ANALYSIS 1
(ENGLISH)
LECTURE 15

PROP $f, g : E \longrightarrow \mathbb{R}$, $x_0 \in E$

$(x_0 - \delta, x_0 + \delta), \exists \delta > 0$

ASSUME f, g ARE CONTINUOUS AT x_0

THEN

$[(\alpha f + \beta g)(x) \doteq \alpha \cdot f(x) + \beta \cdot g(x)]$ "CONTINUOUS"

① $\forall \alpha, \beta \in \mathbb{R}$ $\alpha f + \beta g$ IS C^0 AT x_0

② $f \cdot g$ IS C^0 AT x_0 $[(f \cdot g)(x) \doteq f(x) \cdot g(x)]$

③ IF $g(x) \neq 0$ $\forall x \in E$ $\Rightarrow$ $\dfrac{f}{g}$ IS C^0 AT x_0

$(\forall x \in (x_0 - \delta, x_0 + \delta))$
$\delta > 0$

EXAMPLE $\quad f(x) = \dfrac{\cos(x) - 1}{x}$

$D(f) = \mathbb{R} \setminus \{0\}$

$\exists \tilde{f} : \mathbb{R} \to \mathbb{R}$ WHICH EXTENDS f ? IN A CONTINOUOS WAY

$$\tilde{f}(x) \doteq \begin{cases} f(x) & x \neq 0 \\ a \doteq \lim_{x \to 0} f(x) & x = 0 \end{cases}$$

$\underset{0}{\shortparallel}$

DOES THE LIMIT EXIST ?

NEED TO COMPUTE : $\lim_{x \to 0} \dfrac{\cos(x) - 1}{x}$

$$\lim_{x\to 0} \frac{\cos(x)-1}{x} = \text{"}\left[\frac{0}{0}\right]\text{"}$$

$$1 - \cos^2(x) = \sin^2(x) \qquad (\cancel{x})$$

$$\shortparallel$$

$$(1-\cos(x))\underbrace{(1+\cos(x))}_{\substack{\downarrow\, x\to 0 \\ 2}}$$

$$\lim_{x\to 0} \frac{\overset{-(\sin x)^2}{\cancel{\cos^2(x)-1}}}{x} \cdot \frac{-1}{1+\cos(x)}$$

$$\lim_{x\to 0} \underbrace{\frac{\sin x}{x}}_{\to 1} \cdot \underbrace{\sin x}_{\to 0} \cdot \underbrace{\frac{-1}{1+\cos(x)}}_{\to -\frac{1}{2}} = 0$$

COMPOSITION & CONTINUITY

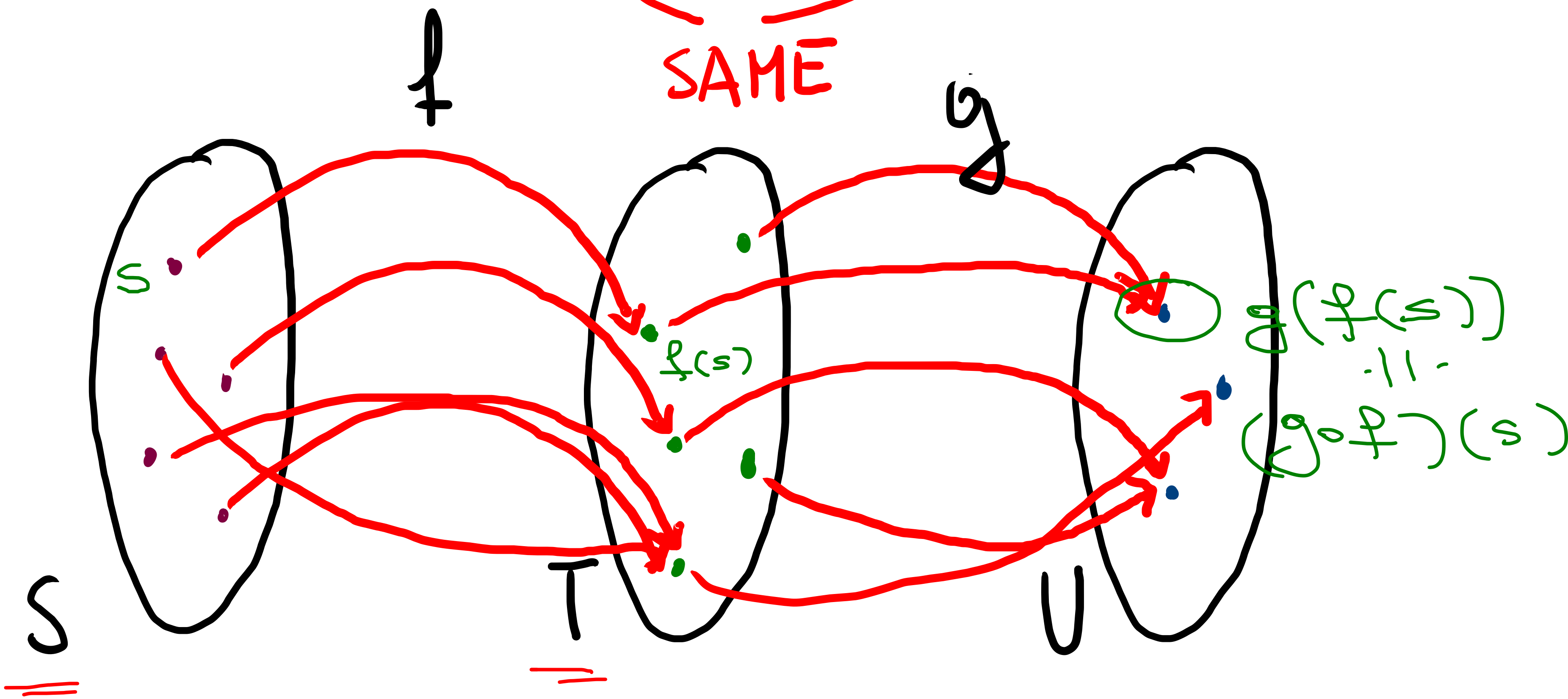
$$f : S \longrightarrow T \qquad g : T \longrightarrow U$$

