ANALYSIS 1

(ENGLISH)

LECTURE 17

UNIFORM CONTINUITY

DEF $f : E \xrightarrow{\subseteq \mathbb{R}} \mathbb{R}$ IS UNIFORMLY CONTINUOUS IF $\forall \varepsilon > 0$, $\exists \delta_\varepsilon > 0$ S.T.

$\forall x, y \in E$, $|x - y| \leq \delta_\varepsilon \Rightarrow$

$$|f(x) - f(y)| \leq \varepsilon$$

$f : E \to \mathbb{R}$ IS UNIF. CONT.

IF $\forall \varepsilon > 0$, WE CAN CHOOSE $\delta_\varepsilon > 0$ IN DEF'N OF CONTINUITY TO BE THE SAME AT EACH PT OF E

EXAMPLE

$$f(x) = \cos(x)$$

FIX $\varepsilon > 0$.

NEED TO FIND $\delta_\varepsilon > 0$ S.T.

$\forall x, y \in \mathbb{R}$, $|x-y| < \delta_\varepsilon$

$$\Longrightarrow |\cos(x) - \cos(y)| < \varepsilon$$

THM $f:[a,b]\to\mathbb{R}$ C^0

THEN f is UNIFORMLY CONTINUOUS

$$f(x)=x^2 \qquad f=g|_{[a,b]}:[a,b]\to\mathbb{R}$$

$\overset{\text{THM}}{\Longrightarrow}$ f is UNIF. C^0

$g(x)=x^2$ $\quad g:\mathbb{R}\longrightarrow\mathbb{R}$

g is C^0 BUT NOT UNIF. C^0

DEF $f : E \to \mathbb{R}$

f IS LIPSCHITZ IF

$\exists \ \underline{\underline{C}} > 0 \ , \ \underline{\underline{C}} \in \mathbb{R}$ S.T.

$\forall x, y \in E$ $\qquad \dfrac{|f(x) - f(y)|}{|x - y|} \leq C$

$x \neq y$

$$\Updownarrow$$

$$\boxed{|f(x) - f(y)| \leq C \cdot |x - y|}$$

EXAMPLE

$f(x) = \lambda x \qquad f : \mathbb{R} \to \mathbb{R}$

f IS LIPSCHITZ, $|\lambda|$ IS THE LIPSCHITZ CONSTANT

$$|f(x) - f(y)| = |\lambda x - \lambda y| \leq |\lambda| \cdot |x - y| \leq C \cdot |x - y|$$

$\forall C \geq |\lambda|$

IF h is a LIPSCHITZ function for some constant $C' > 0$

$$\underset{D(f)}{\forall x, y \in} \quad |f(x) - f(y)| \leq C' \, |x - y|$$

$$\leq C \cdot |x - y| \quad \forall C \geq C'$$

If C' is a constant good for the def. of LIPSCHITZ for $h \Longrightarrow \forall C \in [C', +\infty)$ these are also good constants ∞

LIPSCHITZ CONSTANT of a

LIPSCHITZ CONSTANT λ of a LIPSCH. fct h

is the min of the set

$$\left\{ C \in \mathbb{R}_+^* \;\middle|\; \begin{array}{l} C \text{ is a constant} \\ \text{good for the def'n} \\ \text{of LIPSCHITZ for } h \end{array} \right\}$$

$$\|$$

$$[\lambda, +\infty)$$

$\leq$

$$|f(x) - f(y)| \leq C \cdot |x - y|$$

$$\forall\, C \in (\lambda, +\infty)$$

$\Longrightarrow$ IT holds also for λ

$$\lambda \geq \frac{|f(x) - f(y)|}{|x - y|} \leq \cancel{\alpha} \quad \lambda + \frac{1}{n} \qquad \forall \underset{\in \mathbb{N}^+}{n} > 0$$

$$\forall c \in (\lambda, +\infty)$$

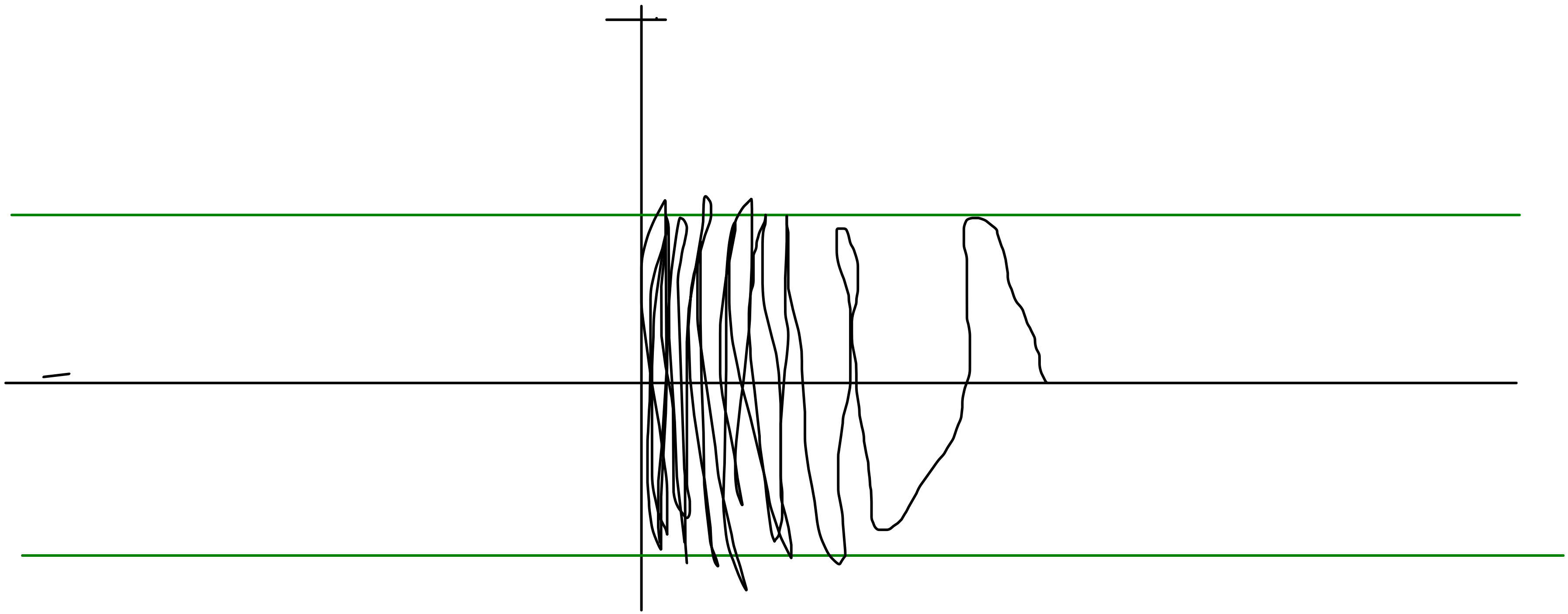
EXAMPLE

$f(x) = \sin(x^{-1})$ $f : \mathbb{R}^* \to \mathbb{R}$

IS IT LIPSCHITZ? NO

NOT UNIFORMLY C^0

$\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ DOES NOT EXIST



$$\underline{\underline{x_n}} = \frac{1}{\frac{\pi}{2} + 2n\pi} \qquad n \in \mathbb{N}$$

$$\sin\left(\frac{1}{x_n}\right) = \sin\left(\frac{\pi}{2}\right) = 1$$

$$\underline{\underline{y_n}} = \frac{1}{\frac{3}{2}\pi + 2n\pi}$$

$$\sin\left(\frac{1}{y_n}\right) = -1$$

$$\left|\sin\left(\frac{1}{x_n}\right) - \sin\left(\frac{1}{y_n}\right)\right| = 2$$

$$|x_n - y_n| \leq |x_n| + |y_n|$$

$$\left|\sin\left(\frac{1}{x_n}\right) - \sin\left(\frac{1}{y_n}\right)\right| = 2$$

$$|x_n - y_n| \leq |x_n| + |y_n| \leq \underline{\underline{\frac{1}{n}}}$$

$\bigwedge\!\!|$ $\delta_1 > 0$ for all $n \gg 0$

$|x_n| \overset{\wedge|}{} \frac{1}{2n}$, $|y_n| \overset{\wedge|}{} \frac{1}{2n}$

$$\cancel{\ast}\ \frac{|f(x_n) - f(y_n)|}{|x_n - y_n|} \geq 2n \quad \forall n \in \mathbb{N}$$

$(-\delta, \delta) \setminus \{0\}$

$\varepsilon = 1$ in the def'n of UNIFC°

$\cancel{\ast}\ \delta_1 \leq \frac{1}{n} \quad \forall n \in \mathbb{N}^*$

THM $f : E \to \mathbb{R}$

INTERVAL

f IS LIPSCHITZ

THEN f IS UNIFORMLY CONTINUOUS

LIPSCHITZ:

$\exists C > 0$ s.t.

$\forall x, y \in E$ $\quad |f(x) - f(y)| \le C \cdot |x-y| \le \frac{\varepsilon}{C} := \delta_\varepsilon$

FIX $\varepsilon > 0$. WANT TO FIND

$\delta_\varepsilon > 0$ s.t.

