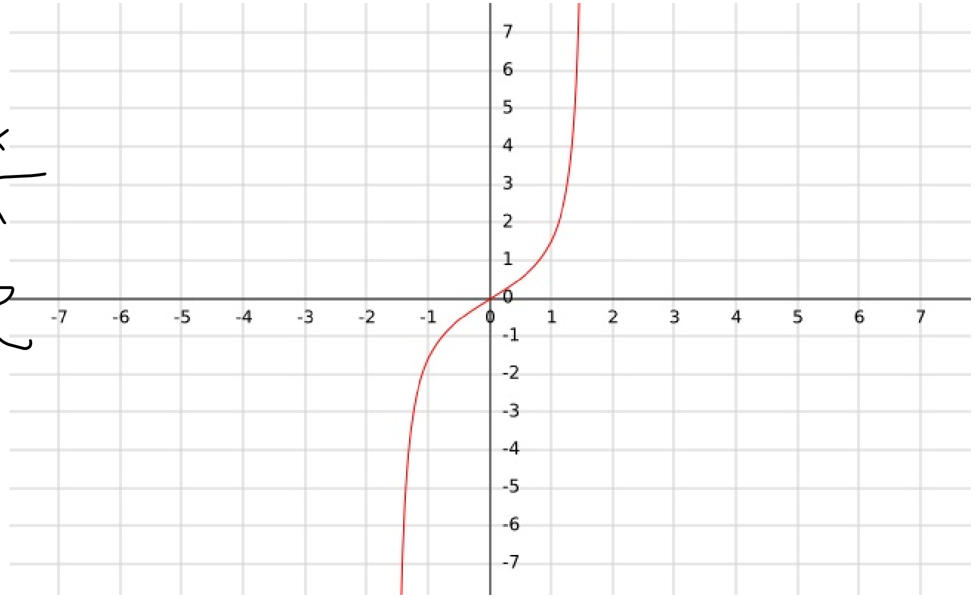
f^{-1} is C^0

$f^{-1}(x) := \arcsin(x)$

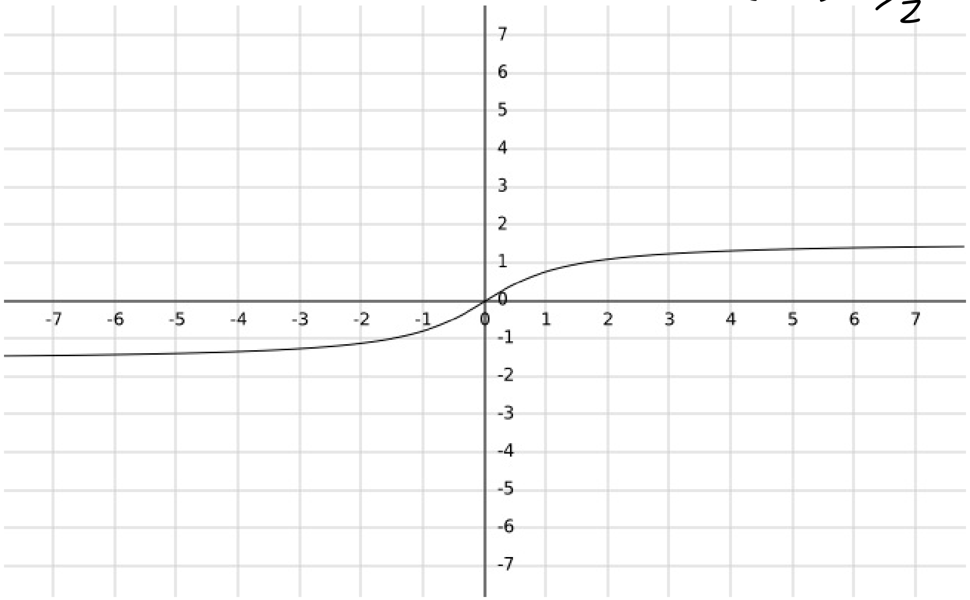
$$f(x) = \tan x = \frac{\sin x}{\cos x}$$

$$f : \left(-\pi/2, \pi/2\right) \to \mathbb{R}$$

STR. INCR.



