ANALYSIS 1

(ENGLISH)

LECTURE 16

ALGEBRA & LIMITS AT ∞

PROP $f, g : E \to \mathbb{R}$, $x_0 \in \mathbb{R}$

ASSUME E CONTAINS A PUNCTURED NEIGHBORHOOD S OF x_0, [$\forall x \in S$]

ASSUME g IS BOUNDED FROM BELOW IN S

* ① IF $\lim_{x \to x_0} f(x) = +\infty \implies \lim_{x \to x_0} f(x) + g(x) = +\infty$

② IF $\lim_{x \to x_0} f(x) = +\infty$, $g(x) \geq \underline{\underline{A}} > 0$ $\forall x \in E$ S

$\lim_{x \to x_0} f(x) \cdot g(x) = +\infty$

③ If $g(x)$ bounded on S

(THAT IS, $\{g(x) \mid x \in S\} \subseteq \mathbb{R}$ is BOUNDED)

& $f(x) \neq 0 \ \forall x \in S \Rightarrow$

$$\lim_{x \to x_0} g(x)/f(x) = 0$$

EXAMPLE $\sum \sin\left(\frac{1}{n^\alpha}\right)$

FOR WHAT VALUES OF $\alpha \in \mathbb{R}$ DOES IT CONVERGE?

No ① $\forall \alpha \in \mathbb{R}$ $\alpha = 0$ $\sin\left(\frac{1}{1}\right) = \sin 1$

$s_n = n \cdot \sin(1) \overset{\vee}{0}$

D

② $\alpha \in (0, +\infty)$ ONLY

③ $\alpha \in (1, +\infty)$ ONLY

Q

$\mathbb{R}/\mathbb{Q}$

④ $\alpha \in (2, +\infty)$ ONLY

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1 \iff \forall x_n \to 0 \Rightarrow \frac{\sin(x_n)}{x_n} \to 1$$

ST

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1 \iff \forall x_n \to 0 \implies \frac{\sin(x_n)}{x_n} \to 1 \quad (*)$$

$$\alpha > 0 \implies \frac{1}{n^\alpha} \to 0 \overset{(*)}{\implies} \boxed{\frac{\sin\left(\frac{1}{n^\alpha}\right)}{\frac{1}{n^\alpha}}} \to 1$$

$$\implies \forall n \gg 0 \quad \frac{1}{2} \le \frac{\sin\left(\frac{1}{n^\alpha}\right)}{\frac{1}{n^\alpha}} \le \frac{3}{2}$$

$$0 \le \sin\left(\frac{1}{n^\alpha}\right) \le \frac{3}{2}\frac{1}{n^\alpha}, \qquad \alpha > 1 \implies \sum \frac{1}{n^\alpha}$$

$$\sum \sin\left(\frac{1}{n^\alpha}\right) \overset{SQ}{\Longleftarrow} \text{CONVERGE}$$

$$\frac{1}{2}\frac{1}{n^\alpha} \le \sin\left(\frac{1}{n^\alpha}\right) \qquad 0 < \alpha \le 1 \implies \sum \frac{1}{n^\alpha} = +\infty$$

$$\sum \sin\left(\frac{1}{n^\alpha}\right) = +\infty \overset{SQ}{\Longleftarrow}$$

x_n

$y_n = c \cdot x_n \qquad c \in \mathbb{R}$

$\sum x_n$ conv. $\Leftrightarrow \sum y_n$ conv.

SQUEEZE AT ∞

PROP $f, g : E \to \mathbb{R}$ $\quad x_0 \in \mathbb{R}$

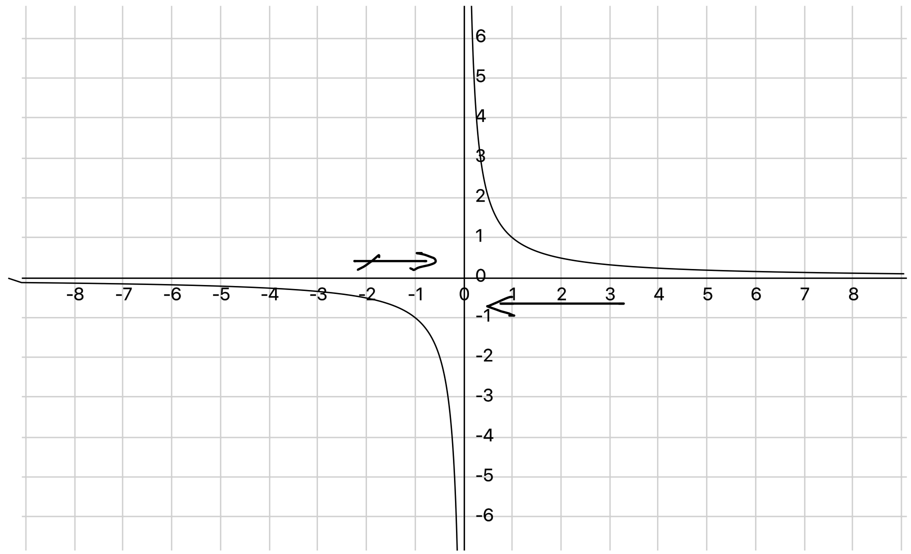
ASSUME E CONTAINS A PUNCT. NEIGHBORHOOD S OF x_0.

ASSUME $f(x) \leq g(x)$ $\forall x \in S$.