SAME

COMPOSITION & CONTINUITY

$f : S \to \underline{\underline{T}} \qquad g : \underline{\underline{T}} \to U$

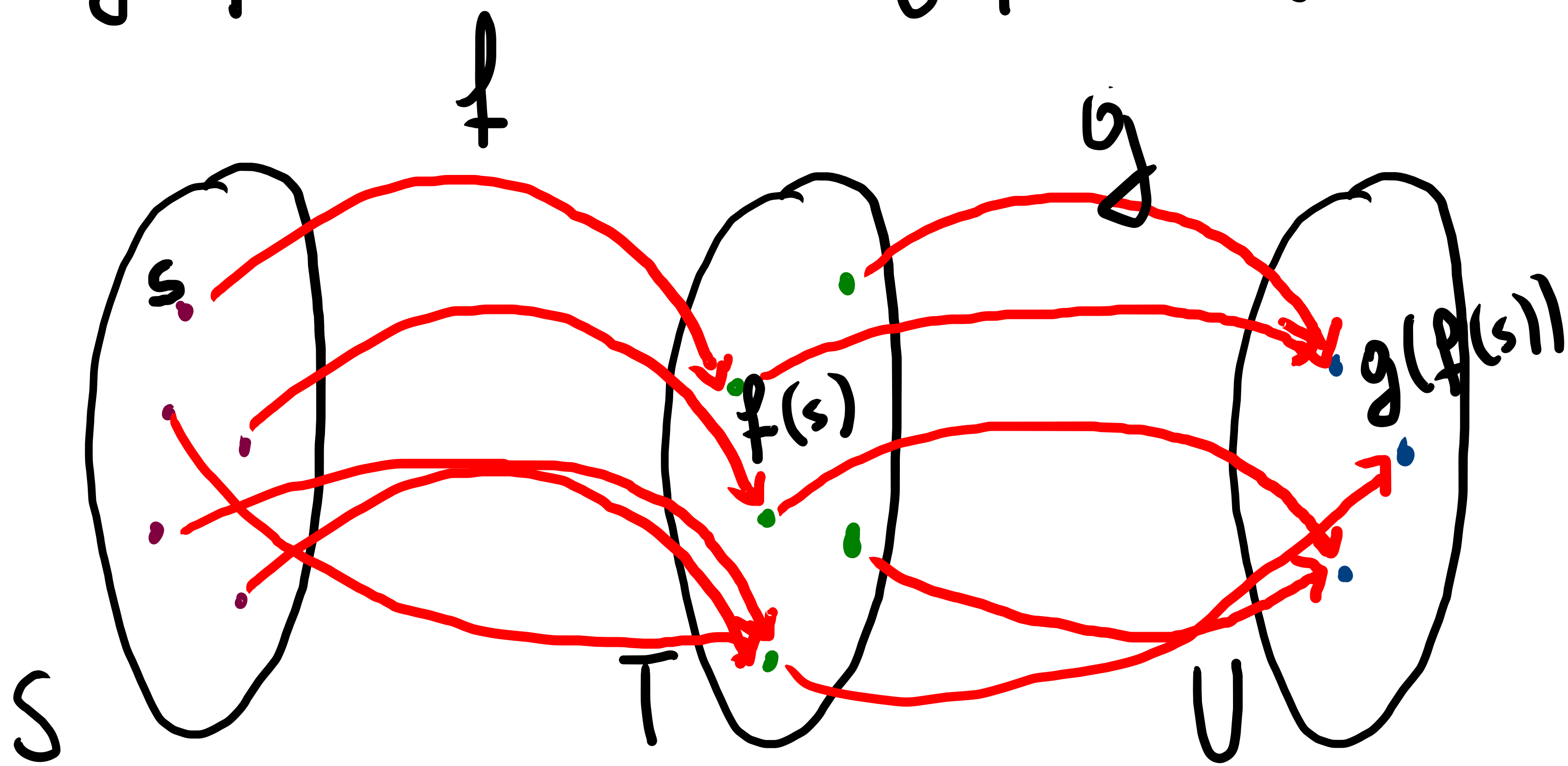
SAME



COMPOSITION & CONTINUITY

$f : S \longrightarrow T$ $\quad$ $g : T \longrightarrow U$

$g \circ f : S \longrightarrow U$ $\quad$ $g \circ f(s) \doteq g(f(s))$



$f : S \longrightarrow T$ $\qquad$ $g : V \longrightarrow U$

$T \supseteq V$

IS $g \circ f$ DEFINED ?