$R(f) = \mathbb{R}$ since $\lim_{x \to \pi/2^-} f(x) = +\infty$

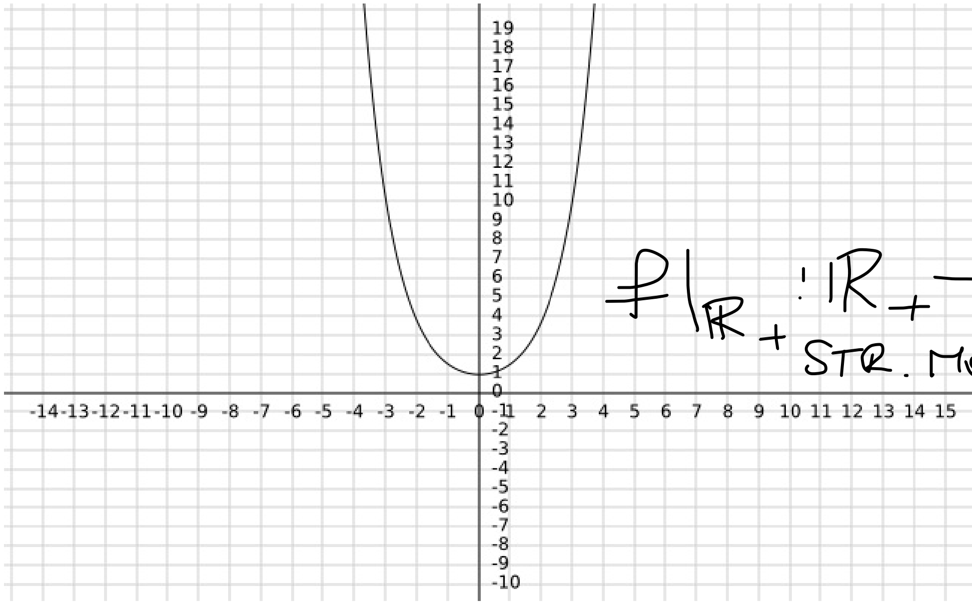


$$f^{-1} : \mathbb{R} \longrightarrow \mathbb{R}$$

$$R(f^{-1}) = \left(-\frac{\pi}{2}, \pi/2\right)$$

$$f^{-1}(x) = \arctan(x)$$

f^{-1} is C^0



$$\cosh(x) = \frac{e^{x} + e^{-x}}{2}$$

EVEN

$f|_{\mathbb{R}_+} : \mathbb{R}_+ \to \mathbb{R}$

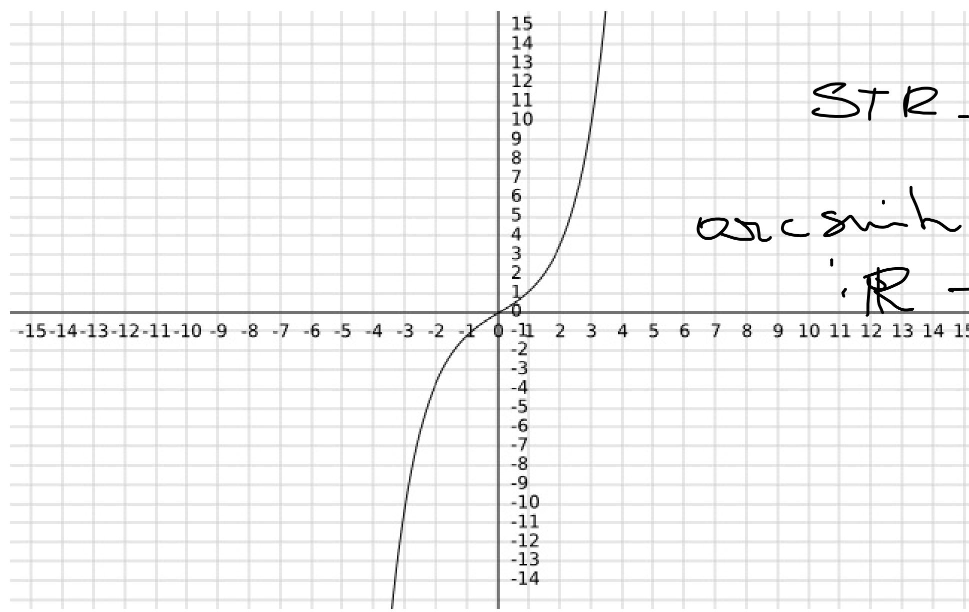
STR. MON.