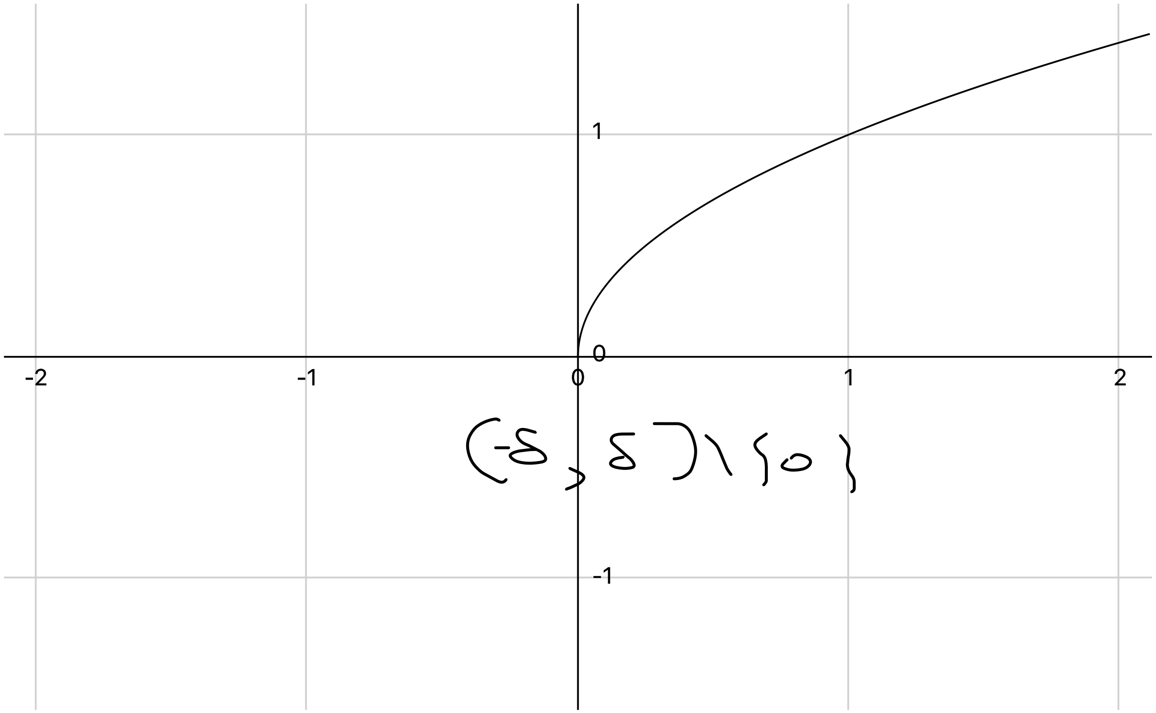
① IF $\lim_{x \to x_0} f(x) = +\infty \Rightarrow \lim_{x \to x_0} \underbrace{g(x)}_{+\infty} = +\infty$

② IF $\lim_{x \to x_0} g(x) = -\infty \Rightarrow \lim_{x \to x_0} \underbrace{f(x)}_{-\infty} = -\infty$.

ONE SIDED LIMITS



$$f(x) = \frac{1}{x}$$



$(-\delta,\delta)\setminus\{0\}$

$f(x)=\sqrt{x}$

$f:\mathbb{R}_+\longrightarrow\mathbb{R}$

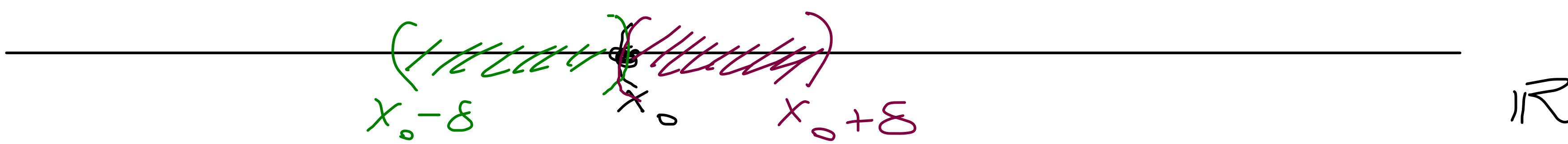
$D(f)$ does not contain any punctured nbhd of 0

IDEA: $D(f)\supseteq \underline{\underline{(0,\delta)}},\ \forall\delta>0$

DEF $f : E \to \mathbb{R}$ $\quad x_0 \in \mathbb{R}$

f IS DEFINED ON THE LEFT / RIGHT OF x_0 IF FOR SOME $\delta > 0$

$$D(f) \supseteq \begin{matrix} (x_0 - \delta,\ x_0) \\ (x_0,\ x_0 + \delta) \end{matrix}$$



DEF $f: E \to \mathbb{R}$ $\quad x_0 \in \mathbb{R}$

$\lim_{x \to x_0^+} f(x) = \ell \in \overline{\mathbb{R}}$ IF:

FROM THE RHS $\qquad \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$

① f IS DEFINED ON THE RIGHT OF x_0; $\quad D(f) \supset (x_0, x_0+\delta)$

② $\forall (y_n) \subseteq E$, $y_n > x_0$ $\forall n \in \mathbb{N}$

IF $y_n \xrightarrow{n\to\infty} x_0 \implies f(y_n) \xrightarrow{n\to\infty} \ell$

DEF $\quad f : E \longrightarrow \mathbb{R} \qquad x_0 \in \mathbb{R}$

$$\lim_{x \to x_0^-} f(x) = \ell \in \overline{\mathbb{R}} \quad \text{IF :}$$

limit from the LHS

① f IS DEFINED ON THE LEFT OF x_0;

② $\forall (y_n) \subseteq E$, $y_n < x_0 \quad \forall n \in \mathbb{N}$

IF $y_n \xrightarrow{n \to \infty} x_0 \Longrightarrow f(y_n) \xrightarrow{n \to \infty} \ell$

PROP $f: E \to \mathbb{R}$, $x_0 \in \mathbb{R}$

[ASSUME f IS DEF'D ON A PCT'D NBD OF x_0]