$f : S \to T \qquad g : V \to U$

$T \supseteq V$

IS $g \circ f$ DEFINED ?

IF $R(f) \subseteq V \Rightarrow \forall s \in S$

$f(s) \in D(g)$

$\Downarrow$

THIS IS THE RIGHT CONDITION

$g \circ f(s) := g(f(s))$ IS WELL DEFINED.

EXAMPLE

$$f, g : \mathbb{R} \longrightarrow \mathbb{R}$$

$$j(y) = y^2 + 1$$

$$h(t) = \log(t)$$

$$h \circ j(y)$$

$$f(x) = x^2 + 6x + 1$$

$$g(y) = \sqrt[3]{y} + \log\left(\underline{y^2 + 1}\right)$$

$$\in (\cancel{0}\,1, +\infty)$$

$$g \circ f(x) = \sqrt[3]{x^2 + 6x + 1} +$$

$$+ \log\left((x^2 + 6x + 1)^2 + 1\right)$$

EXAMPLE

$f : \mathbb{R} \longrightarrow \mathbb{R}$

$g : D(g) \longrightarrow \mathbb{R}$

$D(g) = [0, +\infty)$

$D(\sqrt{y}) \cap D(\log(y+1))$

$[0, +\infty) \quad (-1, +\infty)$

$f(x) = x^2 + 6x + 1$

$g(y) = \sqrt[2]{y} + \log(y^{\cancel{2}} + 1)$

$x^2 + 6x + 1 = x^2 + 2 \cdot 3 \cdot x + 1$

$f(x) = (x+3)^2 - 8$

$f(0) = 0$

$x > y \geq 0$

$\shortparallel$

0

$(x+3) > (y+3) > 0$

$\Downarrow$

$(x+3)^2 > (y+3)^2$

$\Downarrow$

$(x+3)^2 - 8 > (y+3)^2 - 8$

$f(x) = (x+3)^2 - 8 \geqslant (0+3)^2 - 8 = 1 > 0$

ON $\underline{\underline{R(f|_{\mathbb{R}_+})}} \subseteq (0, +\infty)$ ◡̈

$R(f|_{\mathbb{R}_-})$

$x = -2$

$f(-2) = (-2+3)^2 - 8$

$= 1 - 8 = \underline{\underline{-7}}$

$R(f|_{\mathbb{R}_-}) \not\subseteq [0, +\infty)$

$\Rightarrow$ WE CANNOT DEFINE $g \circ f$

BUT WE CAN DEFINE $g \circ (f|_{\mathbb{R}_+})$

EXAMPLE

$$f : \mathbb{R} \longrightarrow \mathbb{R}_+$$

$$g : \mathrm{Dom}(g) \longrightarrow \mathbb{R}$$

$$D(g)$$

$$\{y \geq 0\}$$

$$f(x) = x^2 + 2x + 1$$

$$g(y) = \sqrt[2]{y} + \log\left(y^{\cancel{2}} + 1\right)$$

$$g \circ f(x) = \sqrt{x^2 + 2x + 1} + \log\left(x^2 + 4x + 2\right)$$

$$R(f) = \{ s \in \mathbb{R} \mid s = x^2 + 2x + 1 = (x+1)^2 \geq 0 \} = [0, +\infty)$$

$R(f) = D(g) \Rightarrow g \circ f$ IS

WELL DEF'D

EVERYWHERE

ON $D(f) = \mathbb{R}$

CONTINUOUS FUNCTIONS:

- POLYNOMIALS
- sin, cos, tg, ...

CAVEAT: GOT TO PAY ATTENTION TO DOMAIN OF FUNCTION

- exp, log

$$\exp = e^x \qquad \log = \log(x) = \log_e(x)$$

- $\sqrt[n]{x}$

THM

$x_0 \in S$

$f : S \to \mathbb{R}$

$g : T \to \mathbb{R}$

$R(f) \subseteq D(g)$

ASSUMED

[THEN $g \circ f$ IS WELL DEF'D ON S]