$|x-y| \le \delta_\varepsilon \Rightarrow |f(x)-f(y)| < \varepsilon$

MORE CONSEQUENCES OF B-W

THM $f : [a,b] \to \mathbb{R}$ C^0

THEN $R(f)$ IS A CLOSED INTERVAL

BOUNDED

LAST TIME: $R(f) \subseteq [m, M]$

$m = \min_{x \in [a,b]} f(x)$, $M = \max_{x \in [a,b]} f(x)$

MORE CONSEQUENCES OF B-W

THM $f : [a, b] \to \mathbb{R}$

C^0 $\left[\iff f \; C^0 \text{ at } x_0 \in \,]a, b[\right.$

$\left. \lim\limits_{x \to a^+} f(x) = f(a), \; \lim\limits_{x \to b^-} f(x) = f(b) \right]$

THEN $R(f)$ IS A CLOSED INTERVAL

MORE CONSEQUENCES OF B-W

THM $f : [a,b] \to \mathbb{R}$

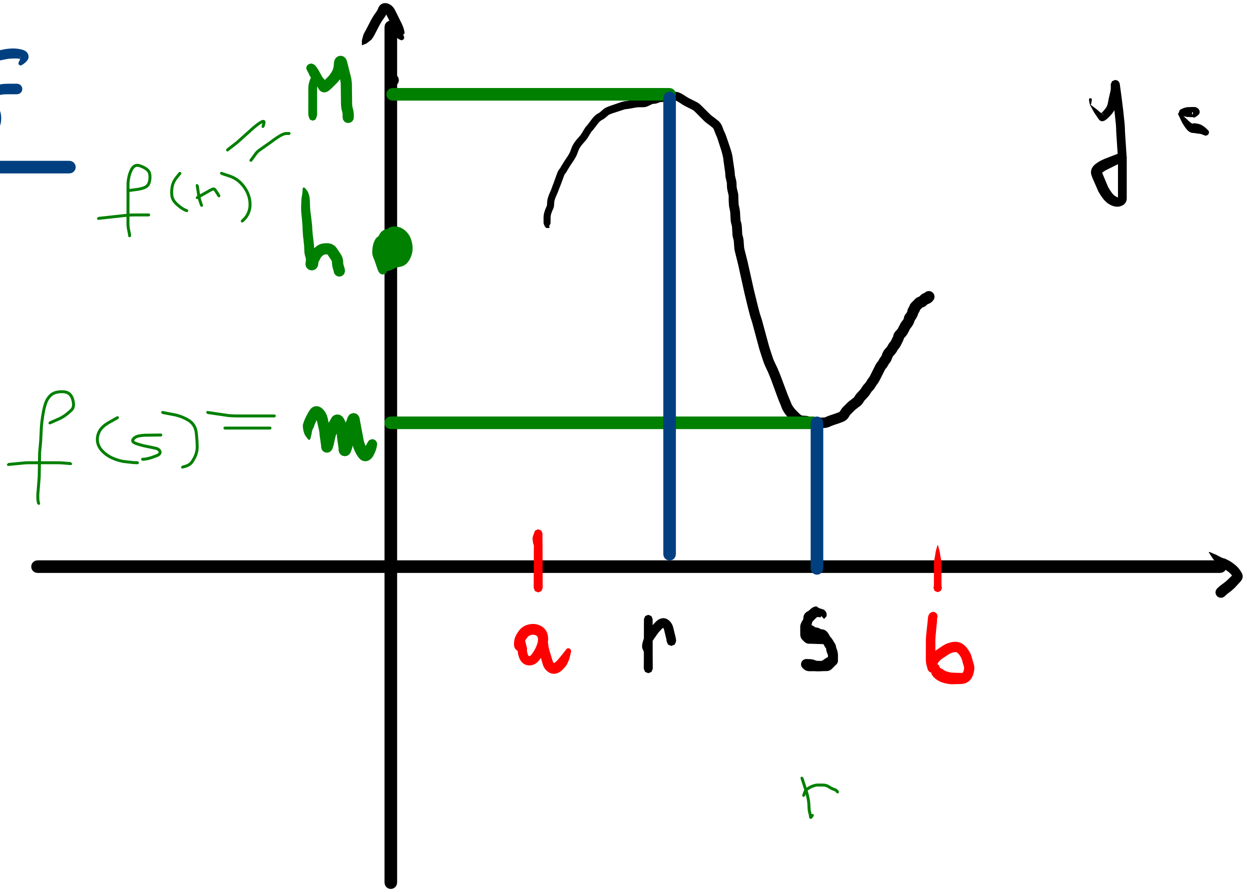
C^0 $\left[\Leftrightarrow f \ C^0 \text{ at } x_0 \in \,]a,b[\quad \lim_{x \to a^+} f(x) = f(a), \ \lim_{x \to b^-} f(x) = f(b) \right]$

THEN $R(f)$ IS A CLOSED INTERVAL

$$[\underset{\substack{\| \\ \min_{x \in [a,b]} f}}{m}, \underset{\substack{\| \\ \max_{x \in [a,b]} f}}{M}]$$

PROOF

$y = f(x)$



$$m = f(s) < h < f(r) = M$$

$$s \in \{ x \in \cancel{\mathbb{R}}\,[r, s] \mid f(x) \leq h \}$$

$$f(s) < h < f(r)$$

$$T \doteq \inf\left\{ \boxed{x \in [r,s] \;\middle|\; f(x) \le h} \right\}$$

BOUNDED

$$T \in [r,s] \qquad \exists\, x_n \downarrow \xrightarrow{n \to \infty} T$$

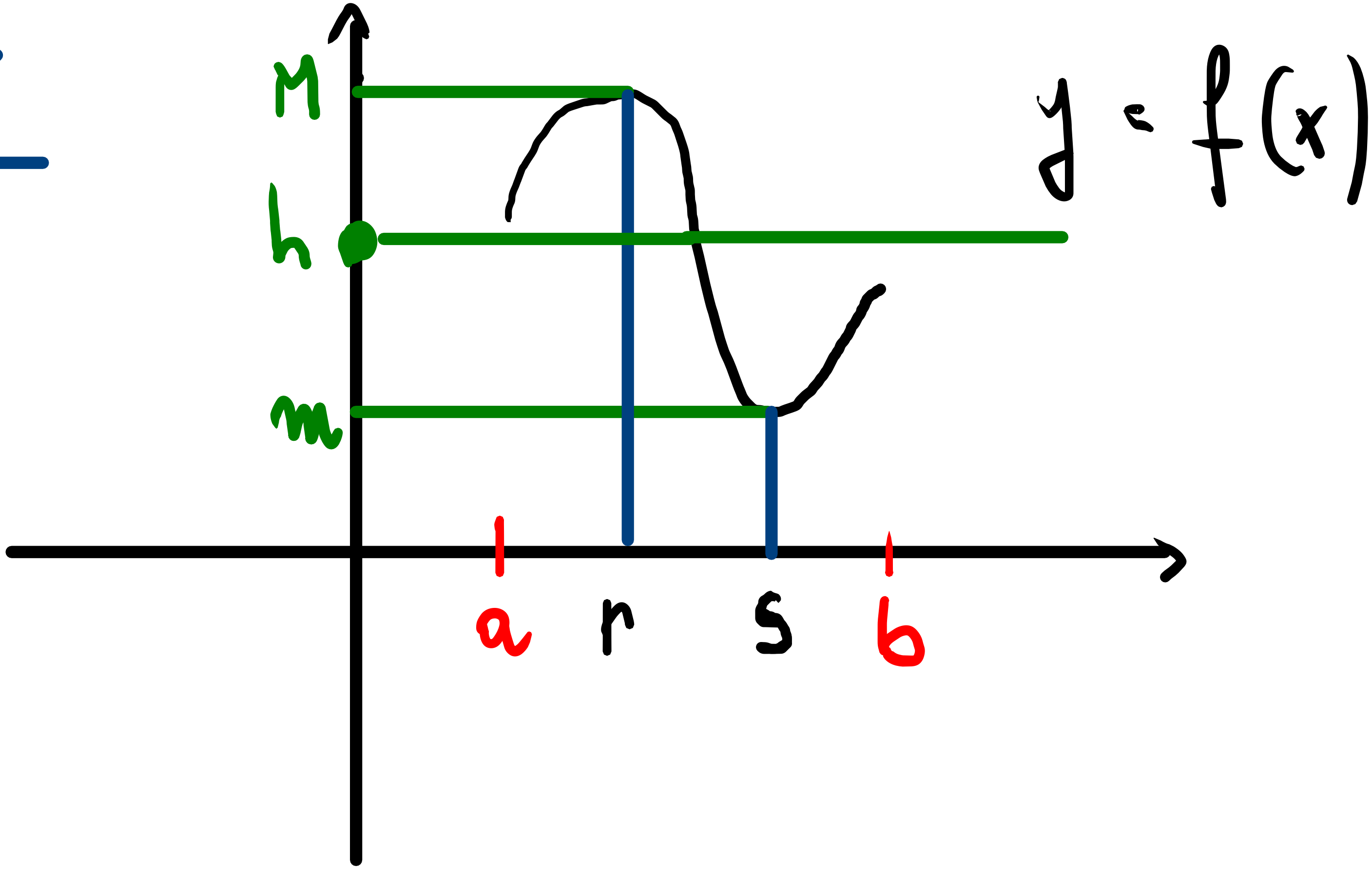
$$\overset{C^0}{\Longrightarrow} \underset{\substack{\wedge\!\!| \\ h}}{f(x_n)} \;\forall n \xrightarrow{n \to +\infty} f(T) \le h$$

COMP. THM

$$\Longrightarrow T > r$$

$f(r) = M$

PROOF



$$f(s) < h < f(r)$$

$$T \doteq \inf \{ x \in [r, s] \mid f(x) \leq h \}$$

$$f(s) < h < f(r)$$

$$T \doteq \text{INF}\{x \in [r,s] \mid f(x) \leq h\}$$

$$T \in [r,s] \qquad \underset{\Downarrow}{\exists}\, x_n \xrightarrow{n \to \infty} T$$

DOES NOT BELONG

$$\Rightarrow f(x_n) \xrightarrow{n \to +\infty} f(T) \leq h$$

$\forall n >> 0$

$$\Rightarrow T > r \qquad f\left(T - \frac{1}{n}\right) > h \quad \forall n$$

$$T - \frac{1}{n} \in [r, s]$$

$\forall n >> 0$

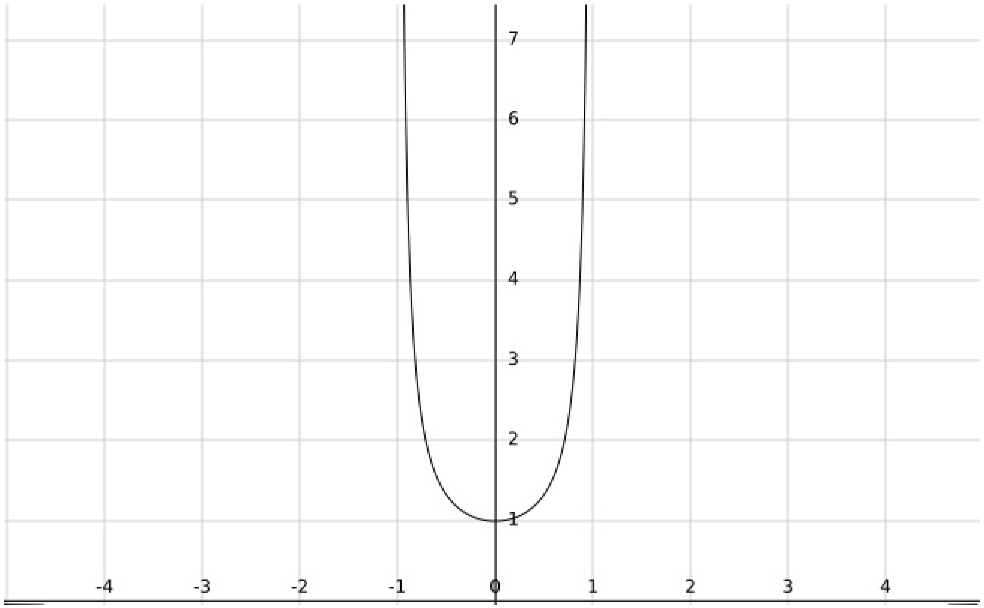
$$T - \frac{1}{n} \to T \Rightarrow \underbrace{f\left(T - \frac{1}{n}\right)}_{\substack{\wedge \\ T}} \to f(T) \quad \text{with } f\left(T-\tfrac{1}{n}\right) > h,\ f(T) \geq h$$