ASSUME $\lim_{x \to x_0^+} f(x) = \ell_1$

$\lim_{x \to x_0^-} f(x) = \ell_2$ $\in \overline{\mathbb{R}}$

THEN

$$\lim_{x \to x_0} f(x) = \ell \iff \ell_1 = \ell_2$$

$$f(x) = \begin{cases} x+1 & x > 0 \\ \frac{1}{x} & x < 0 \end{cases}$$

TO CHECH IF $\cancel{\lim_{x \to 0} f(x)}$ EXISTS

WE CAN CHECK JUST

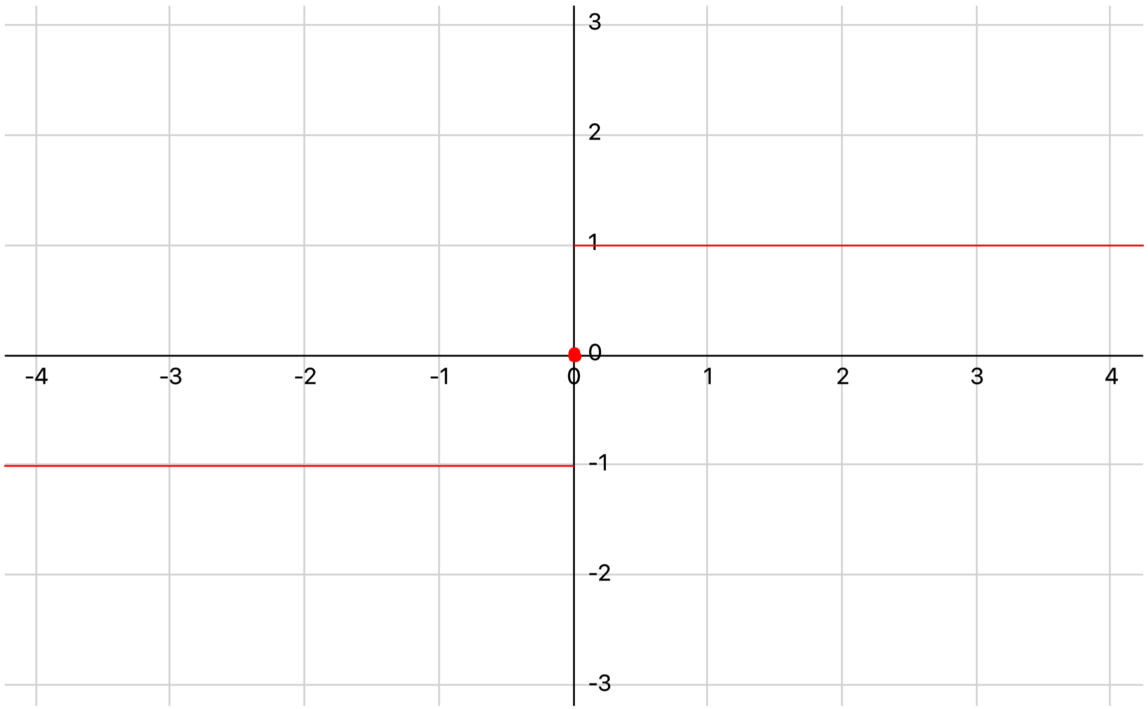
$$\lim_{x \to 0^+} f(x) = 1$$

$$\lim_{x \to 0^-} f(x) = -\infty$$

EXAMPLE

$$f(x) = \mathrm{sgn}(x)$$

$$\mathrm{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases} \overset{DEF}{\iff} \begin{cases} x/|x| & x \neq 0 \\ 0 & x = 0 \end{cases}$$