ASSUME f IS C^0 AT x_0 AND

g IS C^0 AT $f(x_0) \in D(g)$

THM $f: S \to \mathbb{R}$ $\cdot$ $R(f) \subseteq Dom(g)$

$x_0 \in S$ $g: T \to \mathbb{R}$

ASSUMED

ASSUME f IS C^0 AT x_0 AND

g IS C^0 AT $f(x_0)$

$$\Downarrow$$

$g \circ f$ IS C^0 AT x_0

$g \circ f : S \to \mathbb{R}$

EXAMPLE

$f : \mathbb{R} \to \mathbb{R} \qquad f(x) = x^2 \in C^0(\mathbb{R})$

$C^0 \quad g : \mathbb{R} \to \mathbb{R} \qquad g(t) = \begin{cases} \frac{\sin t}{t} & t \neq 0 \\ 1 & t = 0 \end{cases}$

$1 = \lim_{x \to 0} \frac{\sin x}{x}$

$t = x^2$

$g \circ f(x) = \begin{cases} \sin(x^2)/x^2 & x \neq 0 \iff t \neq 0 \\ 1 & x = 0 \iff t = 0 \end{cases}$

IS IT C^0 AT $x = 0$? YES !!

BECAUSE $f \in C^0(\mathbb{R})$ ($\Rightarrow f$ is C^0 AT x_0) +

+ $f(0) = 0$ & g is C^0 AT $t = 0 = f(0)$

$$\lim_{x \to 0} \frac{\sin(x^2)}{x^2} = 1$$

$$g \circ f(x) = \begin{cases} \dfrac{\sin x^2}{x^2} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

$g \circ f$ IS C^0 AT $x = 0$

$$\Downarrow$$

$$\lim_{x \to 0} g \circ f(x) = g \circ f(0) = 1$$

EXAMPLE

$f : \mathbb{R} \to \mathbb{R} \quad f(x) = x^2 \quad C^0$

$C^0 \quad g : \mathbb{R} \to \mathbb{R} \quad g(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$

$$g \circ f(x) = \begin{cases} \sin(x^2)/x^2 & x \neq 0 \\ 1^2 & x = 0 \end{cases}$$

IS IT C^0 AT 0? YES!!

$$\Longrightarrow \lim_{x \to 0} \frac{\sin(x^2)}{x^2} =$$

$f : S \to \mathbb{R}$

$g : T \to \mathbb{R}$

$R(f) \subseteq D(g)$

$x_0 \in S$, $\Longleftrightarrow$ f is C^0 AT $x = 0$

$\wedge$

g is C^0 A $t = f(x_0)$

$\Longleftrightarrow$

$$\lim_{x \to x_0} f(x) = f(x_0)$$

$$\lim_{t \to f(x_0)} g(t) = g(f(x_0))$$

THEN IF $\lim_{x \to x_0} f(x) = \ell$

$$\lim_{t \to \ell} g(t) = s$$

$$\Longrightarrow \lim_{x \to x_0} g(f(x)) = s$$

IN GENERAL:

NOT TRUE THAT

IF WE DO NOT ASSUME C^0

$$\lim_{x \to x_0} f(x) = y_0$$

$$\lim_{y \to y_0} g(y) = l$$

$$\Longrightarrow \lim_{x \to x_0} g \circ f(x) = l$$

IN GENERAL: NOT TRUE THAT

$$\lim_{x \to x_0} f(x) = y_0$$

$$\lim_{y \to y_0} g(y) = \ell \implies \lim_{x \to x_0} g \circ f(x) = \ell$$

$$g(t) = \begin{cases} 1 & t = 0 \\ 0 & t \neq 0 \end{cases} \qquad f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

g is NOT C^0 AT $t = 0$

f is C^0 AT $x = 0$

$$\forall n \in \mathbb{N} \quad g \circ f\left(\frac{1}{n\pi}\right) = g\left(\frac{1}{n\pi} \cdot \underbrace{\sin(n\pi)}_{=0}\right) = g(0) = \underline{\underline{1}}$$

$$\forall n \in \mathbb{N} \quad g \circ f\left(1/(\pi/2 + 2n\pi)\right) = g\left(\frac{1}{\pi/2 + 2n\pi} \cdot \underset{1}{\cancel{\sin(\pi/2 + 2n\pi)}}\right) = \underline{\underline{0}}$$

$$x_n = \frac{1}{n\pi} \longrightarrow 0$$

$$y_n = \frac{1}{\pi/2 + 2n\pi} \longrightarrow 0$$

$$g \circ f(x_n) = 1 \quad \forall n$$

$$g \circ f(y_n) = 0 \quad \forall n$$

$$\Longrightarrow \lim_{x \to 0} g \circ f \quad \text{DOES NOT EXIST.}$$

$$g \circ f(s) := g\big(f(s)\big)$$

$$g(t) = \begin{cases} 1 & t = 0 \\ 0 & t \neq 0 \end{cases}$$

IF $f(s) = 0 \implies g(f(s)) = 1$

IF $f(s) \neq 0 \implies g(f(s)) = 0$

PROP $f, g : E \longrightarrow \mathbb{R}$, C^0 AT $x_0 \in E$.