$R(\cosh) = [1, +\infty)$

$R(\sinh) = \mathbb{R}$

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2}$$

$: \mathbb{R} \longrightarrow \mathbb{R}$

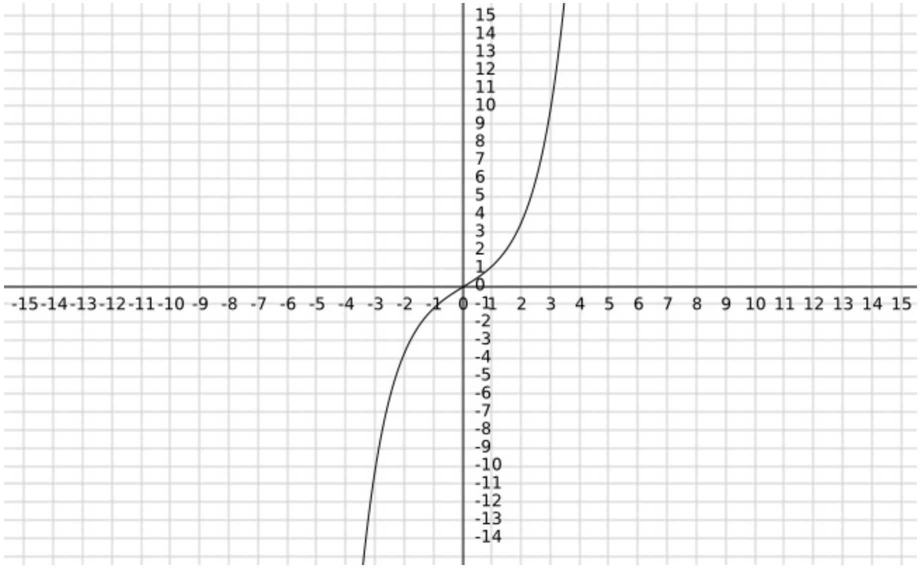


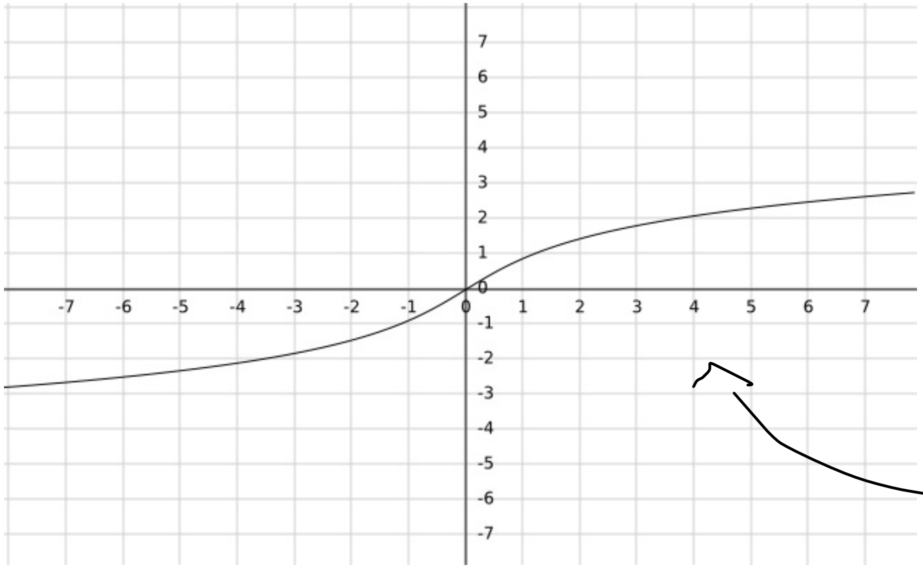
STR. MON.