$$\lim_{x \to 0^+} f = 1$$

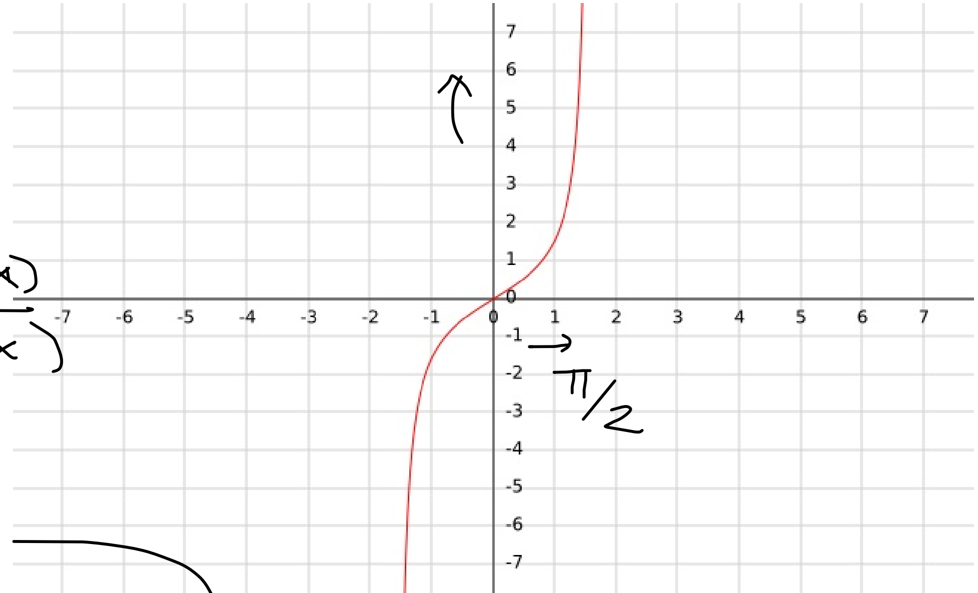
$$\neq$$

$$\lim_{x \to 0^-} f = -1$$

EXAMPLE

$f(x) = \tan(x) = \dfrac{\sin(x)}{\cos(x)}$

$tg(x)$

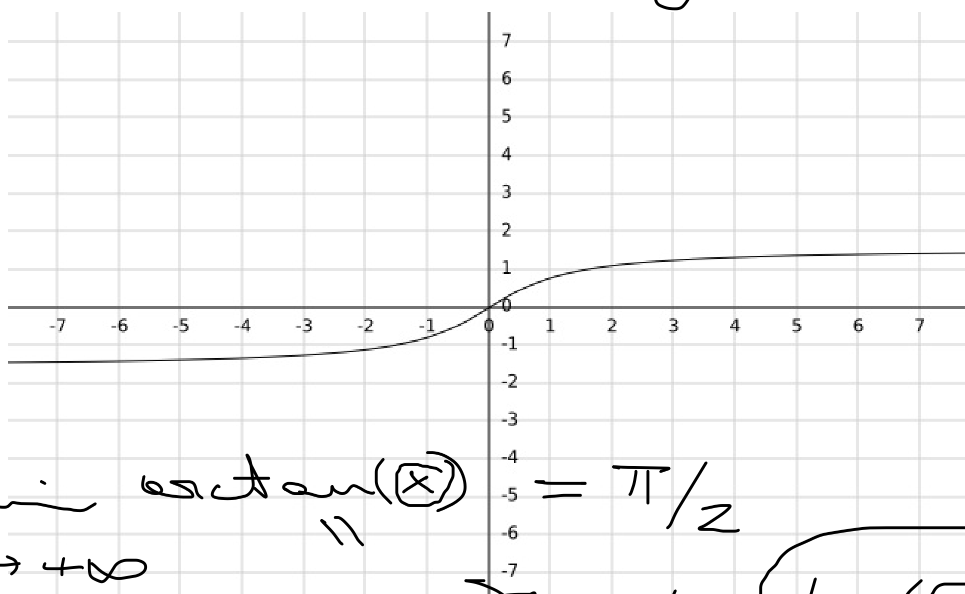


$f : \underline{\underline{(-\pi/2, \pi/2)}} \longrightarrow \mathbb{R}$

BIJECTIVE

C^0

$\lim_{x \to \pi/2^-} \tan x = \dfrac{\sin x}{\cos(x)}$



$f^{-1}(x) = \arctan x$

$\operatorname{arctg} x$

$f^{-1} : \mathbb{R} \longrightarrow (-\pi/2, \pi/2)$

$\lim_{x \to +\infty} \arctan(x) = \pi/2$

$y \in \cancel{\mathbb{R}}\ (-\pi/2, \pi/2)$ s.t. $tg(y) = x$

$$\lim_{x \to \pi/2^-} \tan x = \lim_{x \to \pi/2^-} \frac{\overset{\to 1}{\sin x}}{\underset{\to 0^{+}}{\cos x}} = +\infty$$

$(-\pi/2, 0)$

cos is $\underline{\underline{> 0}}$ on $(0, \pi/2)$

$$\lim_{x \to -\pi/2^+} \tan x = \text{- - -} \frac{\overset{\to -1}{\sin x}}{\underset{\to 0^{+}}{\cos x}} = -\infty$$

DEF $f : E \to \mathbb{R}$, $x_0 \in E$

ASSUME THAT

f DEFINED ON THE RIGHT OF x_0

$\left([x_0, x_0+\delta) \subseteq E \right)$
$\exists \delta > 0$

THEN f IS CONTINUOUS AT x_0 FROM THE RIGHT WHEN

$$\lim_{x \to x_0^+} f(x) = f(x_0)$$

DEF $f: E \to \mathbb{R}$, $x_0 \in E$

ASSUME THAT

f DEFINED ON THE LEFT OF x_0

$\left((x_0 - \delta, x_0] \subseteq E \right)$

$\exists\, \delta > 0$

THEN f IS CONTINUOUS AT x_0 FROM THE LEFT WHEN

$$\lim_{x \to x_0^-} f(x) = f(x_0)$$

PROP $f: E \longrightarrow \mathbb{R}$ $\quad x_0 \in E$

f is C^0 AT x_0

$\Updownarrow$

f IS C^0 AT x_0 FROM BOTH THE LEFT & RIGHT

$$\lim_{x \to x_0^+} f(x) = f(x_0) = \lim_{x \to x_0^-} f(x) \iff \lim_{x \to x_0} f(x) = f(x_0)$$

MONOTONE FCTS & LIMITS

(z_n) INCREASING:

- IF BOUNDED $\Longrightarrow$ $z_n \xrightarrow{n\to\infty} \sup\{z_n \mid n \in \mathbb{N}\}$
- IF NOT BOUNDED $\Longrightarrow$ $z_n \xrightarrow{n\to\infty} +\infty$

WHAT ABOUT MONOTONE FUNCTIONS?

PROP $f : E \longrightarrow \mathbb{R}$, $E =]a, b[$

f is MONOTONE — increasing: $\forall\ x \leq y \in E \Longrightarrow f(x) \leq f(y)$ — decreasing