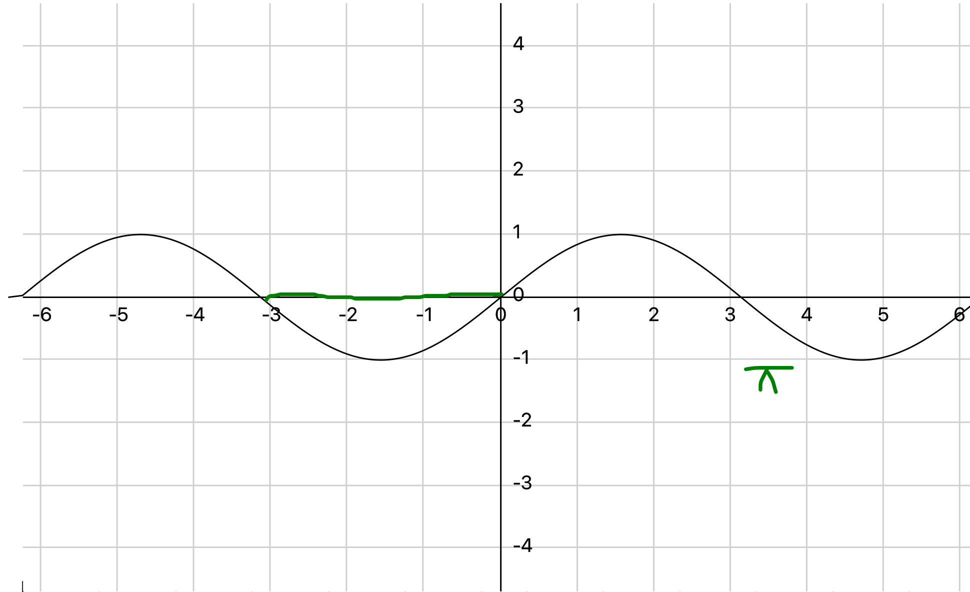
THEN THE FOLLOWING FUNCTIONS ARE C^0 AT x_0

1. $|f|$
2. $\max\{f, g\}$
3. $\min\{f, g\}$
4. $f^+ \doteq \max\{f, 0\}$
5. $f^- \doteq \min\{f, 0\}$

