ANALYSIS 1
(ENGLISH)
LECTURE 13

TODAY

1. MORE BASIC PROPERTIES
2. LIMITS
3. CONTINUITY

BASIC PROPERTIES

$f : S \longrightarrow T$ FUNCTION

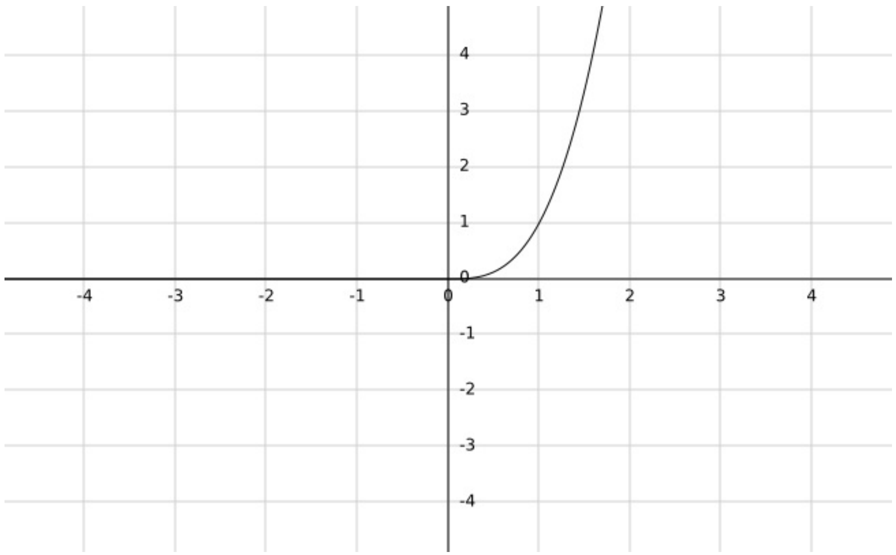
- **INJECTIVE** : $\underset{S}{\overset{\cap}{s_1}} \neq \underset{S}{\overset{\cap}{s_2}} \Rightarrow f(s_1) \neq f(s_2)$

- **SURJECTIVE** : $\forall t \in T, \exists s \in S$ s.t. $f(s) = t$

$R(f) = T \iff$

- **BIJECTIVE** : INJECT. + SURJECT.

$\forall t \in T, \exists! \, s \in S$ s.t. $f(s) = t$



$$f : \underbrace{\mathbb{R}_+}_{D(f)} \longrightarrow \underbrace{\mathbb{R}_+}_{R(f)}$$