① if $x_0 \in E$: $\lim_{x \to x_0^+} f(x) = l_1 \in \mathbb{R}$

$\lim_{x \to x_0^-} f(x) = l_2 \in \mathbb{R}$

f INCREASING / DECREASING

$l_1 = \inf_{(x_0, b)} f(x)$ $\qquad$ $l_1 = \sup_{x \in (x_0, b)} f(x)$

$l_2 = \sup_{(a, x_0)} f(x)$ $\qquad$ $l_2 = \inf_{x \in (a, x_0)} f(x)$

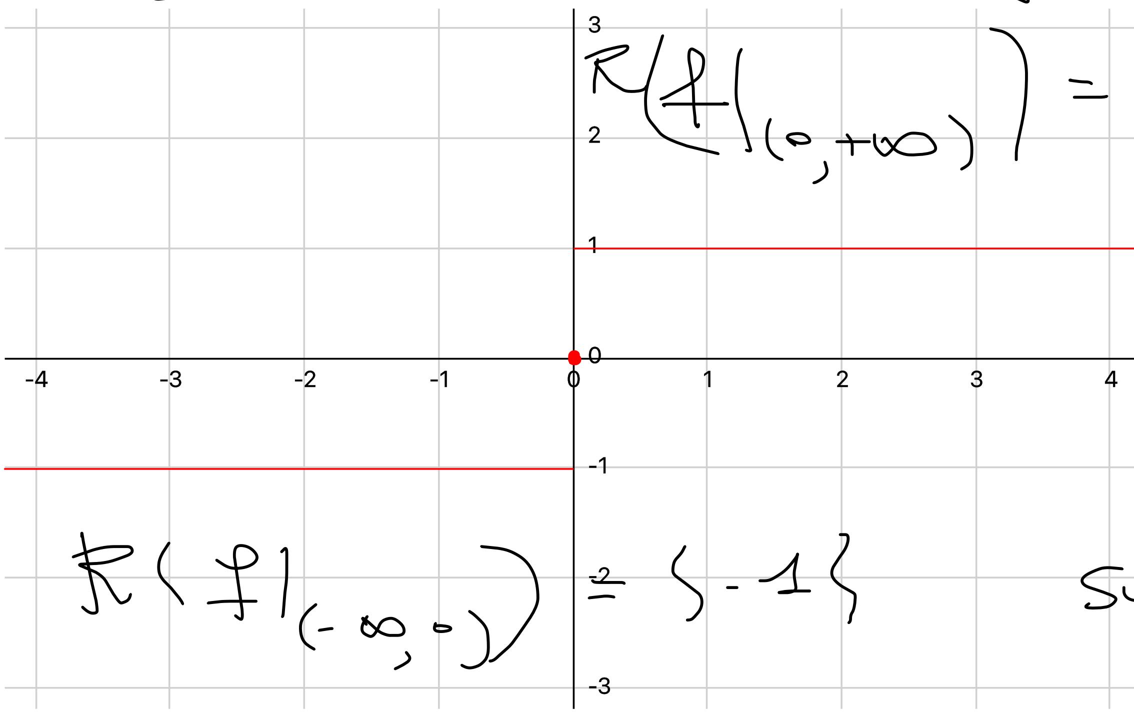
EXAMPLE $\quad f(x) = \operatorname{sgn}(x)$

$$\operatorname{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

increasing (but not strictly)

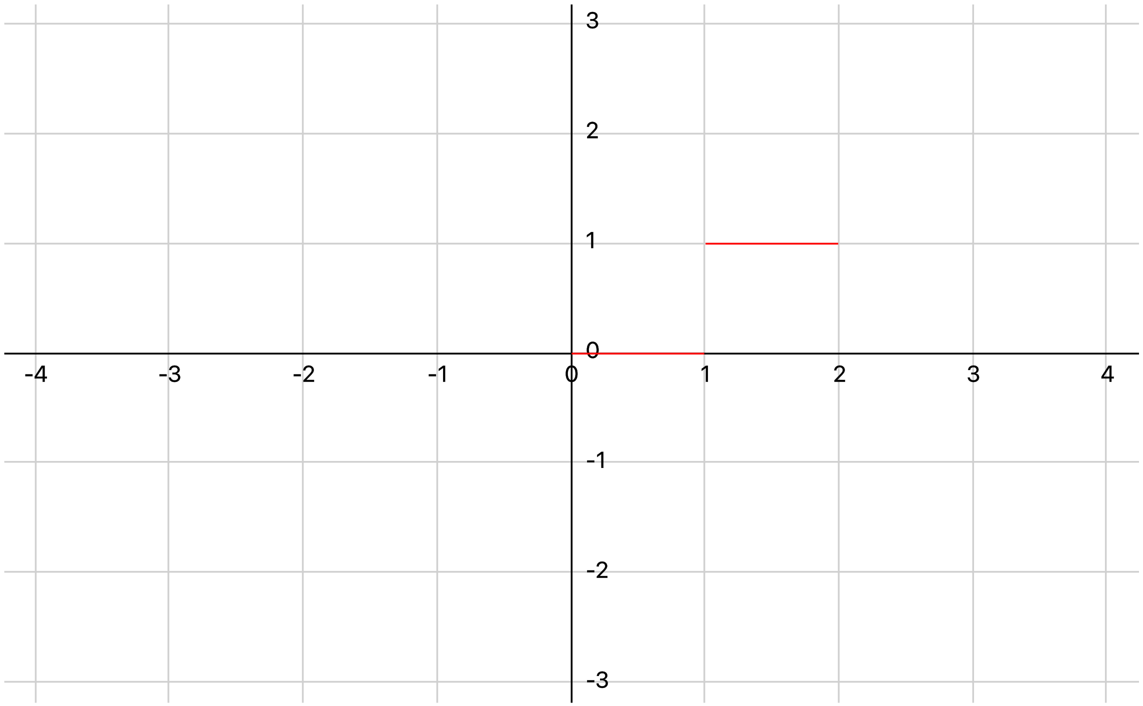
$$\inf R\left(f|_{(0,+\infty)}\right) = \lim_{x \to 0^+} f = 1$$

$$R\left(f|_{(0,+\infty)}\right) = \{1\}$$



$$R\left(f|_{(-\infty,0)}\right) = \{-1\}$$

$$\lim_{x \to 0^-} f = -1 = \sup\left(R\left(f|_{(-\infty,0)}\right)\right)$$



$$f(x) = [x]$$

$$f :]0,2[\to \mathbb{R}$$

increasing
(but not strictly)

$$\lim_{x \to 1^+} f(x) = \inf R\left(f|_{(1,2)}\right) = \{1\} = 1$$

$$\lim_{x \to 1^-} f(x) = \sup \underbrace{R\left(f|_{(0,1)}\right)}_{\{0\}} = 0$$

$f : [a,b] \longrightarrow \mathbb{R}$

Q: HOW DO WE DEFINE CONTINUITY AT a, b?