$$\overbrace{n \le \mathrlap{\underbrace{}_{1^{st}\text{ PART}}}f(T)}^{2^{nd}\text{ PART}} \le n \implies f(T) = n$$

$$T \in [n, s] \subseteq [a, b]$$

EXAMPLE

$$f(x) = \frac{1}{1-x^2} \qquad x \in (-1,1)$$



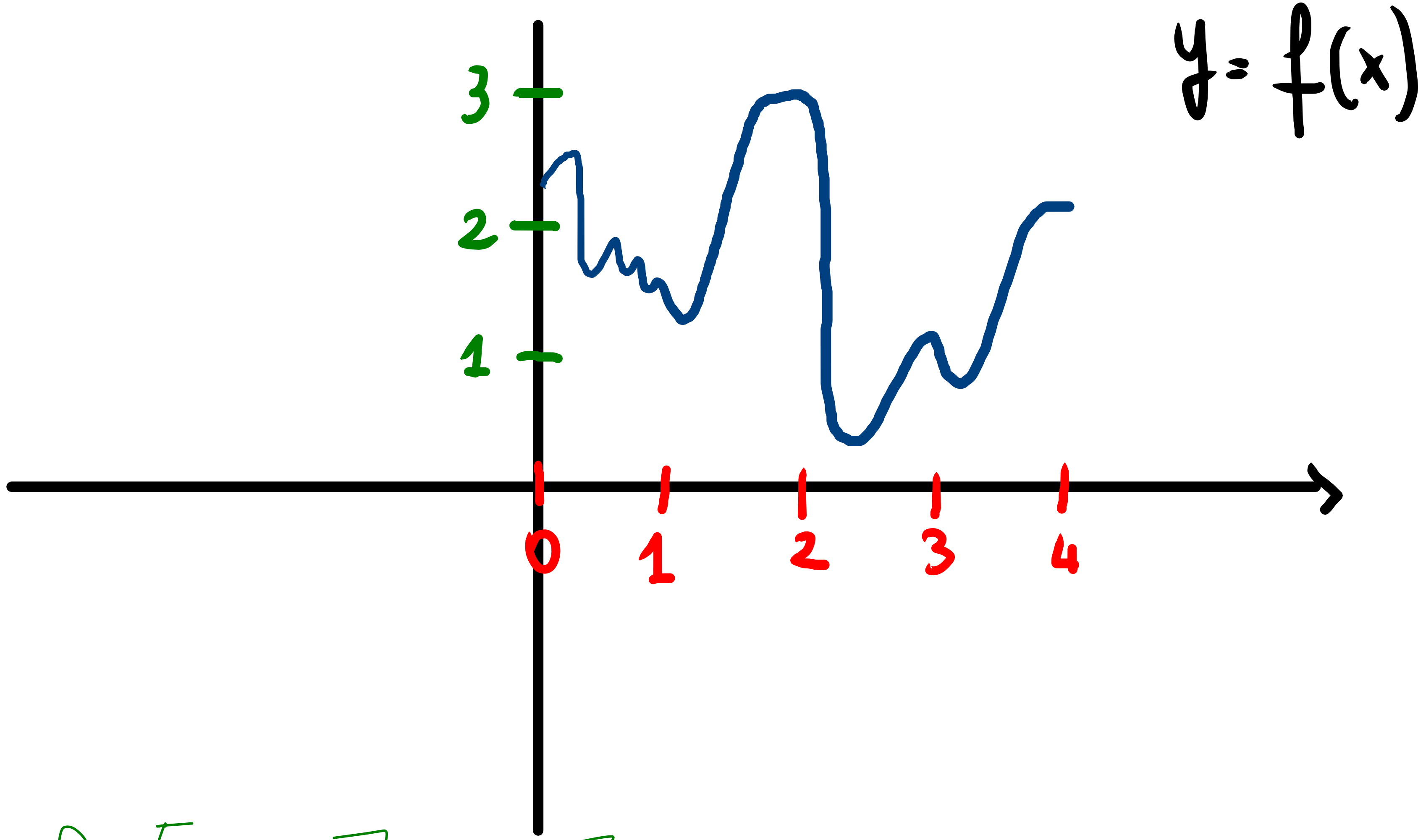
$f : (-1,1) \to \mathbb{R}$

EVEN fct.

$R(f) = [1, +\infty)$

EXAMPLE

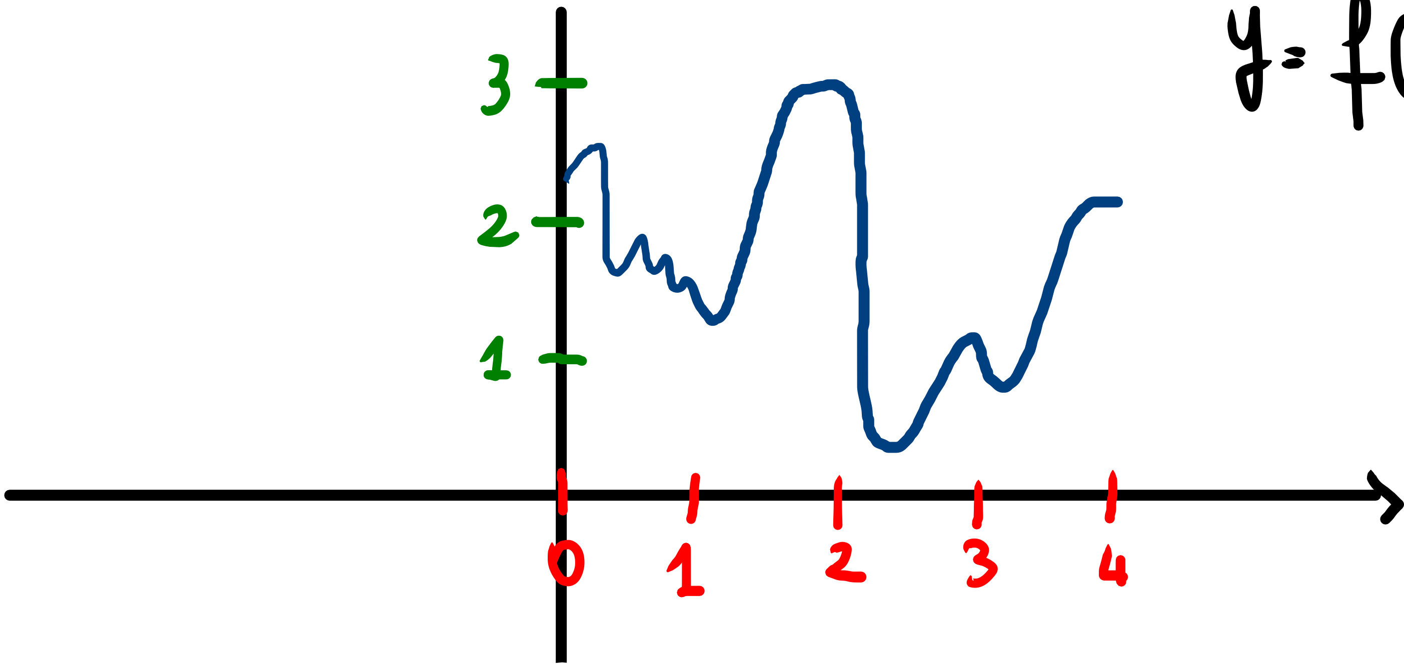
COUNTING SOLUTIONS



$f: [0,4] \to \mathbb{R}$

EXAMPLE COUNTING SOLUTIONS

$y = f(x)$

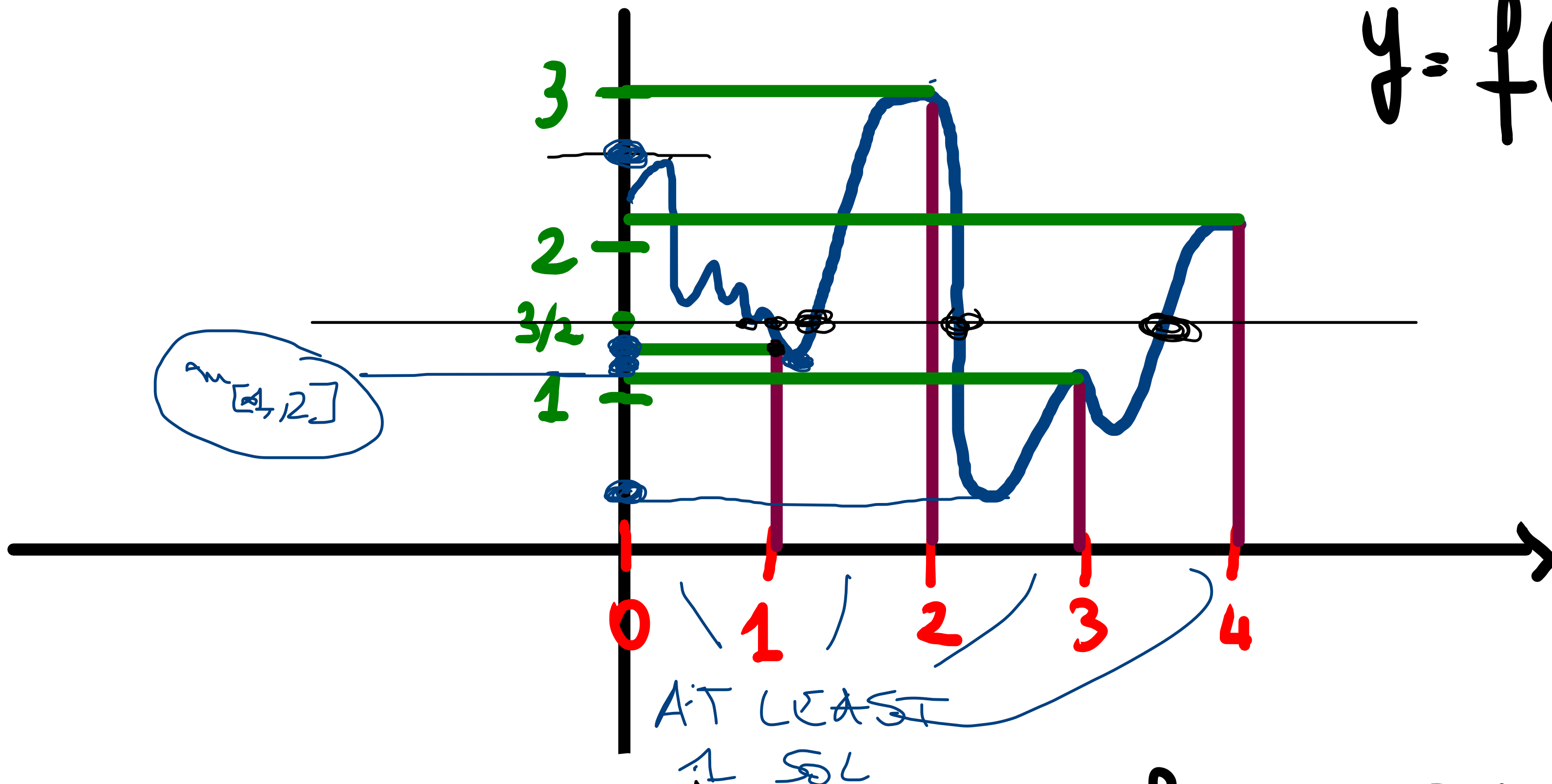


HOW MANY SOLUT'S FOR $f(x) = 3/2$?