$$f(x) = x^3$$

Bij. $\forall y \in R(f)$

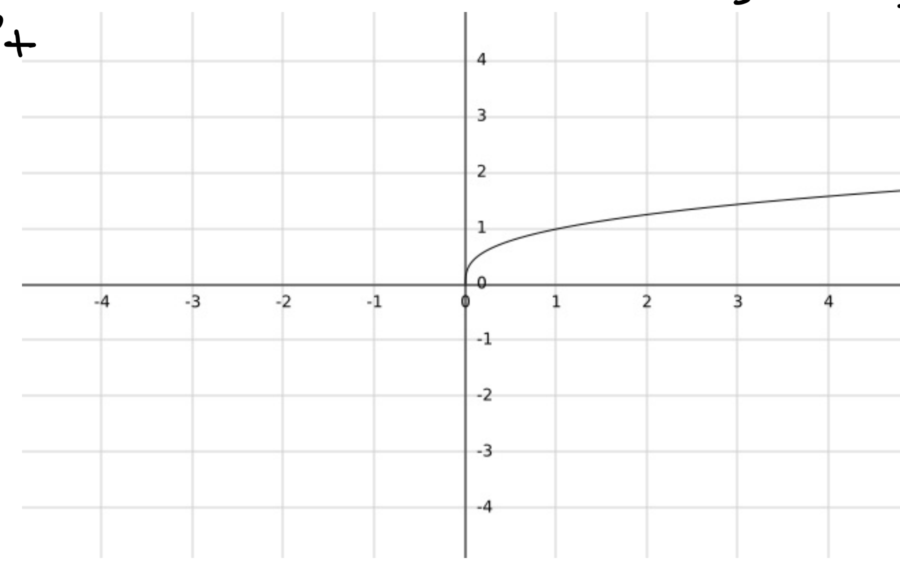
f^{-1}

$\exists!\, x \in D(f)$

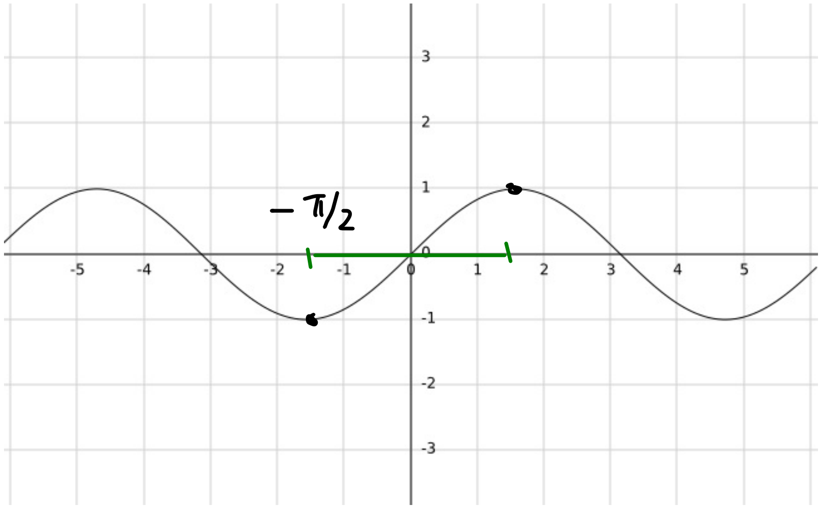
s.t. $f(x) = y$

$$\mathbb{R}_+ = R(f) \qquad D(f) = \mathbb{R}_+ = R(f^{-1})$$

$$f^{-1} : R(f) \to R(f^{-1})$$



$$f^{-1}(y) = \sqrt[3]{y}$$



$f(x) = \sin(x)$

$\overset{\sin}{\cancel{\cos}}(x + 2K\pi) = \overset{\sin}{\cancel{\cos}}(x)$

$\forall x \in \mathbb{R}, \ \forall K \in \mathbb{Z}$

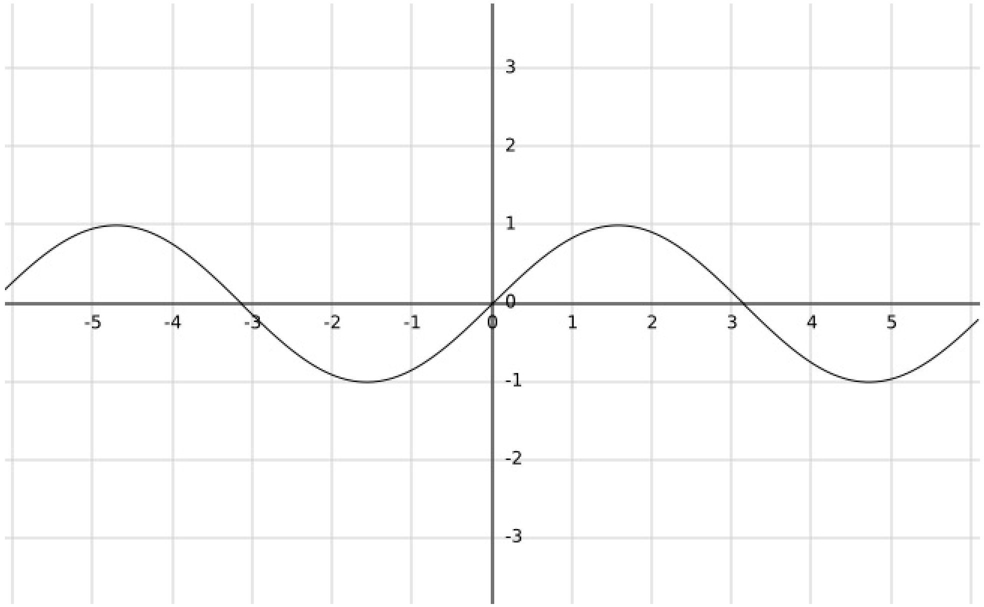
$\overset{\sin}{\cancel{\cos}}(x)$ is NOT INJ.

NOT BIJ.

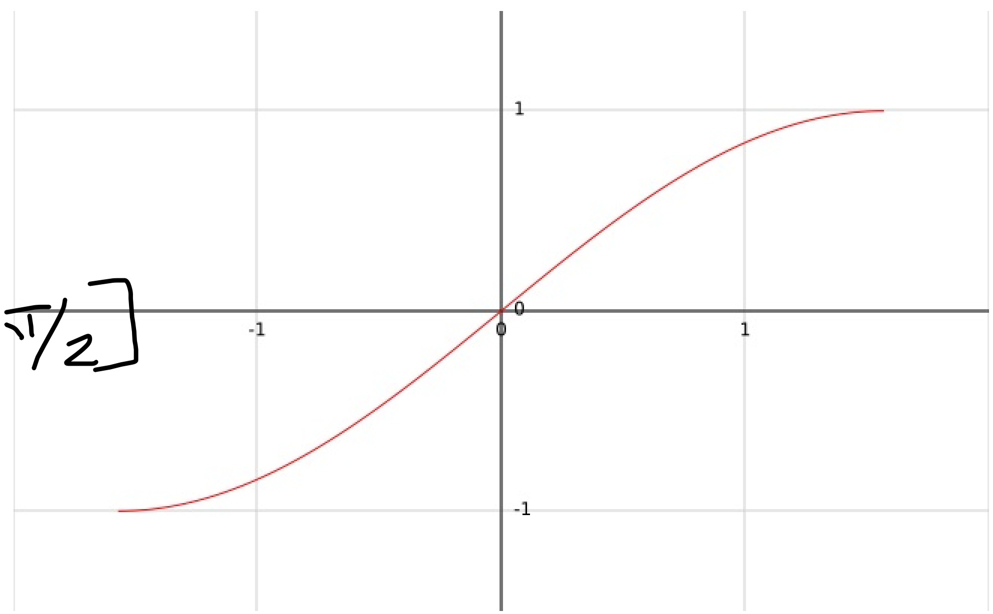
$$g = f\big|_{[-\frac{\pi}{2}, \frac{\pi}{2}]} : [-\pi/2, \pi/2] \longrightarrow \overset{\cancel{\mathbb{R}}}{\underbrace{R(f)}} = [-1, 1]$$