C^0 in (a,b) as usual + C^0 from the LHS at b, RHS at a

Q: WHAT CAN WE SAY ABOUT $R(f)$? IS IT AN INTERVAL? IS IT CLOSED?

CONTINUITY ON CLOSED INTERVALS

CLOSED INTERVAL: $a, b \in \mathbb{R}$

BOUNDED

$[a, b]$ $(a \neq b)$

$[a, b]$ CONTAINS ~~PUNCT.~~ NBD $= (-\delta + x, x + \delta)$

$a \neq b$ $\forall x \in [a, b]$ $x \neq a, b$ $\exists \delta > 0$

$[a, a + \delta')$ $\exists \delta' > 0$

$\subseteq [a, b]$

$(b - \delta'', b]$ $\exists \delta'' > 0$

$f : [a,b] \longrightarrow \mathbb{R}$

f IS DEF'D ON THE RIGHT OF a

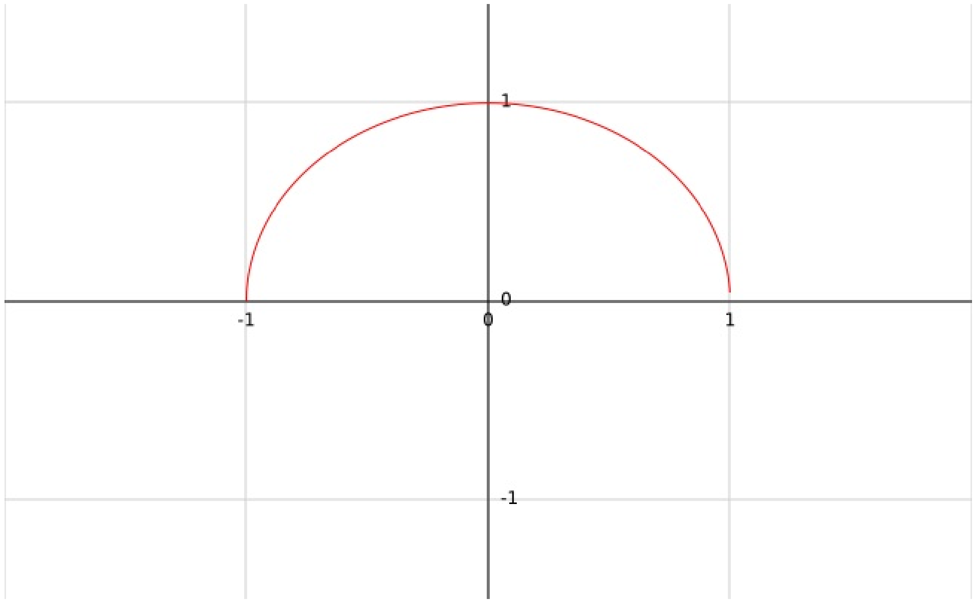
f IS DEF'D ON THE LEFT OF b

DEF f IS C^0 ON $[a,b]$ WHEN

① f IS C^0 AT x, $\forall x \in]a,b[$ &

② f IS C^0 FROM THE RIGHT AT a

③ f IS C^0 FROM THE LEFT AT b



$$f(x)=\sqrt{1-x^2}$$

$f:[-1,1]\to\mathbb{R}$

is C^0 on $[-1,1]$

FOR $x\in(0,1) \implies f$ is C^0 at x

$f(x)=(g\circ h)(x)$ $\quad g\in C^0(\mathbb{R}_+^*)$

$g(t)=\sqrt{t}$ $\quad g:\mathbb{R}_+\longrightarrow\mathbb{R}$ $\quad g$ is C^0 FROM RHS into

$h(x)=1-x^2\in C^0(\mathbb{R})\supseteq C^0((0,1))$

$f(x) = (g \circ h)(x)$ $\quad g \in C^0(\mathbb{R}_+^*)$

$g(t) = \sqrt{t} \quad g : \mathbb{R}_+ \longrightarrow \mathbb{R}$ $\quad g$ is C^0 FROM RHS into

$h(x) = 1 - x^2 \in C^0(\mathbb{R}) \supseteq C^0((\cancel{0}, 1))$

$h\big|_{(0,1)} : (\cancel{0}, 1) \longrightarrow (\cancel{0}, 1)$

$1 > 1 - x^2 > 0$ for $x \in (0, 1) \overset{\text{CONT}}{\Longrightarrow} \overset{f}{\overset{\shortparallel}{(g \circ h)}}$ & COMP IS C^0 on $(0,1)$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \sqrt{1 - x^2}$$

$1 - x_n^2 \longrightarrow 0^+$

$x_n \longrightarrow 0, \ x_n \geq 0 \ \forall n \Longrightarrow \sqrt{1 - x_n^2} \longrightarrow \overset{f(1)}{\overset{\shortparallel}{0^+}}$

THM $f:[a,b]\longrightarrow \mathbb{R}$

CONTINUOUS THEN

$\min_{x\in[a,b]} f(x) = m$ & $\max_{x\in[a,b]} f(x) = M$

EXIST!