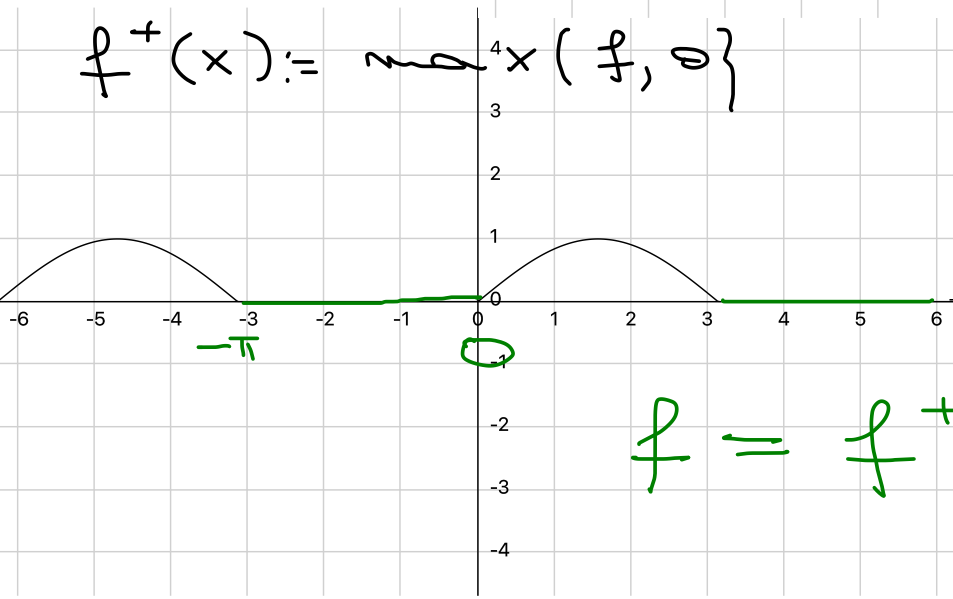
$$\max\{f, g\}(x) := \max(f(x), g(x))$$

$$\min[f, g)(x) := \min(f(x), g(x)]$$

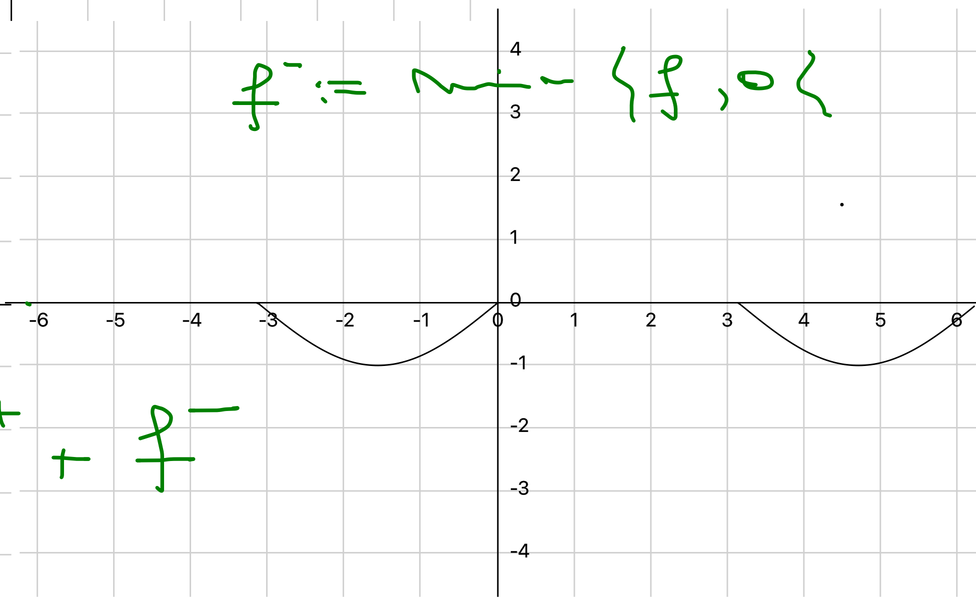
$f(x) = \sin x$



$f^+(x) := \max\{f, 0\}$



$f^- := \min\{f, 0\}$



$f = f^+ + f^-$

NEIGHBORHOODS AT ∞

$x_0 \in \mathbb{R}$

~~PUNCTURED~~ NEIGHBORHOOD:

$\delta > 0$, $\left]-\delta + x_0 , x_0 + \delta\right[\cancel{\setminus \{x_0\}}$

NEIGHBORHOOD OF $+\infty$:

$(a, +\infty) \qquad a \in \mathbb{R}$

NEIGHBORHOOD OF $-\infty$:

$(-\infty, b) \qquad b \in \mathbb{R}$

DEF $f: E \to \mathbb{R}$, $x_0 \in \mathbb{R}$

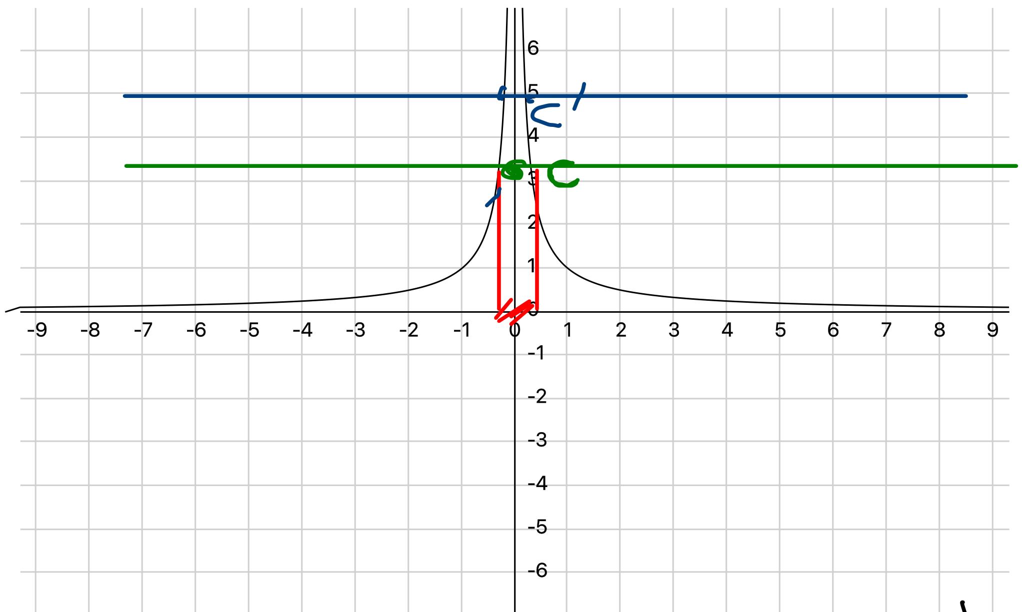
ASSUME E CONTAINS PUNCTURED NEIGHBORHOOD OF x_0

$\lim_{x \to x_0} f(x) = \begin{matrix} +\infty \\ -\infty \end{matrix}$ IF

$\forall\, c \in \mathbb{R}, \exists\, \delta_c > 0$ S.T.