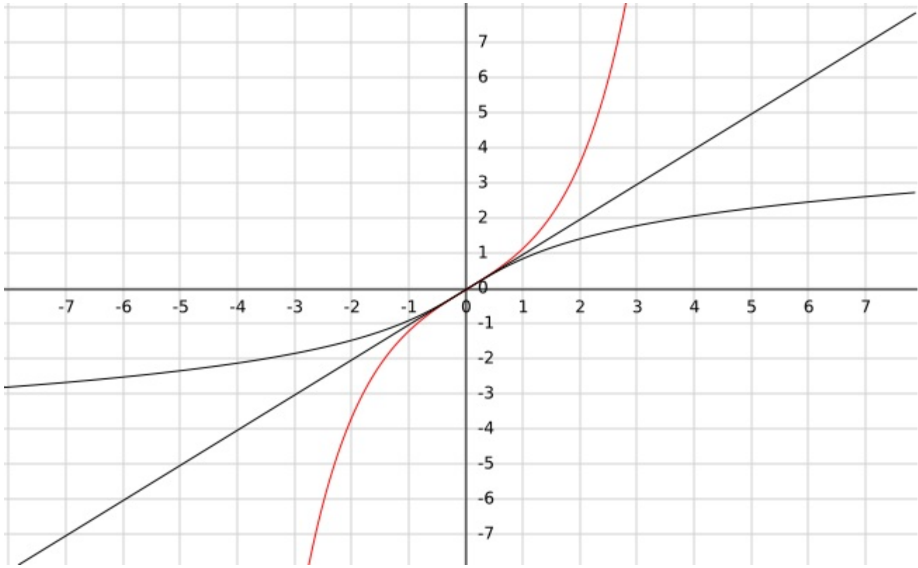
$\operatorname{arcsinh}(x) : \mathbb{R} \to \mathbb{R}$

$$\cosh(x)^2 - \sinh(x)^2 = 1$$

$$x^2 - y^2 = 1$$







$y = \operatorname{arcsinh}(x)$

DIFFERENTIATION

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$

$$f(x) = \cos\left(x^3 + 3x + 16\pi\right) + \pi^2 e^{-\frac{1}{x^2+1}}$$

DIFFERENTIATION

$$f : \mathbb{R} \to \mathbb{R}$$

$$f(x) = \cos(x^3 + 3x + 16\pi) + \pi^2 e^{-\frac{1}{x^2+1}}$$

$$f(0) = \cos(16\pi) + \pi^2 e^{-1} = 1 + \pi^2 e^{-1}$$

Q: $f(x)$? [WHEN $x \approx 0$, FOR EX.]

GENERAL PROBLEM: HOW DO WE APPROXIMATE FUNCTIONS?

NAIVE ATTEMPT: TRY TO USE VERY SIMPLE FUNCTIONS TO APPROXIMATE

SIMPLE FUNCTIONS?

LINEAR FUNCTIONS

$$h(x) = c + dx$$

c, d fixed real numbers

$$f(x) \overset{x \approx 0}{\approx} f(0) + d \cdot x$$

ERROR IN THE APPROX. SHOULD NOT BE TOO LARGE

GENERAL PROBLEM: HOW DO WE APPROXIMATE FUNCTIONS?

NAIVE ATTEMPT: TRY TO USE LINEAR FUNCTIONS TO APPROXIMATE

$$f :]a,b[\to \mathbb{R}, \quad x_0 \in]a,b[$$

FOR $x \approx x_0$

ERROR FCT.