$f:[a,b] \longrightarrow R(f) \subseteq [m,M]$

$\underset{m,\,M}{\overset{\ni}{}}$

THM $f:[a,b] \longrightarrow \mathbb{R}$

CONTINUOUS THEN

$\max_{x\in[a,b]} f(x)$ EXIST & $\min_{x\in[a,b]} f(x)$ EXISTS

$\exists\, c \in [a,b]$
S.T. $f(c) \geq f(x)$
$\forall x \in [a,b]$

$\exists\, d \in [a,b]$ S.T.
$f(d) \leq f(x)$
$\forall x \in [a,b]$

EXAMPLE

IF $D(f)$ IS NOT A $\boxed{\text{CLOSED + BOUNDED}}$ INTERVAL

$\Longrightarrow$ THM NOT TRUE !!

CLOSED:

$$f : (0,1) \longrightarrow \mathbb{R}$$

$$f(x) = \frac{1}{x} \longrightarrow \underline{R(f) = (1, +\infty)}$$

BOUNDED:

$$f : [1, +\infty) \longrightarrow \mathbb{R}$$

$$f(x) = x$$

$R(f) = [1, +\infty)$ NOT BOUNDED

PROOF

STEP 1: $R(f)$ IS BOUNDED

SUPPOSE NOT, FOR EXAMPLE NOT BOUNDED FROM ABOVE.

$\Rightarrow \forall n \in \mathbb{N}, \exists \; x_n \in [a,b]$ S.T.

$f(x_n) \geq n$

PROOF STEP 1: $R(f)$ IS BOUNDED

SUPPOSE NOT, FOR EXAMPLE NOT BOUNDED FROM ABOVE.

$\Rightarrow \forall n \in \mathbb{N}, \exists\, x_n \in [a,b]$ S.T. $f(x_n) \geq n$

$\Rightarrow (x_n)$ BOUNDED SEQ. $(\subseteq [a,b])$

BW

$\Rightarrow \exists\, x_{n_k} \xrightarrow{k\to\infty} x \in [a,b]$

CO

$\Rightarrow f(x_{n_k}) \geq n_k \xrightarrow{k\to\infty} f(x)$

$R(f)$ BOUNDED $\Longrightarrow$ SUP $R(f)$ EXIST
INF $R(f)$

$R(f)$ BOUNDED $\Longrightarrow$ $\sup R(f)$, $\inf R(f)$ EXIST

STEP 2: MAX $R(f)$ EXISTS

SUPPOSE NOT $\Longrightarrow$ $s = \sup R(f) \notin R(f)$

$\Longrightarrow$ $\forall \underline{\underline{n}} \in \mathbb{N}$, $\exists f(x_n) \in R(f)$ S.T.

$$s - \frac{1}{n} \leq f(x_n) < s$$

$(x_n) \in [a, b]$

BOUNDED

$\Downarrow$ BW

$\exists x_{n_k} \longrightarrow x \in [a, b]$

$\Longrightarrow f(x_n) \xrightarrow{n \to \infty} s$

$f(x_{n_k}) \longrightarrow s$

$\overset{C^0}{\Longrightarrow} f(x_{n_k}) \longrightarrow f(x)$

$\Longrightarrow s \in R(f)$ ↯

$R(f)$ BOUNDED $\Longrightarrow$ $\sup R(f)$, $\inf R(f)$ EXIST

STEP 2: MAX $R(f)$ EXISTS

SUPPOSE NOT $\Longrightarrow$ $S = \sup R(f) \notin R(f)$

$\Longrightarrow$ $\forall n \in \mathbb{N}$, $\exists f(x_n) \in R(f)$ S.T.

$$S - \frac{1}{n} < f(x_n) < S$$

$\Longrightarrow$ $f(x_n) \xrightarrow{n \to \infty} S$. $(x_n) \subseteq [a, b]$ BOUNDED.

$\Longrightarrow$ $\exists x_{n_k} \to x$ $\overset{C^0}{\Longrightarrow}$ $f(x_{n_k}) \xrightarrow{k \to \infty} f(x)$

JUST PROVED: $f:[a,b]\to\mathbb{R}$ C^0

$R(f)$ HAS MAX & MIN

$$\Updownarrow$$

$R(f)\subseteq[c,d]$ &

$c, d\in R(f)$

CAN WE SAY MORE ?

[ABOUT $R(f)$]

UNIFORM CONTINUITY

DEF $f : E^{\subseteq \mathbb{R}} \longrightarrow \mathbb{R}$ IS

UNIFORMLY CONTINUOUS

IF $\forall \varepsilon > 0$, $\exists \delta_\varepsilon > 0$ S.T.

$\forall x, y \in E$, $|x-y| \underset{\leq}{<} \delta_\varepsilon \Longrightarrow$

$$|f(x) - f(y)| \underset{\leq}{<} \varepsilon$$

UNIF. CONT.

IF $\forall \varepsilon > 0$, $\exists \delta_\varepsilon > 0$ S.T.

$\delta_\varepsilon = \delta_\varepsilon(\varepsilon)$

$\forall x, y \in E$, $|x-y| < \delta_\varepsilon \Rightarrow |f(x)-f(y)| < \varepsilon$

VS

CONTINUITY AT x_0

$\forall \varepsilon > 0$, $\exists \delta_\varepsilon > 0$ S.T.