IF $0 < |x - x_0| < \delta_c \implies f(x) \begin{matrix} \geq \\ \leq \end{matrix} c$

$$f(x) = \left|\frac{1}{x}\right|$$



$$\lim_{x \to 0} f(x) = +\infty$$

FIX $C \in \mathbb{R}$

IF $C \le 0$:

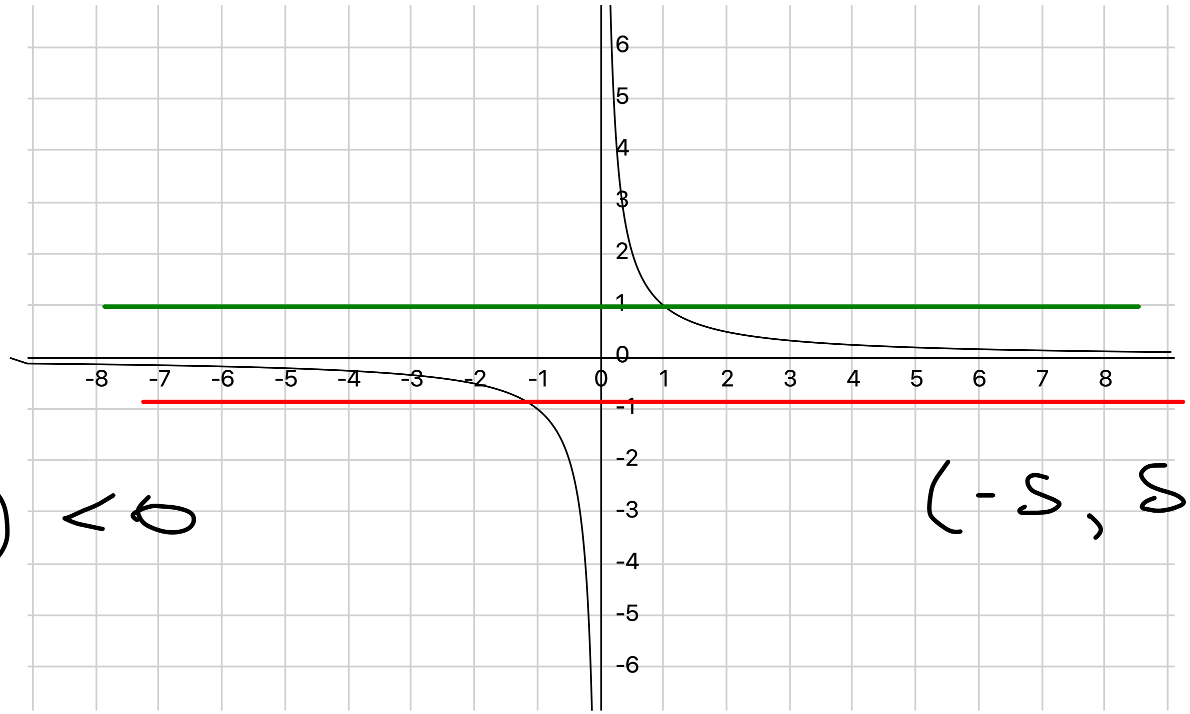
$f(x) > 0 \quad \forall x \in \mathbb{R}^*$

ANY VALUE OF $\delta_c > 0$ WILL DO

IF $C > 0$ $\quad f(x) \ge C \iff \frac{1}{|x|} \ge C \iff |x| \le \frac{1}{C}$

IF $\delta_c \doteq \frac{1}{c}$ THEN $\forall\, 0 < |x - 0| \le \frac{1}{c} \doteq \delta_c \Rightarrow f(x) \ge C$

$f(x) = \frac{1}{x}$



$f|_{\mathbb{R}_-}(x) < 0$

$(-\delta, \delta) \setminus \{0\}$

DOES $\lim_{x \to 0} f(x)$ EXIST? [POLL]

1. $+\infty$
2. $-\infty$
3. 0
4. NONE OF THE ABOVE

THM $f: E \longrightarrow \mathbb{R}$, $E \subseteq \mathbb{R}$, $x_0 \in \mathbb{R}$

ASSUME E CONTAINS A POINTED NEIGHB. OF x_0. TFAE

1. $\lim_{x \to x_0} f(x) = \begin{matrix} +\infty \\ -\infty \end{matrix}$

2. FOR ALL $\boxed{(x_n) \subseteq E \setminus \{x_0\}}$

IF $x_n \xrightarrow{n\to\infty} x_0 \Longrightarrow f(x_n) \xrightarrow{n\to\infty} \begin{matrix} +\infty \\ -\infty \end{matrix}$