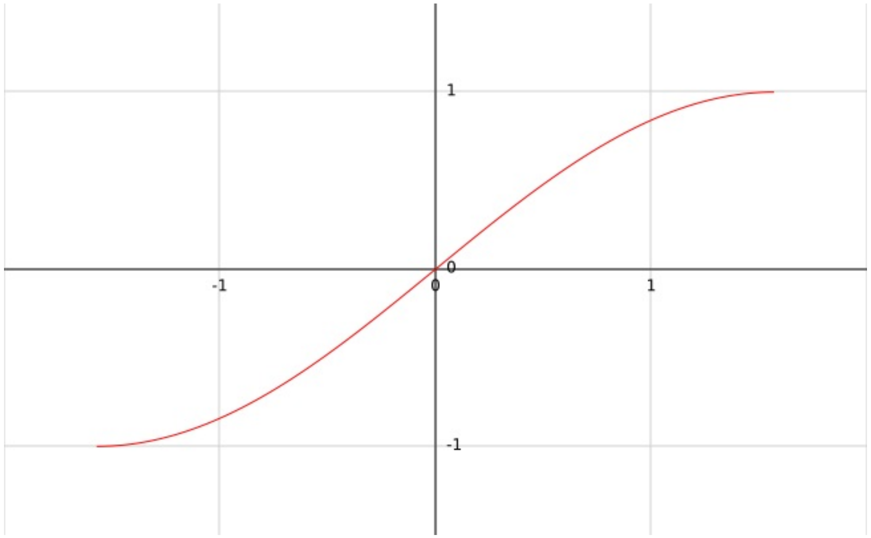
INJECTIVE, SURJECTIVE $\Longrightarrow$ BIJECTIVE

$$\exists \, g^{-1} : [-1, 1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$