(1) NONE (2) 1 (3) 2 (4) ≥ 3 [POLL]

EXAMPLE COUNTING SOLUTIONS

$y = f(x)$



HOW MANY SOLUT'S FOR $f(x) = 3/2$?

COROLLARY

interval

$f : E \longrightarrow \mathbb{R}$ C^0

(a, b)

$(a, b]$

$[a, b)$

$[a, b)$

$(a, +\infty)$

$[a, +\infty)$

$(-\infty, b]$

$(-\infty, b]$

$a, b \in \mathbb{R}$

$\exists\, c, d \in E$ s.t.

$f(c) > 0$

$f(d) < 0$

$\Longrightarrow \exists\, T \in [c, d]$

$[d, c]$

s.t. $f(T) = 0$

PF $f|_{[c,d]} : [c, d] \longrightarrow \mathbb{R}$

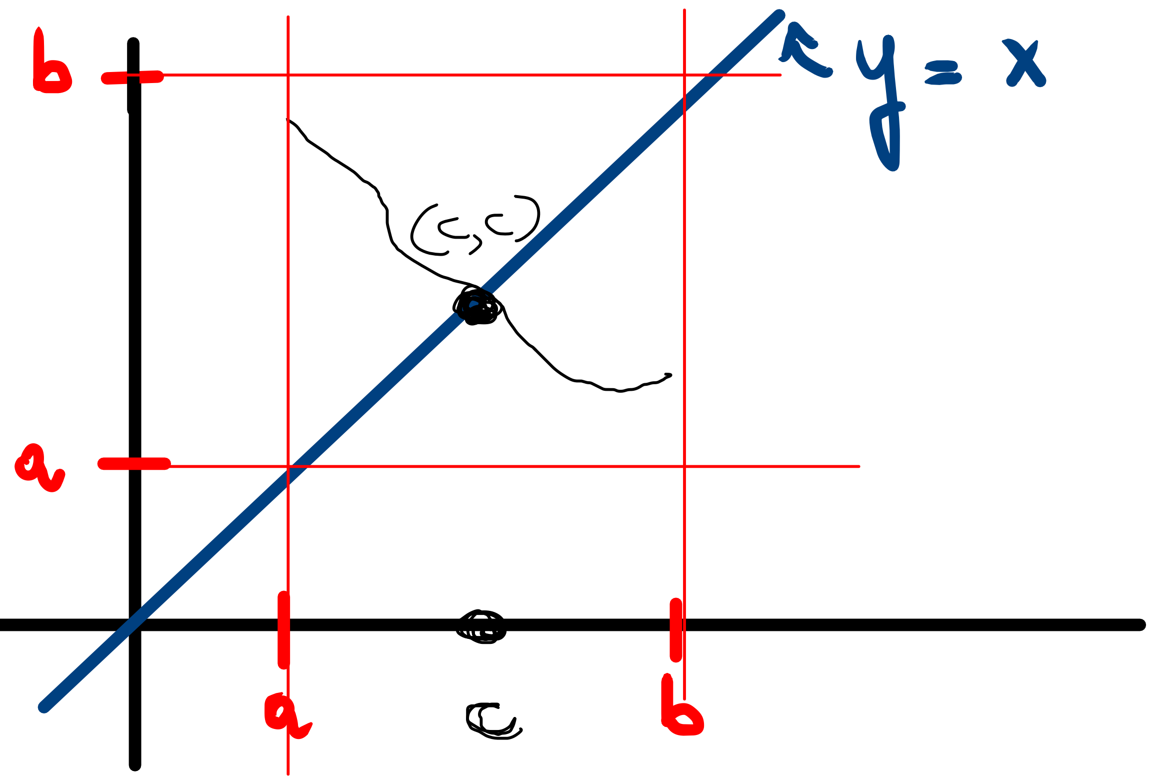
$R(f|_{[c,d]}) = [m, M] \ni 0$

THM $f : [a, b] \longrightarrow [a, b]$ $[R(f) \subseteq [a, b]]$

CONTINUOUS

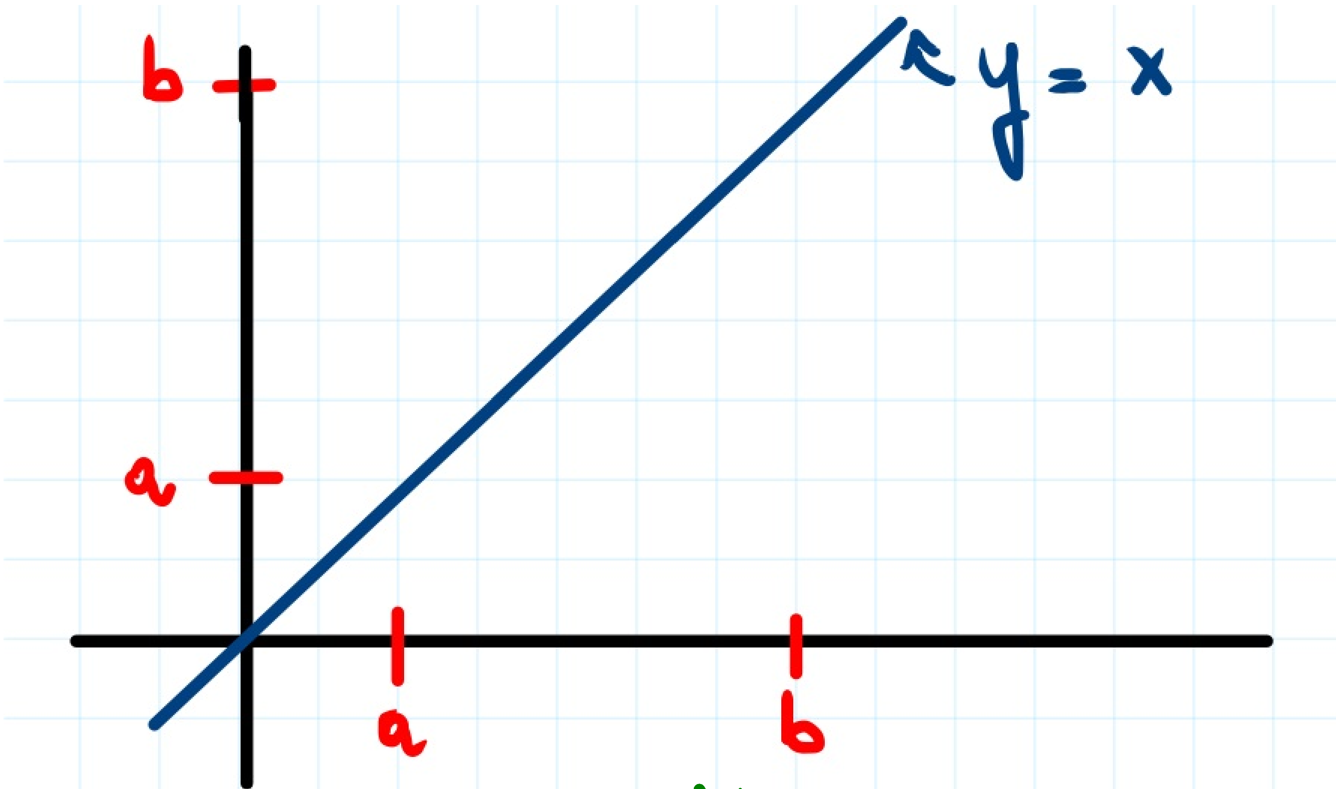
$\Longrightarrow \exists\, c \in [a, b]$ S.T. $f(c) = c$

FIXED PT.



PROOF WE ARE LOOKING FOR SOL'S

OF $f(x) = x \in [a, b]$ $\quad f([a, b]) \subseteq [a, b]$

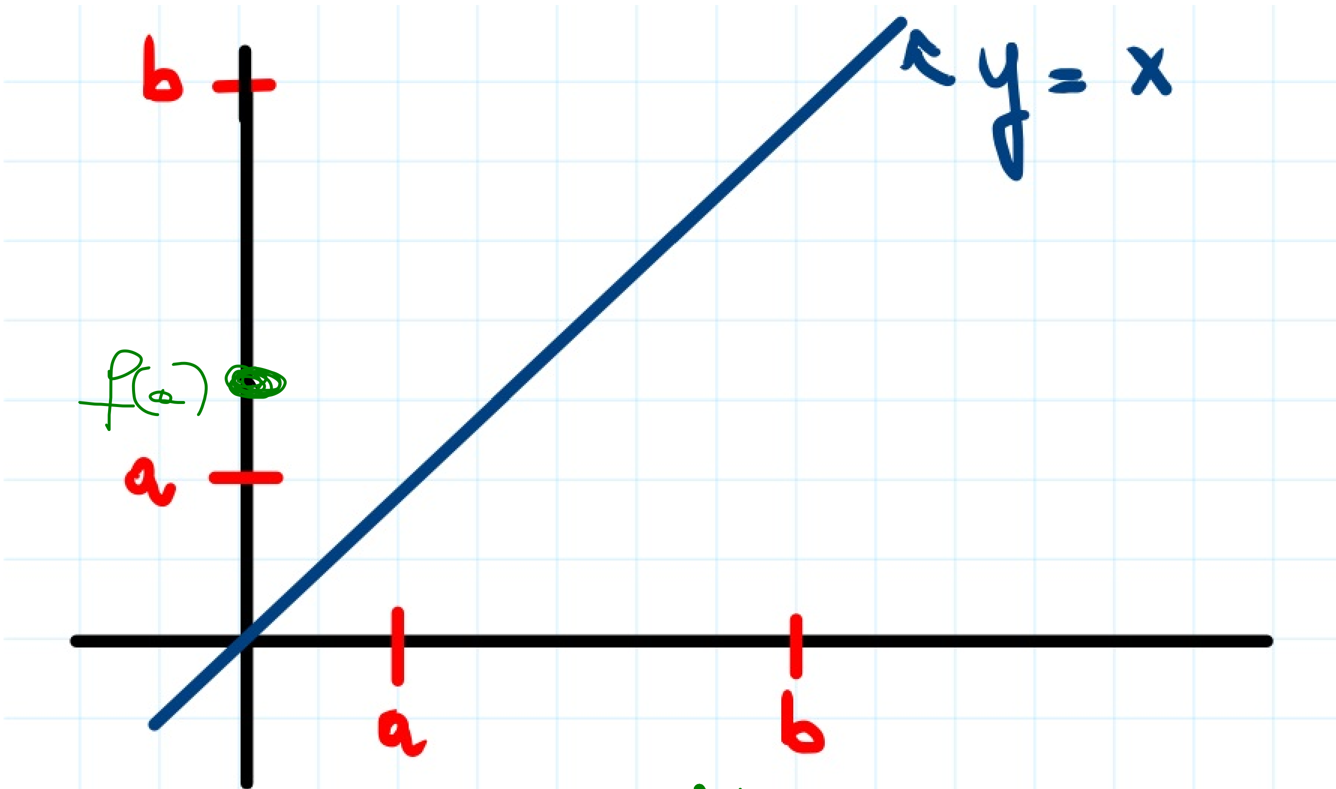


$f(a) = a$?