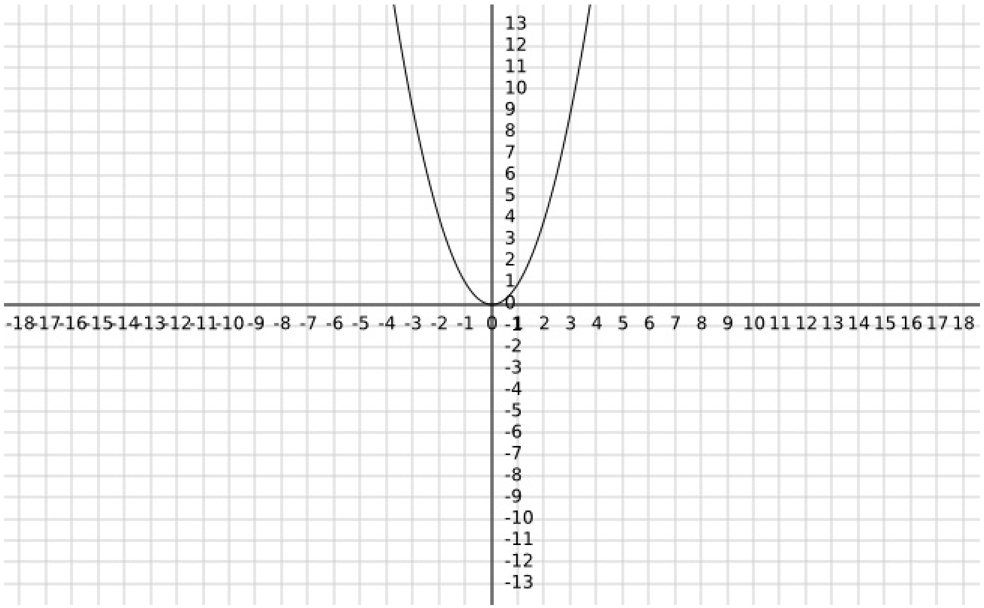
$\delta_\varepsilon = \delta_\varepsilon(\varepsilon, x_0)$

$\forall x \in E$, $|x - x_0| < \delta_\varepsilon \Rightarrow |f(x)-f(x_0)| < \varepsilon$

$f: E \to \mathbb{R}$ IS UNIF. CONT.

IF $\forall \varepsilon > 0$, WE CAN CHOOSE $\delta_\varepsilon > 0$ IN DEF'N OF CONTINUITY TO BE THE SAME AT EACH PT OF E

$f : \mathbb{R} \longrightarrow \mathbb{R}$



$f(x) = x^2$

TAKE $\varepsilon = 1$

$x_0 \in \mathbb{R}$

δ_1 at x_0 in DEF'N OF C^0

$$|x - x_0| \leq \delta_1 \implies |x^2 - x_0^2| \leq 1$$

$$|x^2 - x_0^2| = \underbrace{|x + x_0|}_{\geq 0}\,\underbrace{|x - x_0|}_{\leq \delta_1}$$

$\delta_1 < 1$ FOR $x_0 \gg 0$

$$\left|(x_0+1)^2 - x_0^2\right|$$

$$\shortparallel$$

$$\left|x_0^2 + 1 + 2x_0 - x_0^2\right|$$

$$\shortparallel$$

$$\left|1 + 2x_0\right|$$

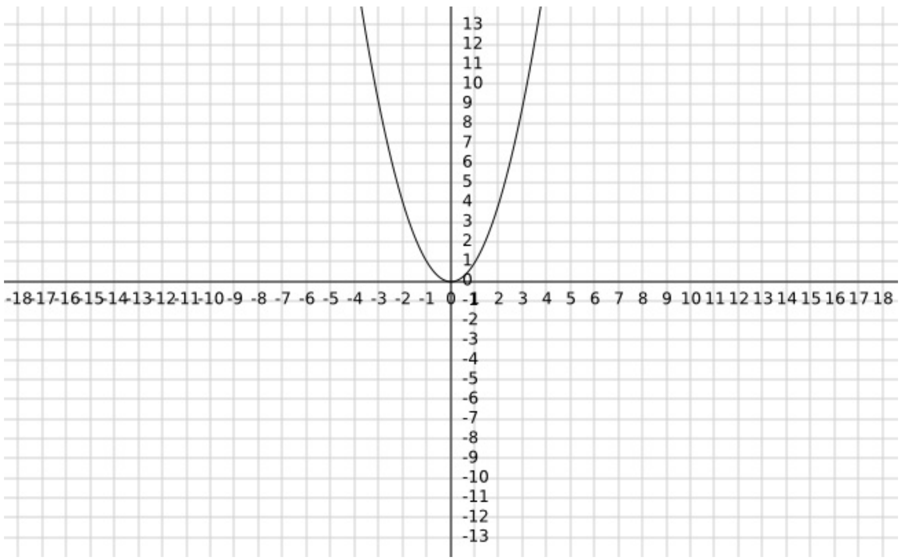
$$\shortparallel$$

$$1 + 2x_0 > \overset{\varepsilon}{\overset{\shortparallel}{1}} \quad \text{FOR } x_0 \gg 1$$

$$\delta_1 = \frac{1}{n}$$

$$\left|\left(x_0 + \frac{1}{n}\right)^2 - x_0^2\right| = \left|\cancel{x_0^2} + \frac{1}{n^2} + 2\cdot\frac{x_0}{n} - \cancel{x_0^2}\right|$$

$$= \frac{1}{n^2} + 2\frac{x_0}{n} \overset{x_0 > n}{>} 2$$



$f(x) = x^2$

TAKE $\varepsilon = 1$

$x_0 \in \mathbb{R}$

δ_1 AT x_0?

$$|x - x_0| < \delta_1 \Rightarrow |x^2 - x_0^2| < 1$$

$$\| \quad |x + x_0| \cdot |x - x_0|$$

EXAMPLE

$$f(x) = \cos(x)$$

EXAMPLE

$$f(x) = \cos(x)$$

FIX $\varepsilon > 0$.

NEED TO FIND $\delta_\varepsilon > 0$ S.T.

$\forall x, y \in \mathbb{R}$, $|x-y| < \delta_\varepsilon$

$\implies |\cos(x) - \cos(y)| < \varepsilon$

THM $f : [a,b] \to \mathbb{R}$ C^0

THEN f is UNIFORMLY CONTINUOUS

$f(x) = x^2$ $\quad f|_{[a,b]} : [a,b] \to \mathbb{R}$