$$g = \cos(x)\Big|_{[-\pi/2,\, \pi/2]}$$

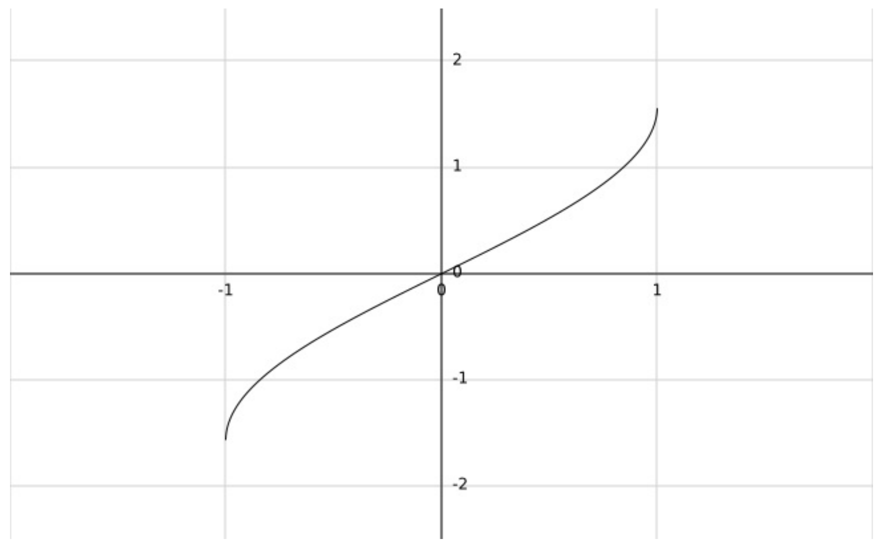




$$g = \cancel{\cos}\ \sin(x)\Big|_{[-\pi/2,\ \pi/2]}$$

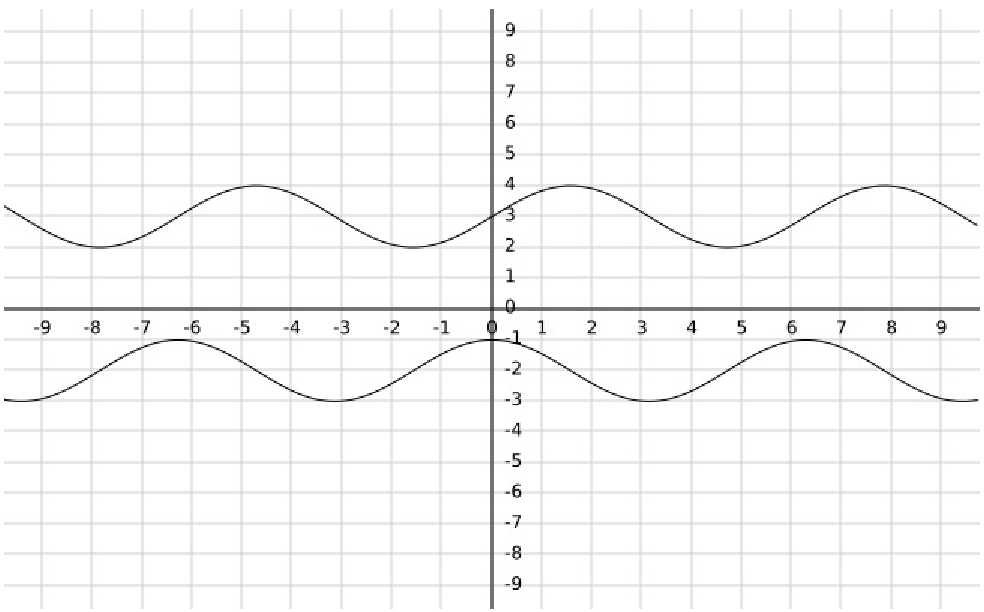
$$g^{-1}(x) = \arcsin(x)$$

BIJ.



$f, g : S \longrightarrow \mathbb{R}$

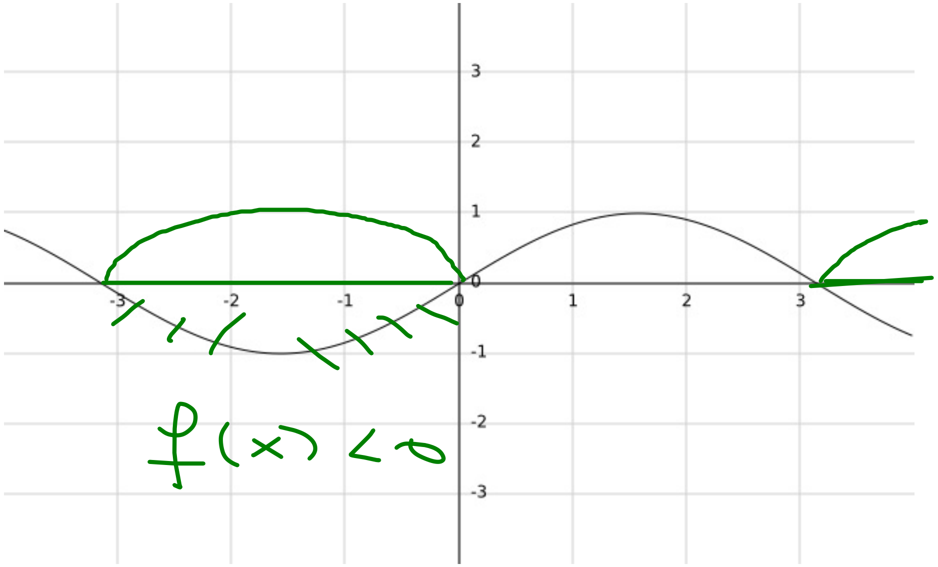
$f \geq g$ DEF: $f(s) \geq g(s)$
$\forall s \in S$