YES

NO

PROOF WE ARE LOOKING FOR SOL'S OF $f(x) = x$

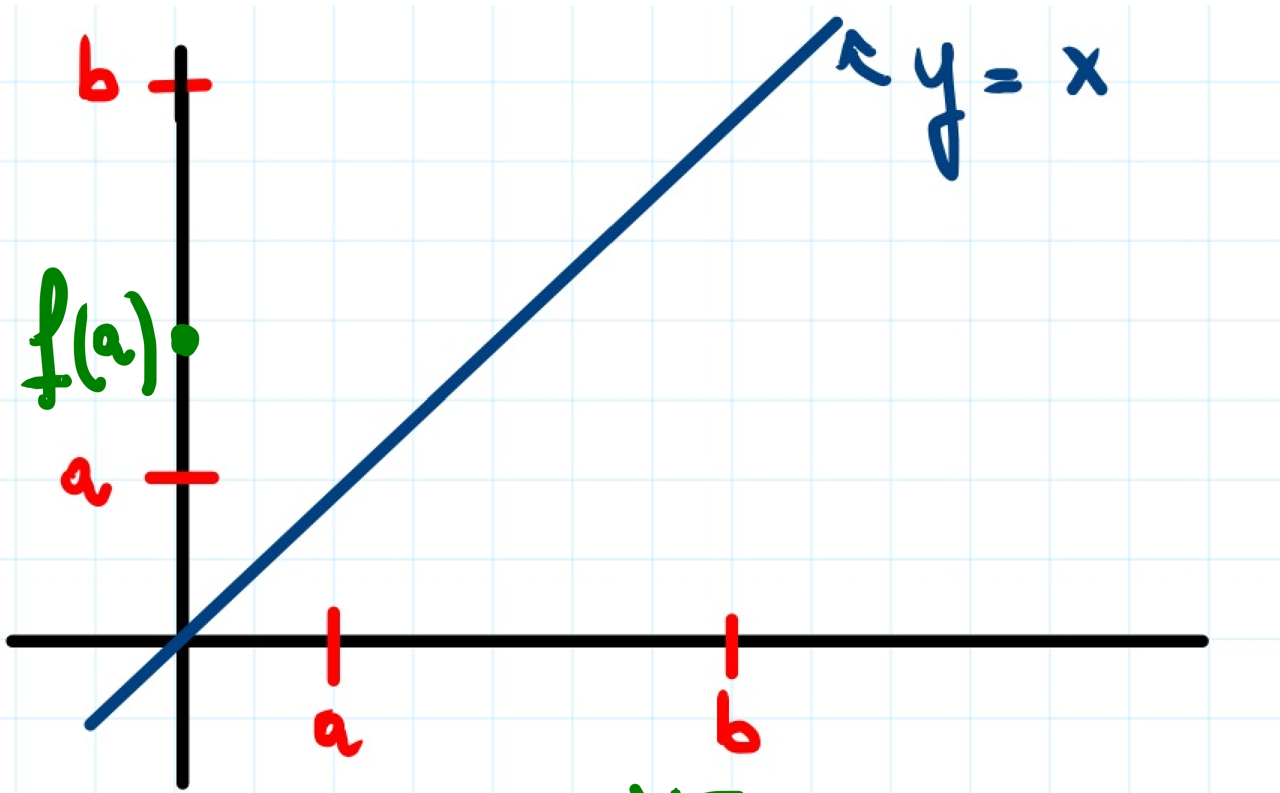


$f(a) = a$?

YES ☺

NO $f(a) > a$ $[R(f) \subseteq [a,b]]$

PROOF WE ARE LOOKING FOR SOL'S OF $f(x) = x$

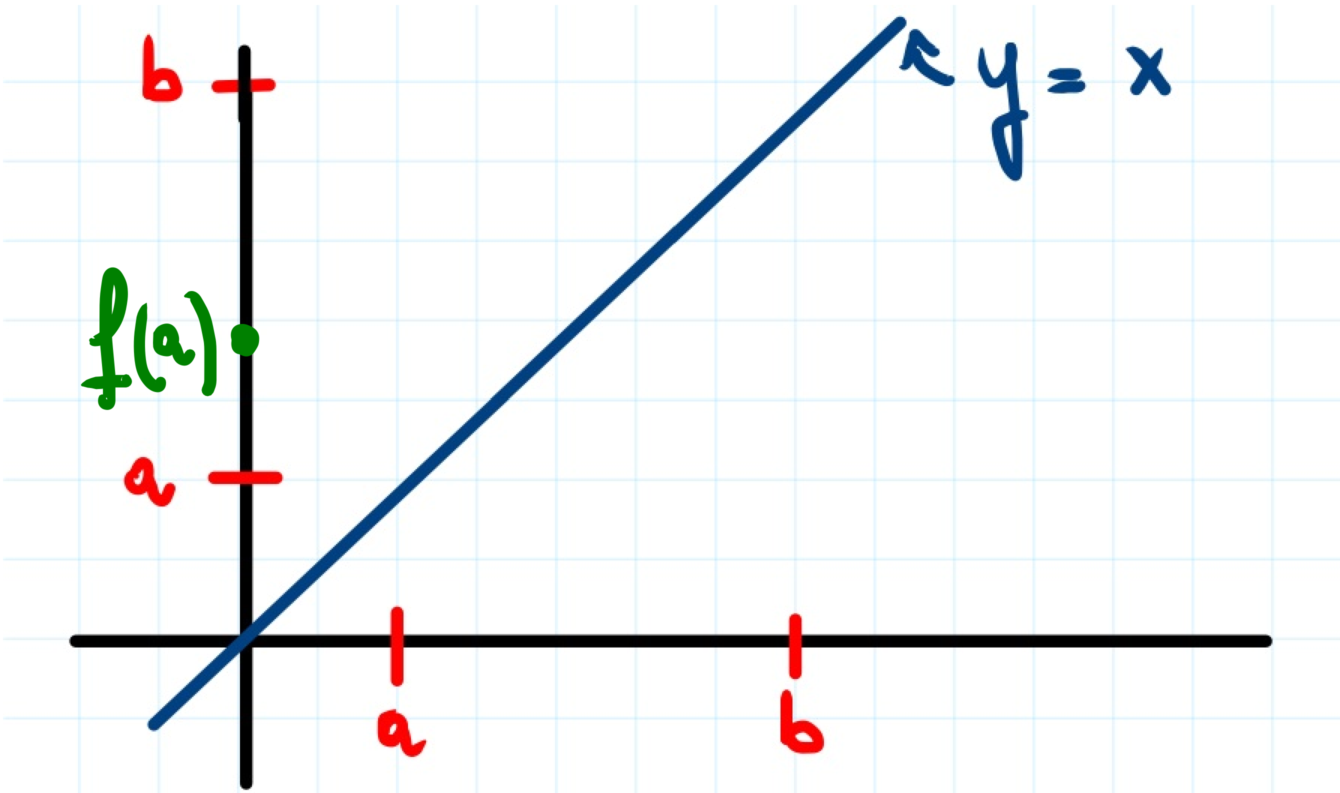


$f(a) = a$?

YES ☺

NO $f(a) > a$

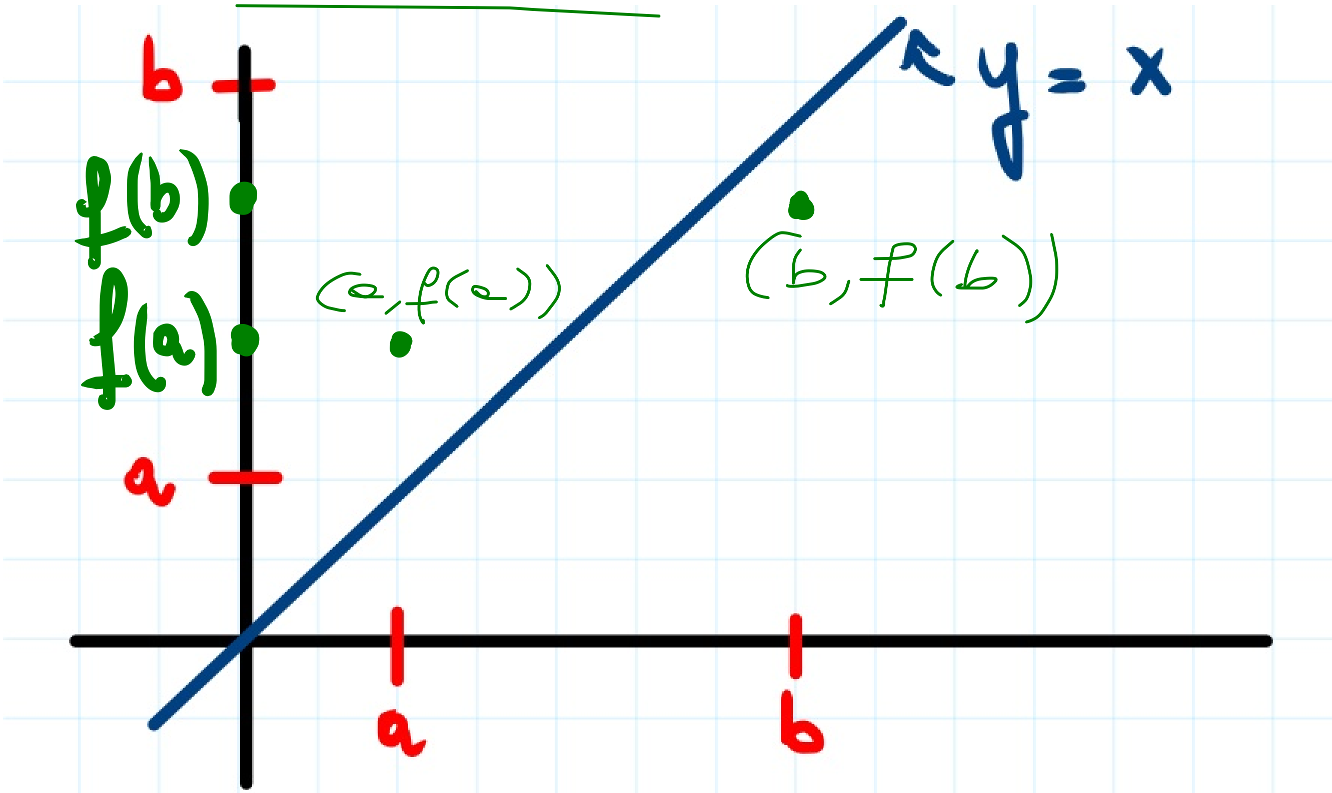
PROOF WE ARE LOOKING FOR SOL'S OF $f(x) = x$



$f(a) > a$

$f(b) = b$? YES / NO

PROOF WE ARE LOOKING FOR SOL'S OF $f(x) = x$



$f(a) > a$

$f(b) = b$?