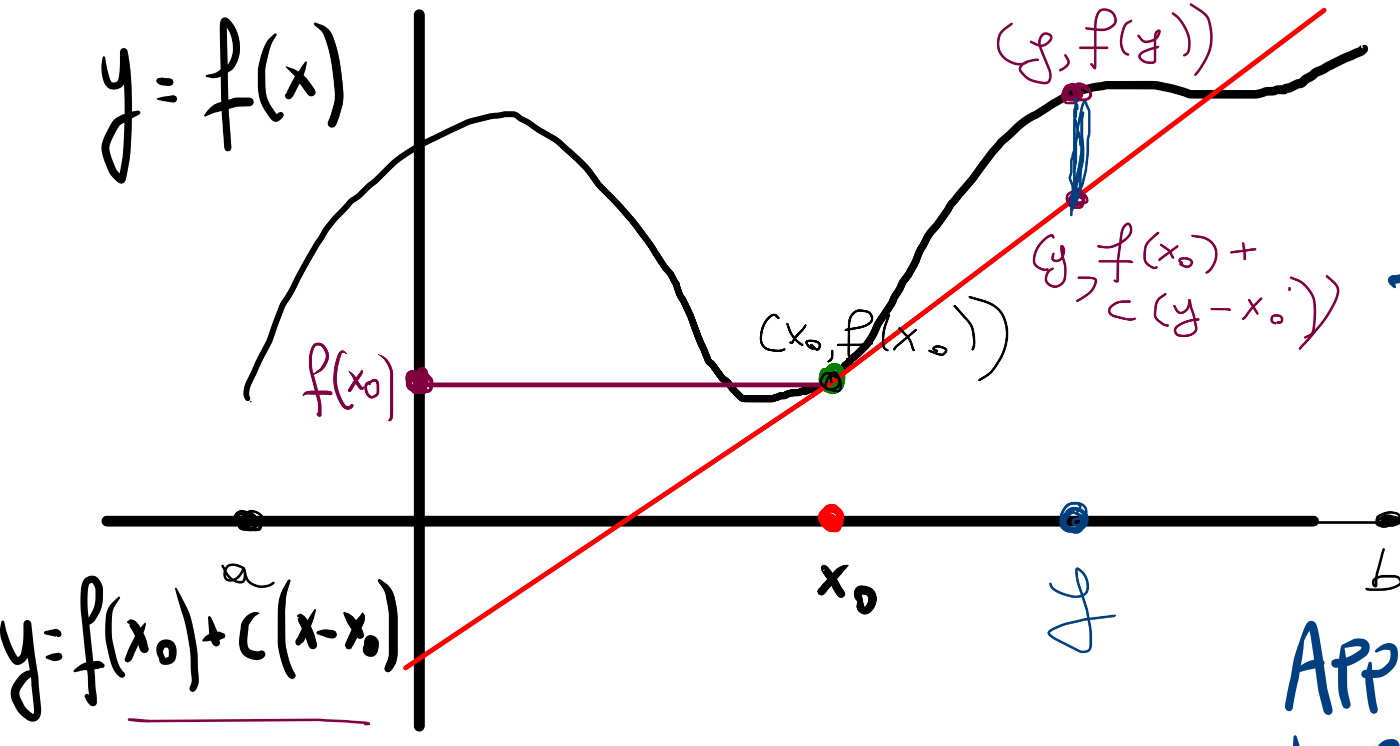
$$f(x) = f(x_0) + \underline{\underline{c}}\,\overbrace{(x - x_0)}^{\text{distance BTW } x \,\&\, x_0} + r(x)$$

$$r(x) = f(x) - f(x_0) - c(x - x_0)$$

$$f :]a, b[\to \mathbb{R}$$

ERROR

$$f(x) = f(x_0) + c(x - x_0) + r(x)$$



WE ARE TRYING TO FIND THE BEST LINEAR APPROXIMATION FOR f AROUND x_0

$f :]a,b[\to \mathbb{R}$

ERROR

$$f(x) = f(x_0) + \underline{\underline{c}}(x - x_0) + r(x)$$

WOULD LIKE $\displaystyle\lim_{x \to x_0} \frac{r(x)}{(x - x_0)} = 0$

$\Updownarrow$

near x_0 $\quad |r(x)| \leq \varepsilon |x - x_0|$

$\forall \varepsilon > 0$

Q: HOW TO COMPUTE c, $r(x)$?

$$f :]a, b[\to \mathbb{R}$$

ERROR

$$(\pi)\quad f(x) = f(x_0) + c(x - x_0) + r(x)$$

WOULD LIKE $\displaystyle\lim_{x \to x_0} \frac{r(x)}{(x - x_0)} = 0 \quad (*)$

$$c \text{ " = " } \frac{f(x) - f(x_0) - r(x)}{(x - x_0)}$$

TAKING LIMIT $x \to x_0$:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} - \frac{r(x)}{(x - x_0)}$$

$$\frac{f(x) - f(x_0)}{x - x_0} \to f'(x_0), \qquad \frac{r(x)}{(x - x_0)} \to 0$$

$f :]a, b[\to \mathbb{R}$

ERROR

$$f(x) = f(x_0) + c(x - x_0) + r(x)$$

WOULD LIKE $\boxed{\lim_{x \to x_0} \frac{r(x)}{(x - x_0)} = 0}$

TAKING LIMIT $x \to x_0$: c "=" $\lim_{x \to x_0} \frac{f(x) - f(x_0) - r(x)}{(x - x_0)}$

$$= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{(x - x_0)} - \cancel{\lim_{x \to x_0} \frac{r(x)}{(x - x_0)}}$$

SO IF WE CAN FIND c S.T.

$$f(x) \approx f(x_0) + c\,(x - x_0) \text{ FOR } x \approx x_0$$

& THE ERROR IS LESS THAN LINEAR

$$\Longrightarrow \quad c = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

IF THIS LIMIT EXISTS

WELL DEF'D

$$f'(x_0) := \lim_{x \to x_0} \boxed{\frac{f(x) - f(x_0)}{x - x_0}}$$

DEF $f : E \longrightarrow \mathbb{R}$, $x_0 \in E$

ASSUME f IS DEFINED ON A NEIGHBORHOOD OF x_0 $\left[(x_0 - \delta, x_0 + \delta),\ \delta > 0\right]$

f IS DIFFERENTIABLE AT x_0

IF $f'(x_0) \doteq \lim\limits_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$

EXISTS.