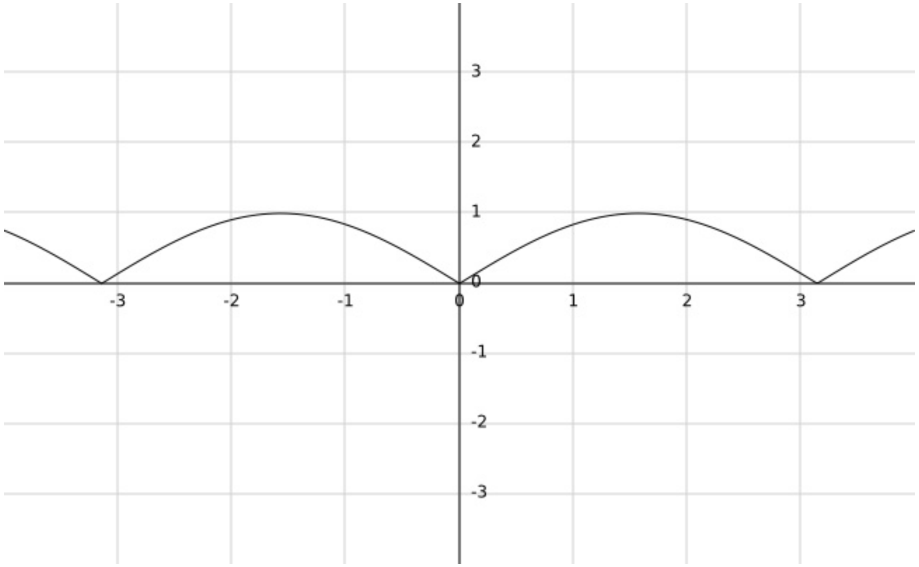
$$f : S \longrightarrow \mathbb{R}$$

$$|f| : S \longrightarrow \mathbb{R}$$

$$|f|(s) \doteq |f(s)| \qquad \forall s \in S$$



$$f(x) = \sin(x)$$



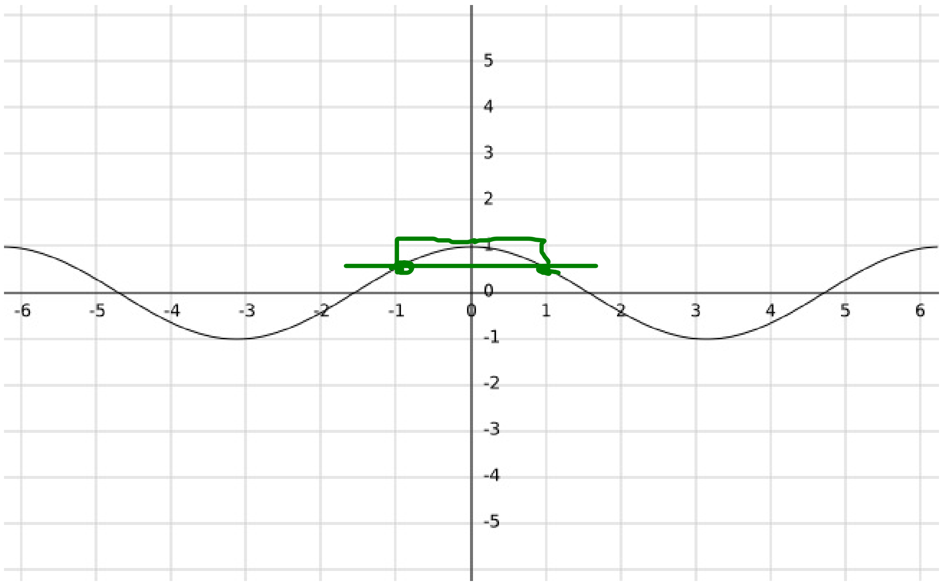
$$|f|(x) = |\sin(x)|$$

$$f : E \longrightarrow \mathbb{R}$$

$E \subseteq \mathbb{R}$

REQUIRES: if $x \in E \Rightarrow -x \in E$

- EVEN: $f(x) = f(-x)$



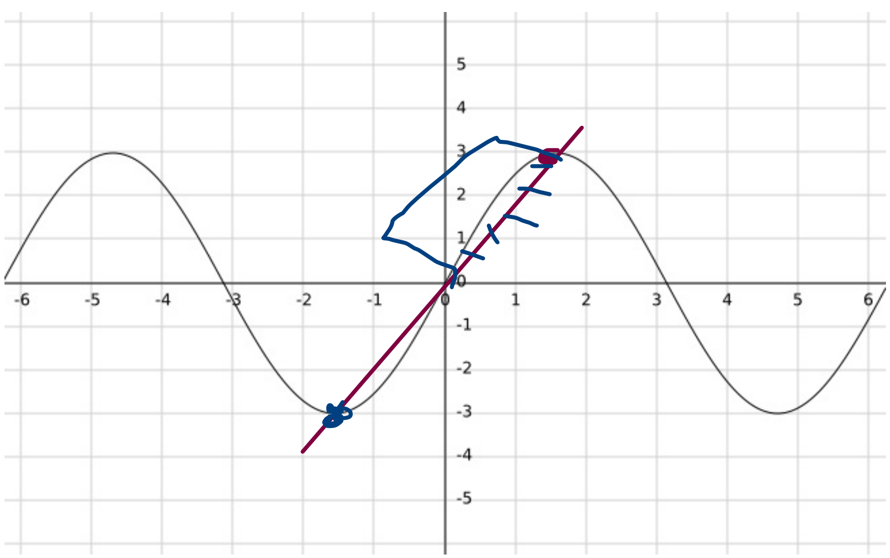
$$f(x) = \cos(x)$$

$$\cos(x) = \cos(-x)$$

GRAPH IS SYMMETRIC W.R.T. Y AXIS

$$f : \underset{\mathbb{R}}{\underset{\cap}{E}} \longrightarrow \mathbb{R}$$

- EVEN : $f(x) = f(-x)$
- ODD : $f(x) = -f(-x)$



$f(x) = 3 \sin(x)$

$\sin(x) = -\sin(-x)$

ODD

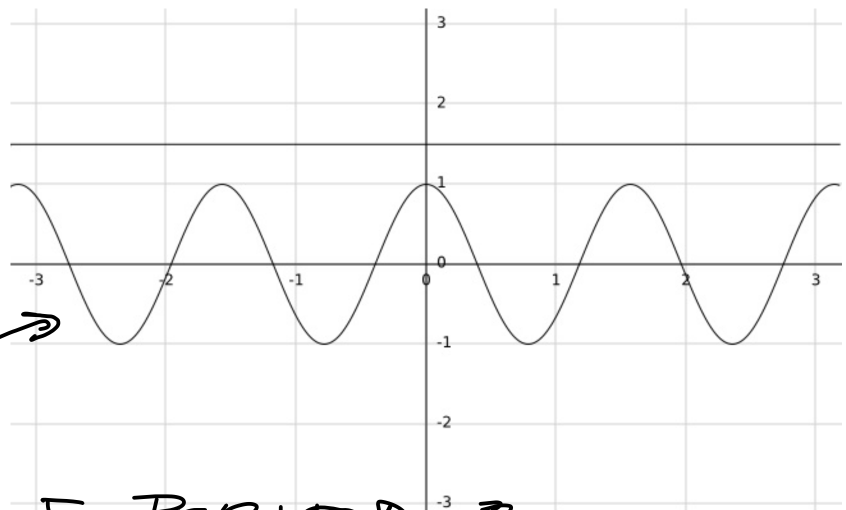
graph (f) is SYMM. THRU THE ORIGIN

$f : \mathbb{R} \longrightarrow \mathbb{R}$ PERIODIC

$f(x) = f(x + kC) \quad \forall k \in \mathbb{Z}$ PERIOD $= C \in \mathbb{R}$

$\Updownarrow$

$$f(x+C) = f(x) \quad \forall x \in \mathbb{R}$$



$f(x) = \frac{1}{2}$

CONST. FCT.