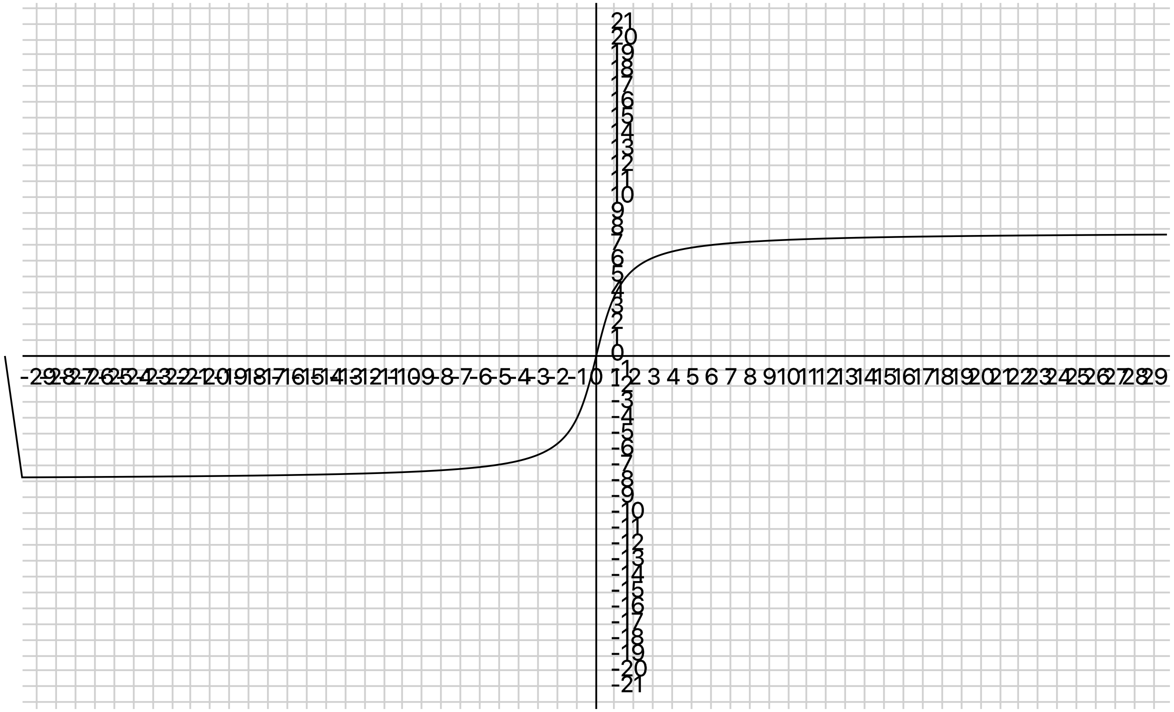
SEQUENCE

DEF $f: E \to \mathbb{R}$, $x_0 \in \mathbb{R}$

ASSUME E CONTAINS NEIGHBORHOOD OF $+\infty$

$$\lim_{x \to +\infty} f(x) = \ell$$

$\forall \varepsilon > 0$,



$f(x) = 5\arctan(x)$

THM $f : E \longrightarrow \mathbb{R}$, $E \subseteq \mathbb{R}$

ASSUME E CONTAINS A NEIGHBORHOOD OF $+\infty$ / $-\infty$

$[E \supseteq (a, +\infty)]$ / $(-\infty, b)$. TFAE

① $\lim_{x \to +\infty / -\infty} f(x) = \ell \in \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$

② FOR ALL $\boxed{(x_n) \subseteq E}$

IF $x_n \xrightarrow{n \to \infty} +\infty$ / $-\infty$ $\Longrightarrow f(x_n) \xrightarrow{n \to \infty} \ell$

SEQUENCE

ALGEBRA & LIMITS AT ∞

PROP $f, g : E \to \mathbb{R}$, $x_0 \in \mathbb{R}$

ASSUME E CONTAINS A PUNCTURED NEIGHBORHOOD S OF x_0 $\left[g(x) \geq C, \forall x \in S\right]$, $C \in \mathbb{R}$

ASSUME g IS BOUNDED FROM BELOW IN S

$*$ ① IF $\lim\limits_{x \to x_0} f(x) = +\infty \implies \lim\limits_{x \to x_0} f(x) + g(x) = +\infty$

② IF $\lim\limits_{x \to x_0} f(x) = +\infty$, $g(x) \geq \underline{\underline{A}} > 0$ $\forall x \in S$

$\lim\limits_{x \to x_0} f(x) \cdot g(x) = +\infty$

③ If $g(x)$ bounded on S

(THAT IS, $\{g(x) \mid x \in S\} \subseteq \mathbb{R}$ is BOUNDED)

& $f(x) \neq 0 \quad \forall x \in S \Rightarrow$

$$\lim_{x \to x_0} g(x)/f(x) = 0$$

EXAMPLE

$f(x) = \left|\frac{1}{x}\right| : \mathbb{R} \to \mathbb{R}$

$$g(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

g IS NOWHERE C^0

g IS BOUNDED

$$\lim_{x \to 0} f(x) + g(x) = +\infty$$

$$f \cdot g \begin{cases} \frac{1}{|x|} & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} / \mathbb{Q} \end{cases}$$

$$\lim_{x \to 0} f(x) = +\infty$$

$\lim_{x \to 0} f(x) \cdot g(x)$ DOES NOT EXIST

SQUEEZE AT ∞

PROP $f, g : E \longrightarrow \mathbb{R}$ $\quad x_0 \in \mathbb{R}$

ASSUME E CONTAINS A PUNCT. NEIGHBORHOOD S OF x_0.

ASSUME $f(x) \leq g(x) \quad \forall x \in S$.

1. IF $\lim_{x \to x_0} f(x) = +\infty \implies \lim_{x \to x_0} g(x) = +\infty$
2. IF $\lim_{x \to x_0} g(x) = -\infty \implies \lim_{x \to x_0} f(x) = -\infty$.