SATISFIES $\lim\limits_{x \to x_0} \dfrac{r(x)}{(x - x_0)} = 0$

$\left[f(x) = f(x_0) + f'(x_0) \cdot (x - x_0) + r(x) \right]$

$$f :]a, b[\to \mathbb{R}$$

ERROR

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + r(x)$$

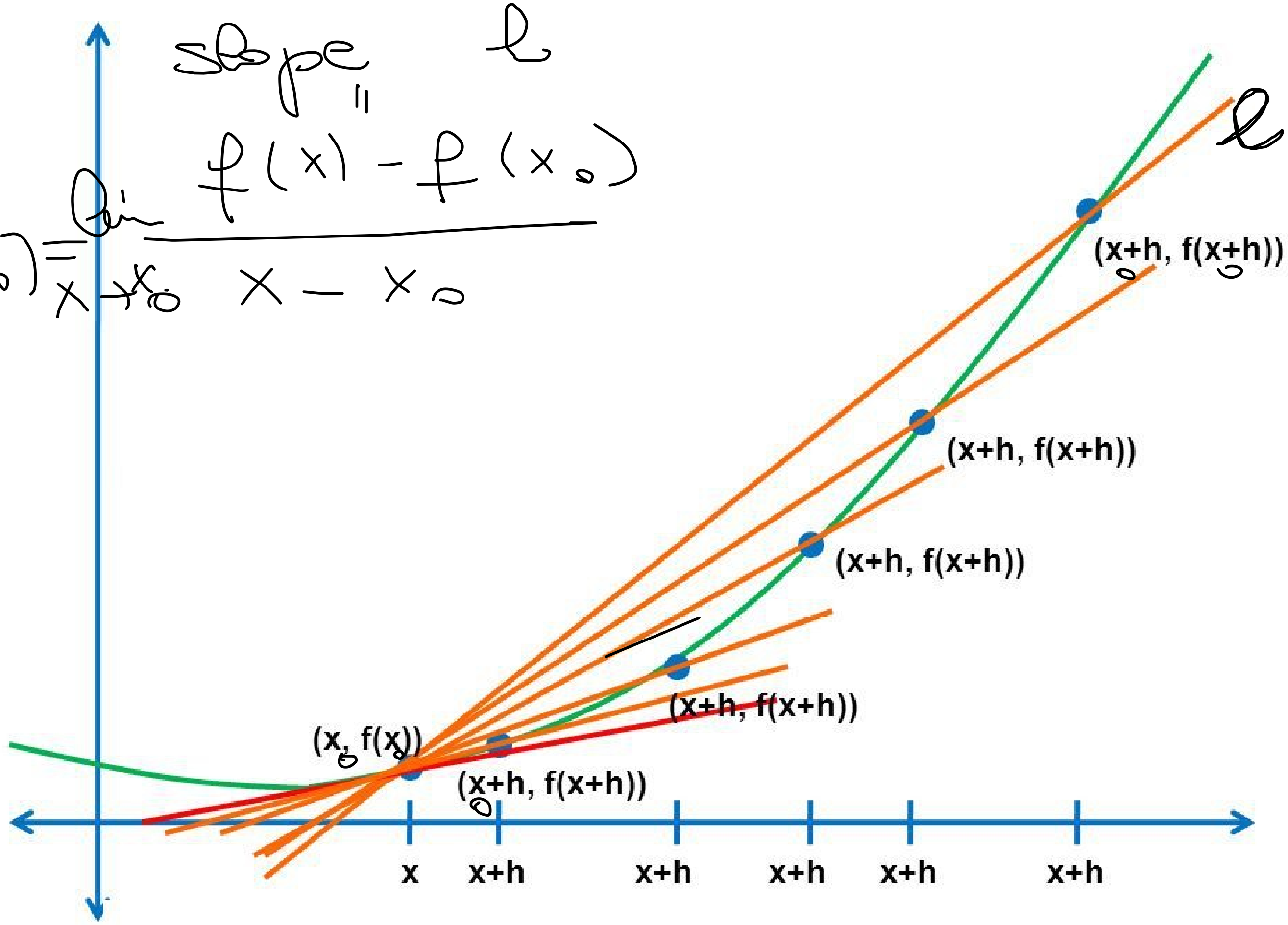
slope ℓ

$\parallel$

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

$$y - f(x_0) = \underline{\underline{c}} \cdot (x - x_0)$$

ℓ



$$f : E \longrightarrow \mathbb{R}$$
$$x \longmapsto f(x_0)$$

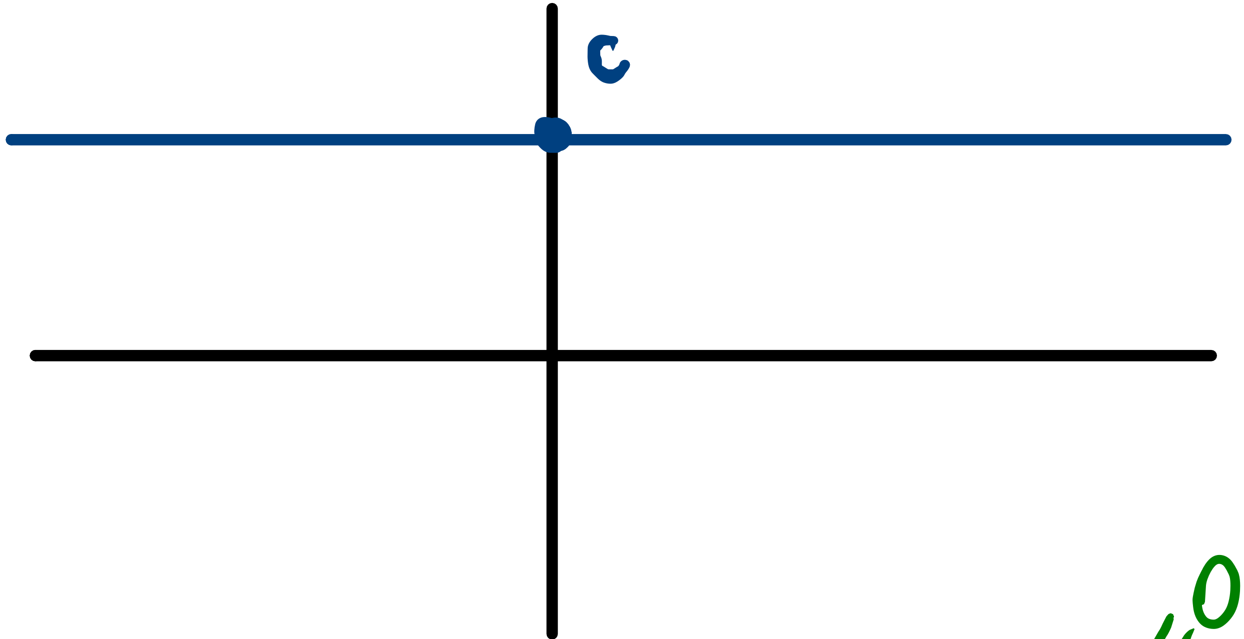
$$\rightsquigarrow$$

$$f' : D(f') \subseteq E \longrightarrow \mathbb{R}$$
$$x \longmapsto f'(x)$$

$$D(f') \doteq \{ x \in \underset{D(f)}{\underset{\shortparallel}{E}} \mid f \text{ IS DIFFER. AT } x \}$$

EXAMPLE

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = c$$



$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{\overbrace{c - c}^{= 0 \text{ ALWAYS}}}{x - x_0} = 0 \quad \forall x_0$$

EXAMPLE

$f : \mathbb{R} \longrightarrow \mathbb{R}, \quad f(x) = x^n \quad n \in \mathbb{N}$

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \underset{x = x_0 + h}{=} \lim_{h \to 0} \frac{(x_0+h)^n - x_0^n}{(x_0+h) - x_0}$$

EXAMPLE

$$f:\mathbb{R}\longrightarrow\mathbb{R},\quad f(x)=x^{n\in\mathbb{N}}$$

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=\lim_{h\to 0}\frac{(x_0+h)^n-x_0^n}{(x_0+h)-x_0}$$