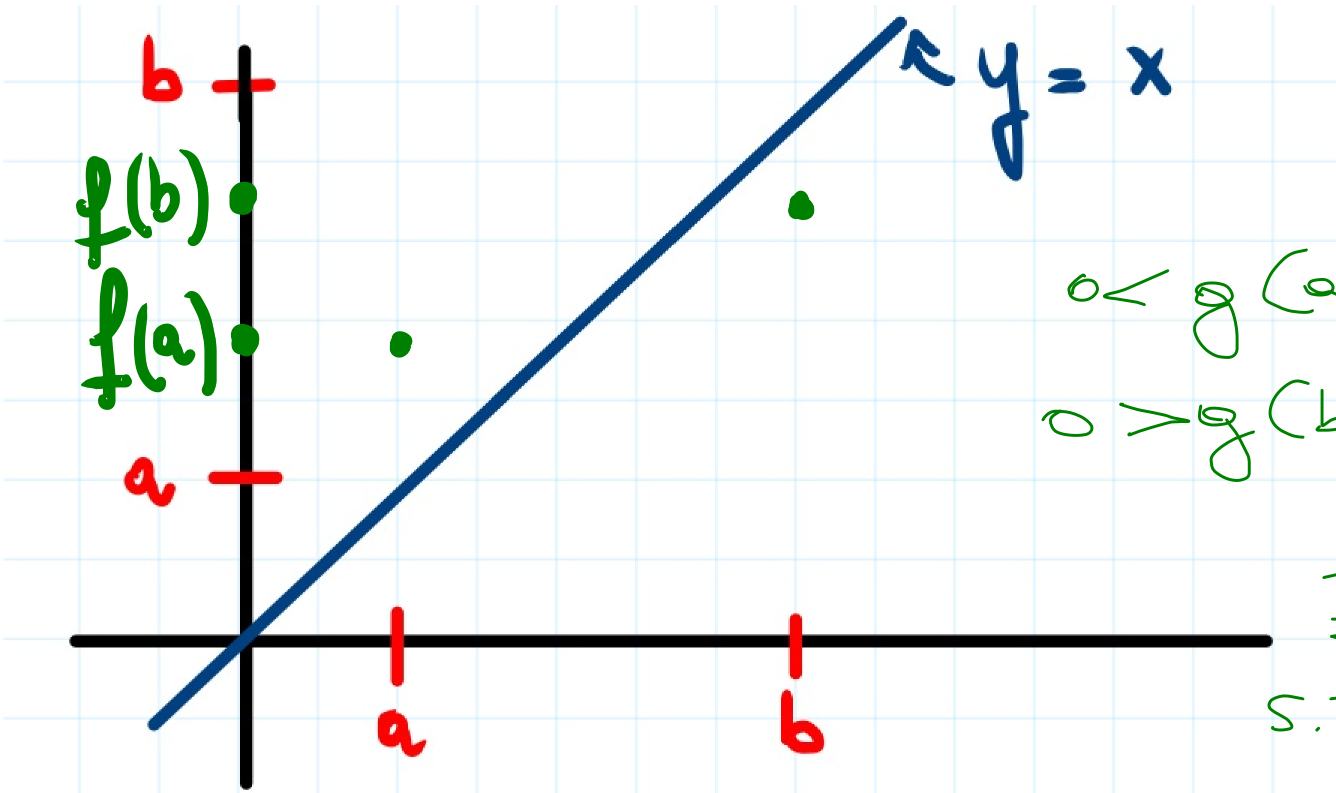
YES :)

NO $f(b) < b$

PROOF WE ARE LOOKING FOR SOL'S

OF $f(x) = x \iff g(x) = 0$

$f(x) - x = 0$



$0 < g(a) = f(a) - a$

$0 > g(b) = f(b) - b$

$\Downarrow$

$\exists\, c \in [a, b]$

s.t. $g(c) = 0$

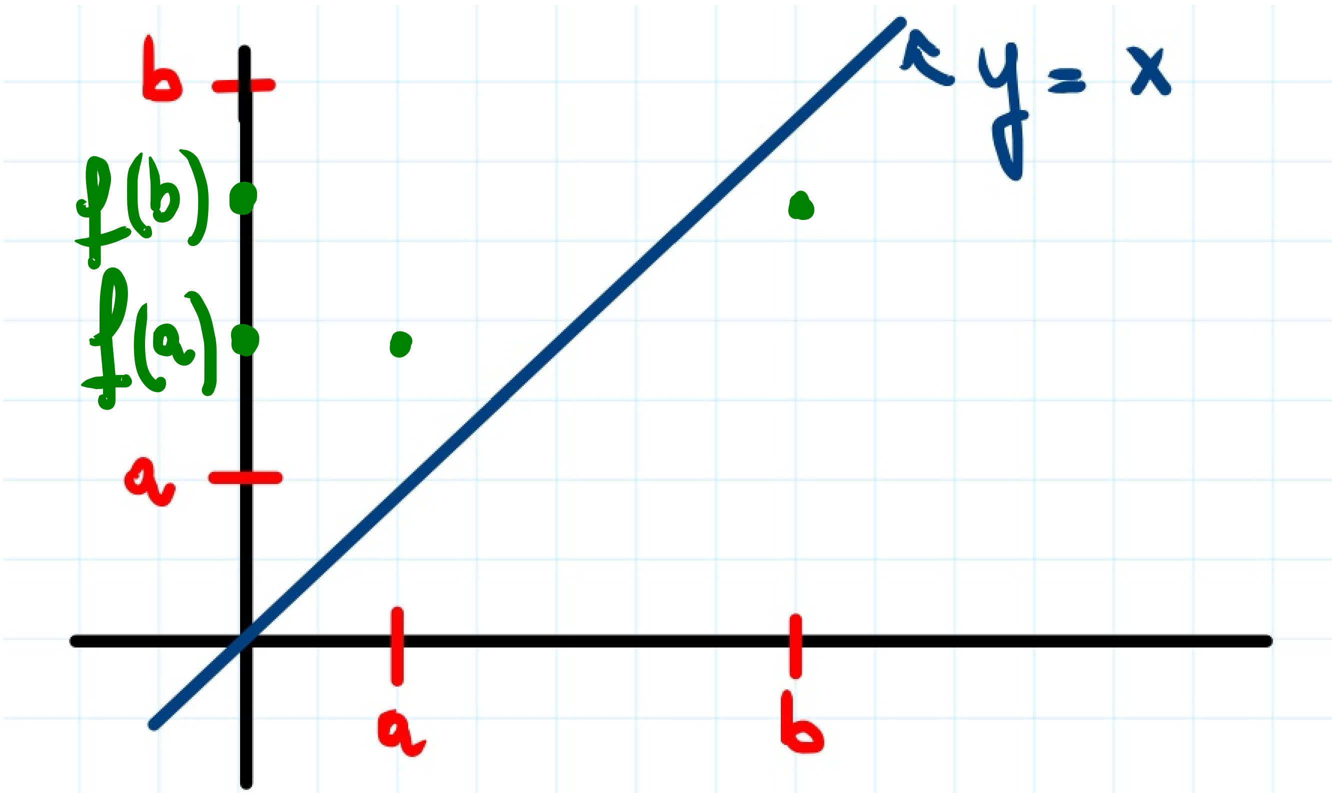
$\Updownarrow$

$f(c) = c$

$f(a) > a$ $\quad$ $f(b) < b$ $\quad$ C^0 $\quad$ C^0

DEFINE $g(x) \doteq f(x) - x$ $\quad$ $g \in C^0$

PROOF WE ARE LOOKING FOR SOL'S OF $f(x) = x \iff g(x) = 0$



$f(a) > a$ $f(b) < b$

DEFINE $g(x) \doteq f(x) - x$

EXAMPLE

$h(x) : \mathbb{R} \to \mathbb{R}$

$$h(x) \doteq \frac{\sin(x^2 - 3x + 1)}{2} + \pi/2 - \frac{1}{x^2+1}$$

$R \subseteq \left[-\frac{1}{2}, \frac{1}{2}\right]$

$R = (0, 1]$

DOES $h(x) = x$ HAS SOL'S?

$$R(h) \subseteq \left[\pi/2 - \frac{3}{2}, \frac{1}{2} + \pi/2\right] = E$$

$$h\Big|_{\left[\pi/2 - \frac{3}{2},\ \pi/2 + \frac{1}{2}\right] = E} : E \longrightarrow E \overset{\text{THM}}{\Longrightarrow} \exists\, x \in E$$

s.t. $h(x) = x$

STRICTLY MONOTONE FCTS

RECALL: STRICTLY MONOT.

STRICTLY INCREAS.

STRICTLY DECREAS.

$f : E \longrightarrow \mathbb{R}$

STRICTLY INCR.

STRICTLY DECR.