PERIODIC OF PERIOD $= C \ \forall C \in \mathbb{R}$

$g(x) = \cos(x)$

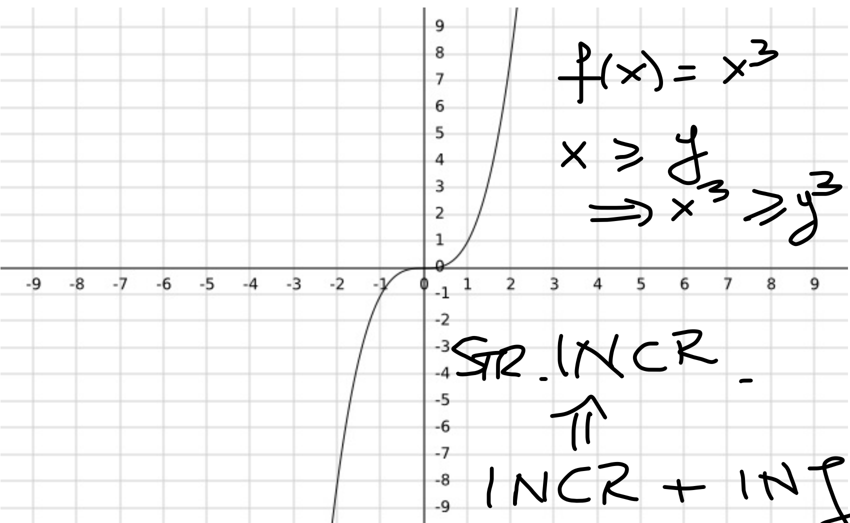
PERIODIC OF PERIOD $= 2\pi$

$$f : E \longrightarrow \mathbb{R}$$

$$E \subseteq \mathbb{R}$$

• STR. INCREASING : IF $\forall s_1, s_2 \in E$

$$s_1 \underset{>}{\geq} s_2 \Longrightarrow f(s_1) \underset{>}{\geq} f(s_2)$$



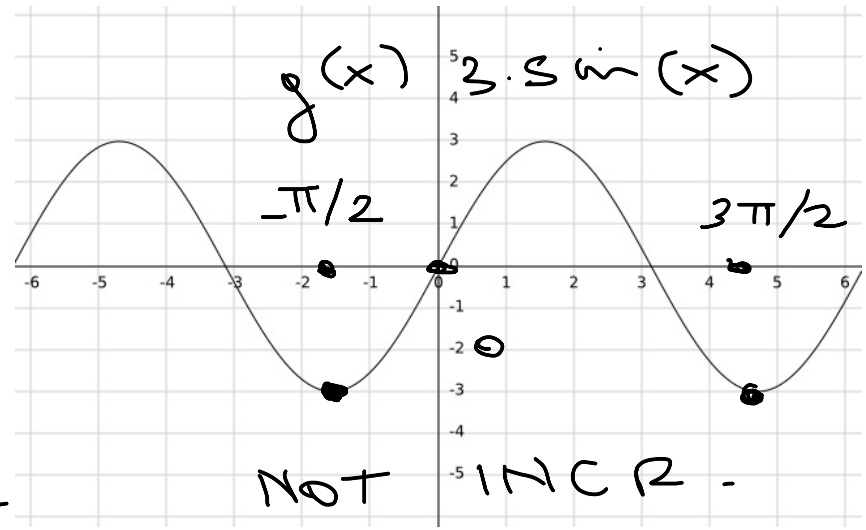
$f(x) = x^3$

$x \geq y$
$\Rightarrow x^3 \geq y^3$

STR. INCR.

$\Uparrow$

INCR + INJ.



$g(x) = 3 \cdot \sin(x)$

$-\pi/2$ $\quad$ $3\pi/2$

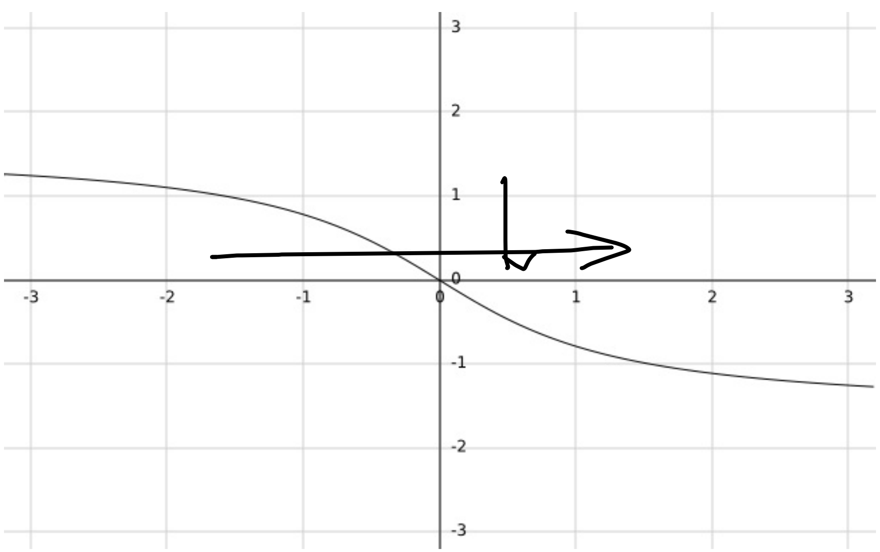
NOT INCR.