$x=x_0+h$

$x\to x_0 \Longleftrightarrow h\to 0$

$$=\lim_{h\to 0}\frac{\cancel{x_0^n}+n\cdot \cancel{h}\,x_0^{n-1}+\sum_{i=2}^{n}\binom{n}{i}h^{i-1}x_0^{n-i}-\cancel{x_0^n}}{\cancel{h}}$$

$$\lim_{h\to 0}\underline{\underline{n\cdot x_0^{n-1}}}+h\sum_{i=2}^{n}\binom{n}{i}\overbrace{h^{i-2}\underline{x_0^{n-i}}}^{\to 0}$$

$$=n\cdot x_0^{n-1}$$

EXAMPLE

$$f(x) = \cos(x)$$

$$\lim_{h \to 0} \frac{\cos(x_0 + h) - \cos(x_0)}{h} =$$

$$\lim_{h \to 0} \frac{\cos(x_0)\cos(h) - \sin(x_0)\sin(h) - \cos(x_0)}{h} =$$

EXAMPLE

$$f(x) = \cos(x)$$

$$\lim_{h \to 0} \frac{\cos(x_0+h) - \cos(x_0)}{h} =$$

$$\lim_{h \to 0} \frac{\cos(x_0)\cos(h) - \sin(x_0)\sin(h) - \cos(x_0)}{h} =$$

$$\lim_{h \to 0} \frac{\cos(x_0)(\cos(h)-1)}{h} - \sin(x_0) \cdot \frac{\sin(h)}{h}$$

$$\lim_{h \to 0} \frac{\sin(h)}{h} \to 1$$

$$-\sin(h)^2 = \cos^2(h) - 1$$

$$\frac{\cos^2(h) - 1}{h(\cos h + 1)} \to 0$$

$$\to 0 \qquad \parallel \qquad 1$$

$$-\sin(x_0)$$

$$f(x) = \cos(x)$$

$$\Rightarrow f'(x) = -\sin(x)$$

SIMILARLY: $g(x) = \sin(x)$

$$\Rightarrow g'(x) = \cos(x)$$

CONTINUITY VS DIFFERENTIABILITY

$$f : E \to \mathbb{R} \,,\quad x_0 \in E$$

- f IS CONTINUOUS AT x_0 IF

$$\lim_{x \to x_0} f(x) = f(x_0)$$

- f IS DIFFERENTIABLE AT x_0

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{(x - x_0)} \text{ EXISTS}$$

$$\lim_{x \to x_0} r(x) = 0$$

$$\Uparrow$$

$$\lim_{x \to x_0} \frac{r(x)}{(x - x_0)} = 0$$

$$\Updownarrow$$

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} \underline{f(x_0)} + \cancel{f'(x_0)} \cdot \cancel{(x - x_0)}_{\to 0} + \cancel{r(x)}_{\to 0}$$

PROP $f : E \to \mathbb{R}$, $x_0 \in E$

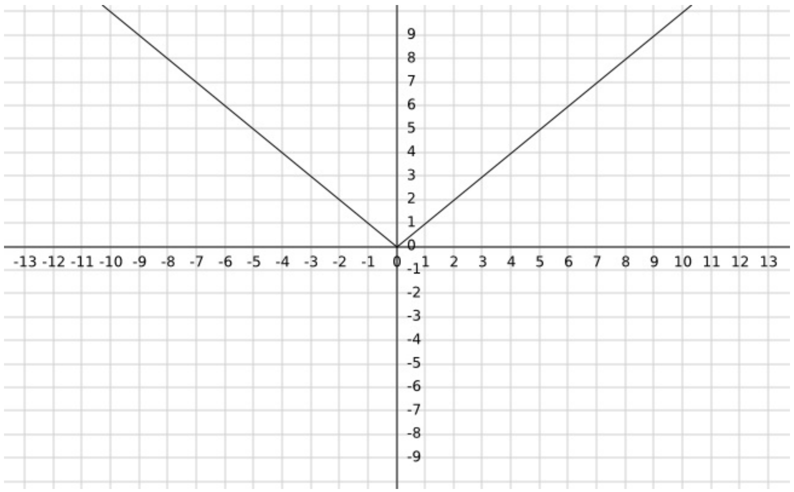
IF f IS DIFFERENTIABLE AT x_0

THEN f IS C^0 AT x_0

DOES THE VICEVERSA HOLD?

$f : E \to \mathbb{R}$, $x_0 \in E$

f C^0 AT x_0 $\overset{?}{\Longrightarrow}$ f DIFFERENTIABLE AT x_0



$x_0 = 0$

$|x|$ is C^0 AT 0

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x}$$

DOES NOT EXIST