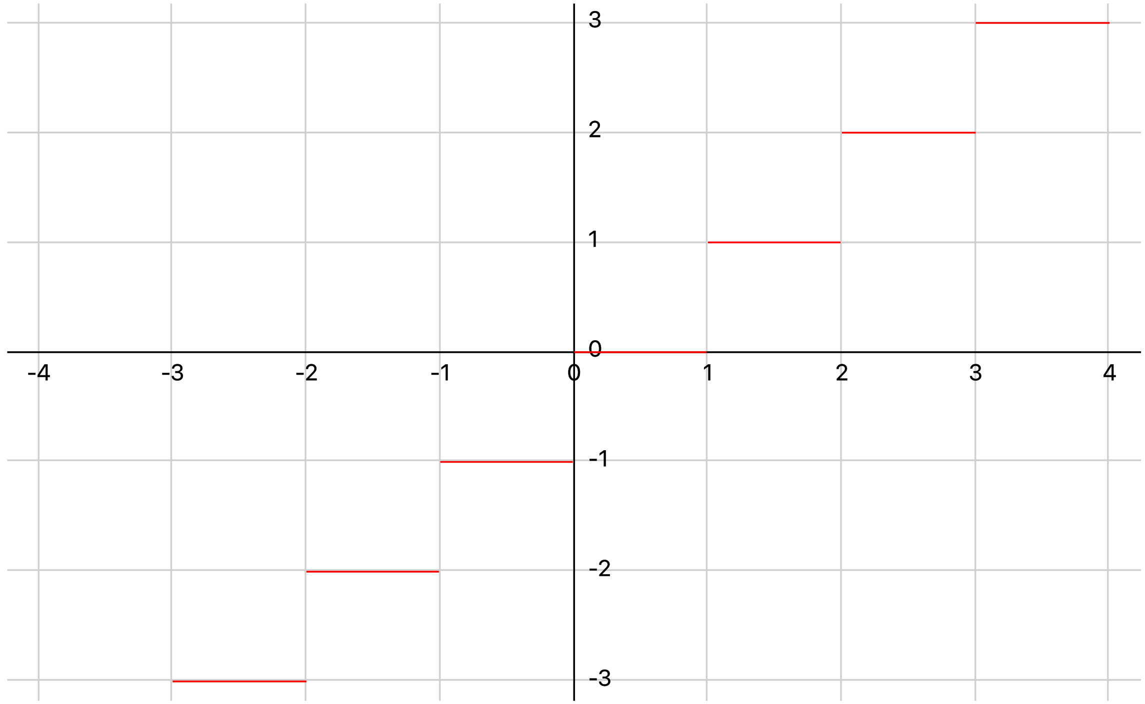
WHEN $\forall x, y \in E$, $x > y \Rightarrow f(x) \underset{<}{>} f(y)$

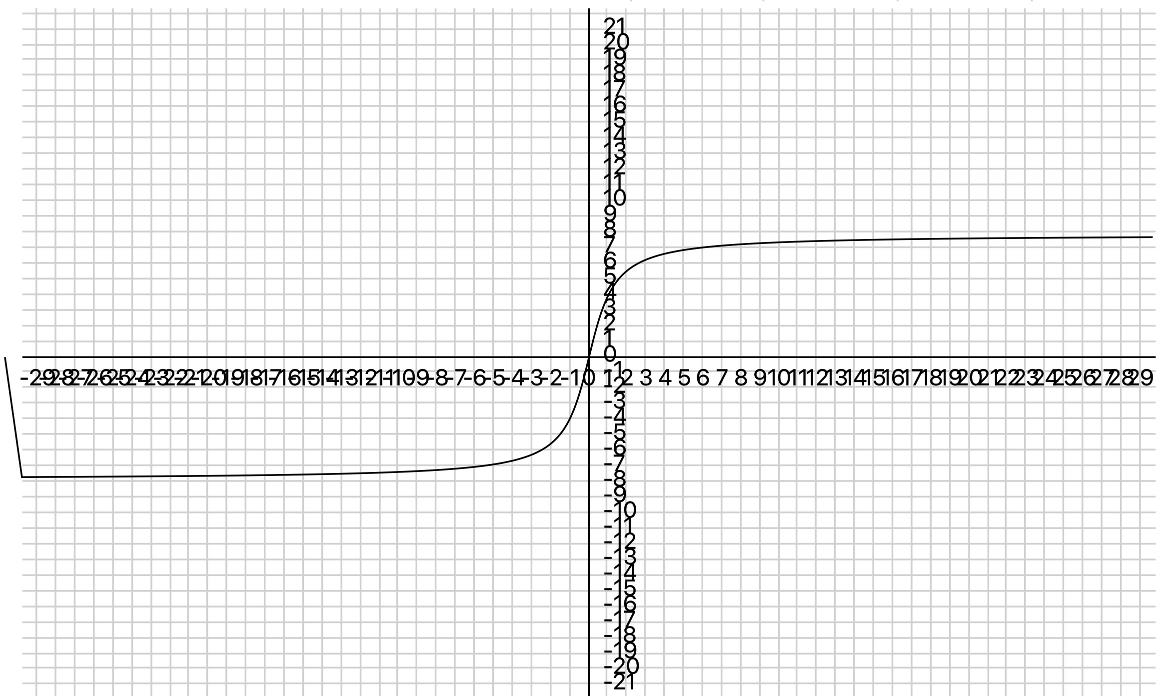
EXAMPLE

$f(x) = \lfloor x \rfloor$

INCR.

NOT STR.





$f(x) =$

$g : \arctan(x)$

STR. INCR.

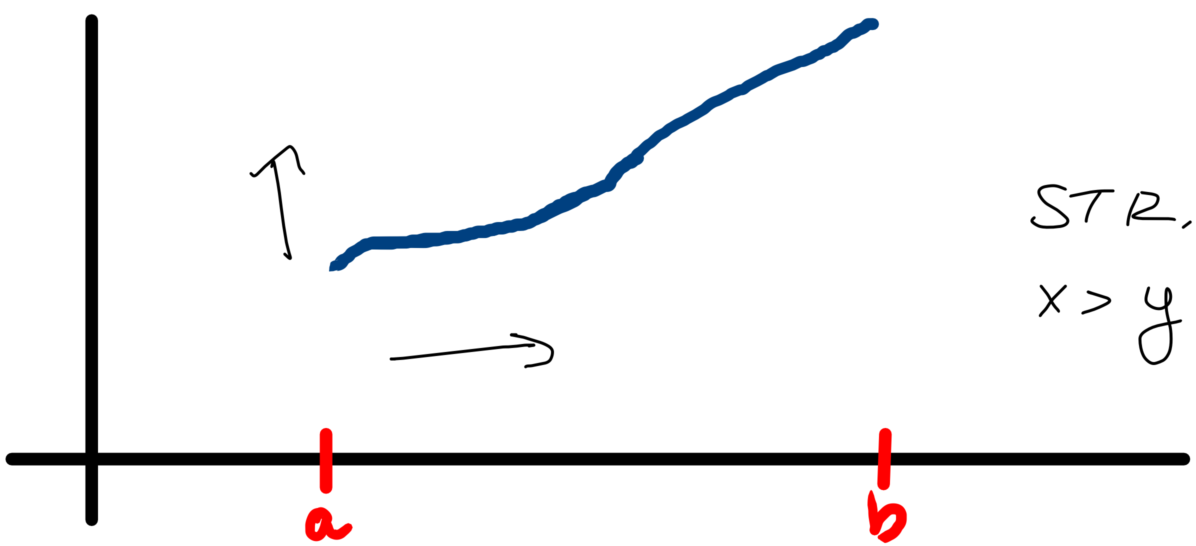
THM $f :]a,b[\longrightarrow \mathbb{R}$

CONTINUOUS &

STRICTLY MONOTONE

$\Longrightarrow$ $R(f)$ IS AN OPEN INTERVAL & f is SURJ.

$f : (a,b) \to R(f)$



STR.