$$g(-\pi/2) = -1 \leq g(0) = 1 \geq \underset{\underset{-1}{\parallel}}{g(3\pi/2)}$$

$$f : E \longrightarrow \mathbb{R}$$

$$E \subseteq \mathbb{R}$$

- **INCREASING**: $s_1 \geq s_2 \Longrightarrow f(s_1) \geq f(s_2)$ $\quad \forall s_1, s_2 \in D(f)$
- **DECREASING**: $\underline{s_1 \geq s_2} \Longrightarrow \underline{f(s_1) \leq f(s_2)}$



$y = \arctan(x)$

f STR. INCR/DECR.

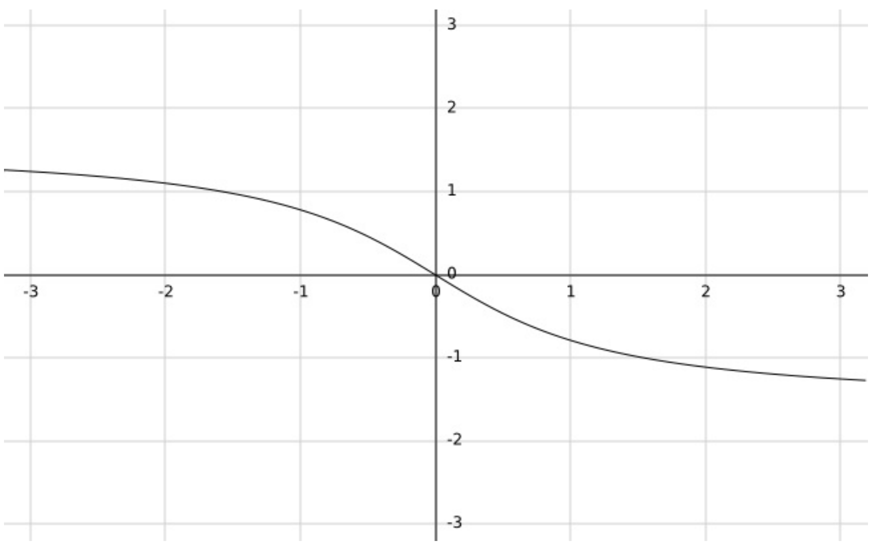
$\Downarrow$

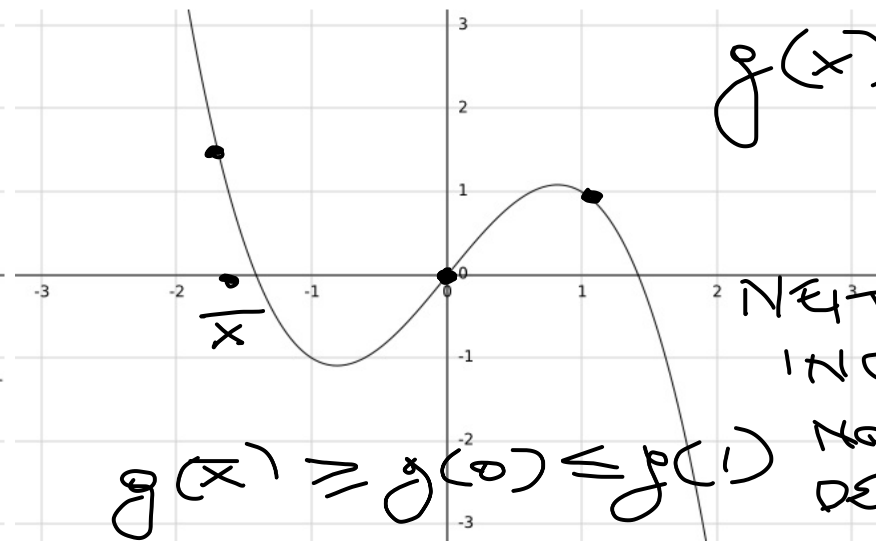
f injective

$$f : E \longrightarrow \mathbb{R}$$

$$E \subseteq \mathbb{R}$$

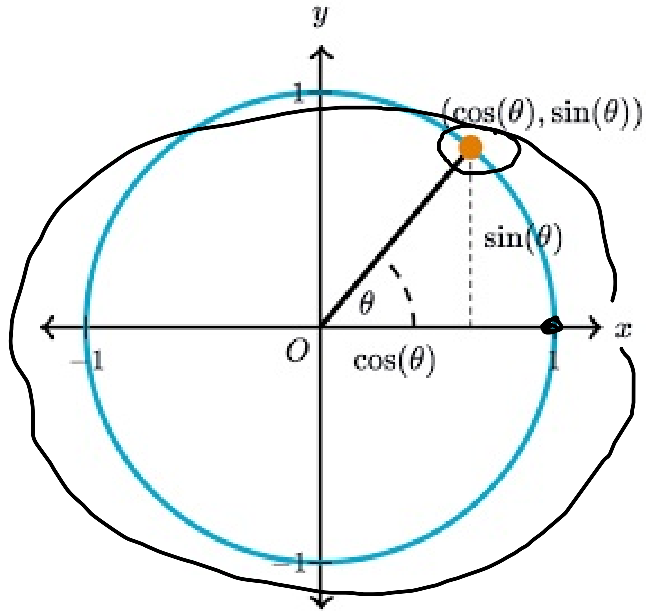
- **INCREASING**: $s_1 \geq s_2 \implies f(s_1) \geq f(s_2)$
- **DECREASING**: $s_1 \geq s_2 \implies f(s_1) \leq f(s_2)$





$g(\bar{x}) \geq g(0) \leq g(1)$

NEITHER INCR. NOR DECR.



$\cos(\theta)$ = x-coordinate of pt on S^1 given by θ-rad. angle

$\sin(\theta)$ = y-coordinate of pt on S^1 given by θ-rad angle

PERIODIC of PERIOD 2π

$\tan(\theta) = \dfrac{\sin\theta}{\cos\theta}$ $\quad$ $\cotan(\theta) = \dfrac{1}{\tan}$

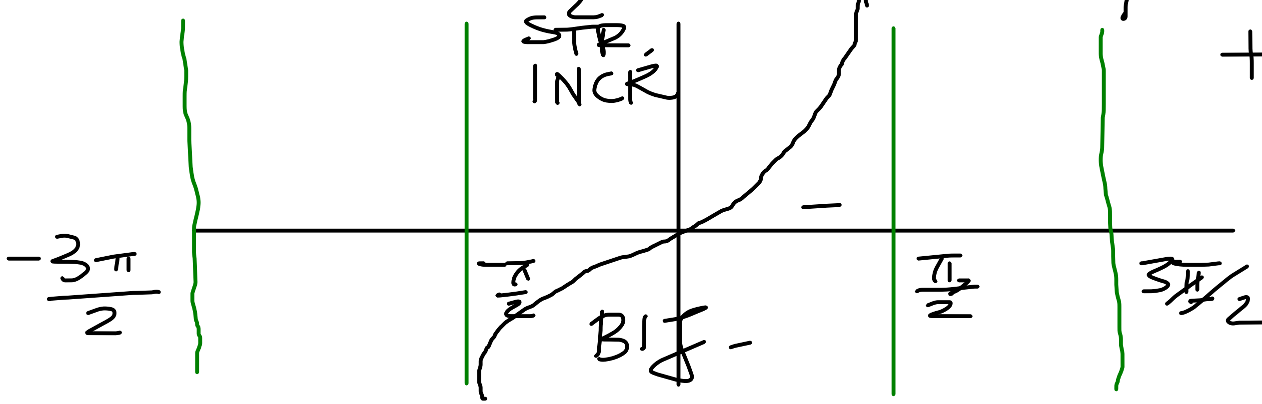