$x > y \implies f(x) > f(y)$

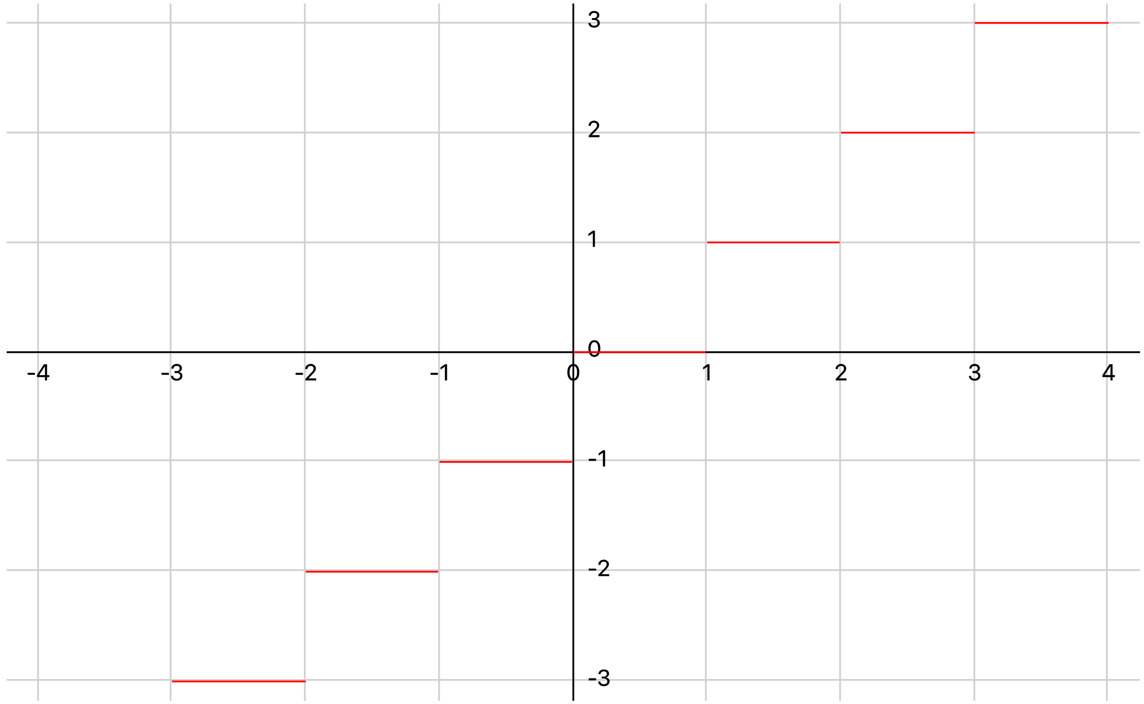
STRICTLY MONOTONE FCTS

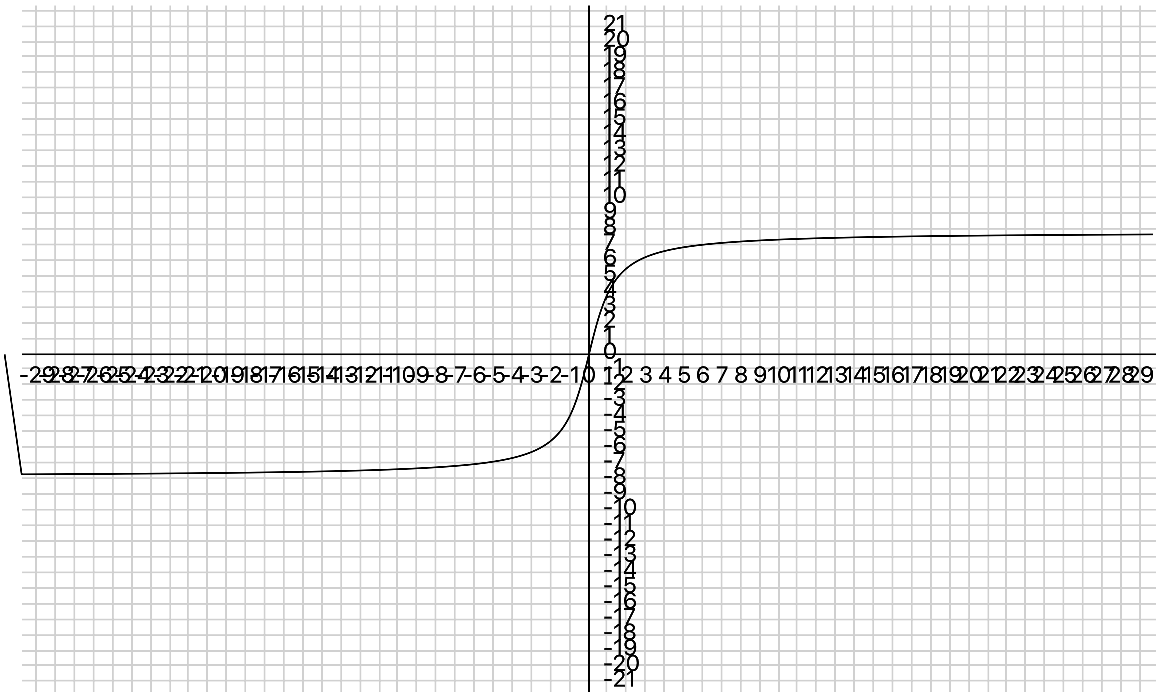
RECALL: STRICTLY MONOT. — STRICTLY INCREAS. / STRICTLY DECREAS.

$f : E \longrightarrow \mathbb{R}$ STRICTLY INCR. / STRICTLY DECR.

WHEN $\forall x, y \in E \,,\; x > y \Rightarrow f(x) \substack{> \\ <} f(y)$

EXAMPLE



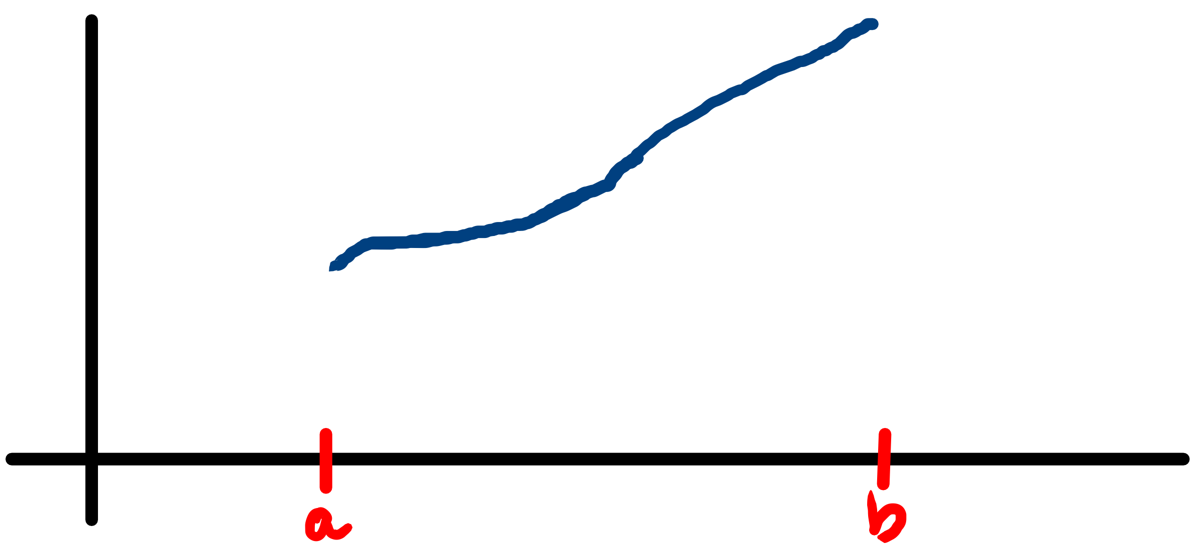


THM $f :]a, b[\to \mathbb{R}$

CONTINUOUS &

STRICTLY MONOTONE

$\Longrightarrow$ $R(f)$ IS AN OPEN INTERVAL



PROOF

(a, b)

ASSUME f IS STR. INCR.

$a_n \doteq a + \frac{1}{n}$ $\downarrow$ STR. $b_n \doteq b - \frac{1}{n}$ $\uparrow$ STR.

$a_n \to a$, $b_n \to b$

$f(a_n)$ STRICTLY DECREASING

$f(b_n)$ STRICTLY INCREASING

$\Longrightarrow$

PROOF

$$a_n \doteq a + \frac{1}{n} \qquad b_n \doteq b - \frac{1}{n}$$

$$a_n \to a \, , \quad b_n \to b$$

$f(a_n)$ STRICTLY DECREASING
$f(b_n)$ STRICTLY INCREASING

STR. MON.

$$\Longrightarrow \lim f(a_n) = \begin{cases} c \\ -\infty \end{cases}$$

PROOF $a_n \doteq a + \frac{1}{n}$ $b_n \doteq b - \frac{1}{n}$

$a_n \to a$, $b_n \to b$

$f(a_n)$ STRICTLY DECREASING

$f(b_n)$ STRICTLY INCREASING

$$\Longrightarrow \lim_{n\to\infty} f(a_n) = \begin{cases} c \\ -\infty \end{cases}$$

$c < d$

$$\lim_{n\to\infty} f(b_n) = \begin{cases} d \\ +\infty \end{cases}$$

PROOF

$$a_n \doteq a + \frac{1}{n} \qquad b_n \doteq b - \frac{1}{n}$$

$$a_n \longrightarrow a \,, \quad b_n \longrightarrow b$$

$f(a_n)$ STRICTLY DECREASING

$f(b_n)$ STRICTLY INCREASING

$$\Longrightarrow \lim_{n \to \infty} f(a_n) = \begin{cases} \boxed{c} \\ -\infty \end{cases}$$

$$\lim_{n \to \infty} f(b_n) = \begin{cases} \boxed{d} \\ +\infty \end{cases}$$

PROOF $\quad a_n \doteq a + \frac{1}{n} \qquad b_n \doteq b - \frac{1}{n}$

$a_n \to a \quad , \quad b_n \to b$

$f(a_n)$ STRICTLY DECREASING

$f(b_n)$ STRICTLY INCREASING

$$\Longrightarrow \lim_{n\to\infty} f(a_n) = \begin{cases} \boxed{c} \\ -\infty \end{cases}$$

$$\lim_{n\to\infty} f(b_n) = \begin{cases} \boxed{d} \\ +\infty \end{cases}$$

WE WANT TO SHOW THAT

$R(f) =]c, d[$

$$\lim a_n = c \qquad f \text{ STR. INCR}$$

$$\lim b_n = d \qquad f \text{ IS } C^0$$

$$f(a_n) = c_n \longrightarrow c \qquad a_n \downarrow a$$

$$f(b_n) = d_n \longrightarrow d \qquad b_n \uparrow b$$

$$R\left(f|_{[a_n, b_n]}\right) = [c_n, d_n]$$

$$R(f) = \bigcup_{n \in \mathbb{N}^*} [c_n, d_n]$$

$$\overset{!}{=} \bigcup_{n \in \mathbb{N}^*} R\left(f|_{[a_n, b_n]}\right) = R\left(f|_{\cup [a_n, b_n]}\right)$$

$$R(f) = \bigcup_{n\in\mathbb{N}^*} [c_n, d_n] \qquad R(f|_{(a,b)})$$

$$\overset{?}{=} \bigcup_{n\in\mathbb{N}^*} R(f|_{[a_n, b_n]}) = R(f|_{\cup[a_n, b_n]})$$

$$\bigcup [a_n, b_n] = (a, b)$$

$$a < c < b$$

$$\exists n \quad a < a + \frac{1}{n} < c < b - \frac{1}{n} < b$$

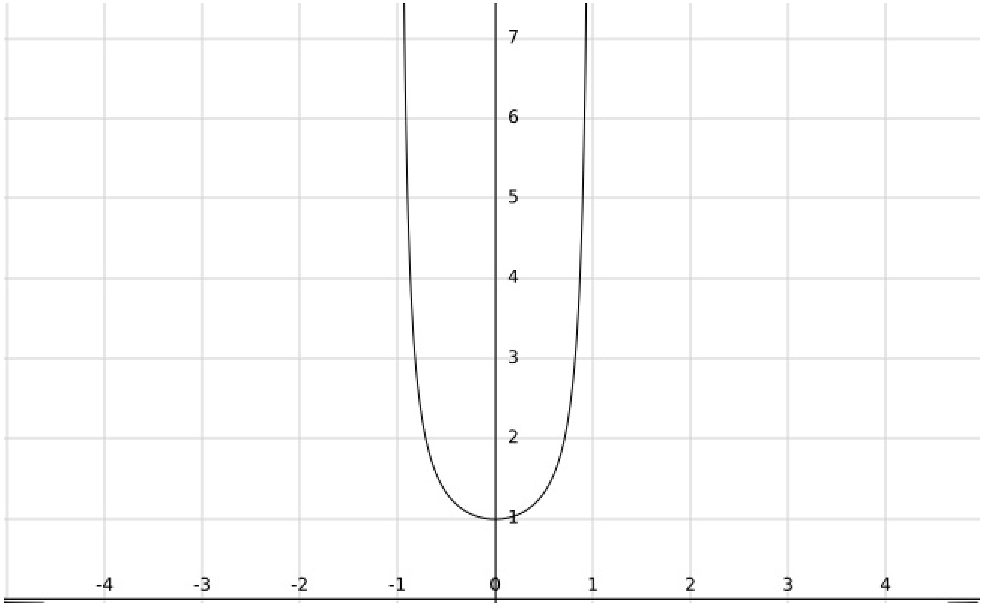
$$[a_n, b_n] \quad \exists n$$

$c_n \to c$

$d_n \to c$

$$\bigcup [c_n, d_n] = (c, d)$$

EXAMPLE



$$f(x) = \frac{1}{1 - x^2}$$

EXAMPLE

$f(x) =$ [illegible] (x)