$$\tan\theta = \frac{\sin(\theta)}{\cos(\theta)} = \mathrm{tg}(x)$$

$$\mathrm{cotan}\,\theta = \frac{\cos(\theta)}{\sin(\theta)} = \mathrm{ctg}(x)$$

$$\cos(x) = 0 \quad \forall x = \pi/2 + k\pi, \quad k \in \mathbb{Z}$$

ODD fct.

$$\tan x : \mathbb{R} \setminus \left\{ \frac{\pi}{2} + k\pi \;\middle|\; k \in \mathbb{Z} \right\} \longrightarrow \mathbb{R}$$



$$\tan x \big|_{(-\pi/2,\, \pi/2)} : (-\pi/2, \pi/2)$$
$$\downarrow$$
$$\mathbb{R} = R(\tan)$$

$g = \tan x|_{(-\pi/2,\, \pi/2)}$

IS BIJ ONTO $\mathbb{R} = R(g)$

$g^{-1}(y): \mathbb{R} \longrightarrow (-\pi/2, \pi/2)$

" $\arctan(y)$

GOOD OLD FRIENDS

$$f : \underset{\substack{\parallel \\ D(f)}}{\overset{\text{Domain}}{E}} \longrightarrow \mathbb{R}$$

[NOTATION]

GOOD OLD FRIENDS

$$f : E \longrightarrow \mathbb{R}$$

$$\overset{\shortparallel}{D(f)} \qquad \text{[NOTATION]}$$

GIVEN $G \subseteq E \rightsquigarrow f|_G : G \to \mathbb{R}$

$$f|_G(x) := f(x)$$

GOOD OLD FRIENDS

$$f : \underset{\overset{\parallel}{D(f)}}{E} \longrightarrow \mathbb{R}$$

[NOTATION]

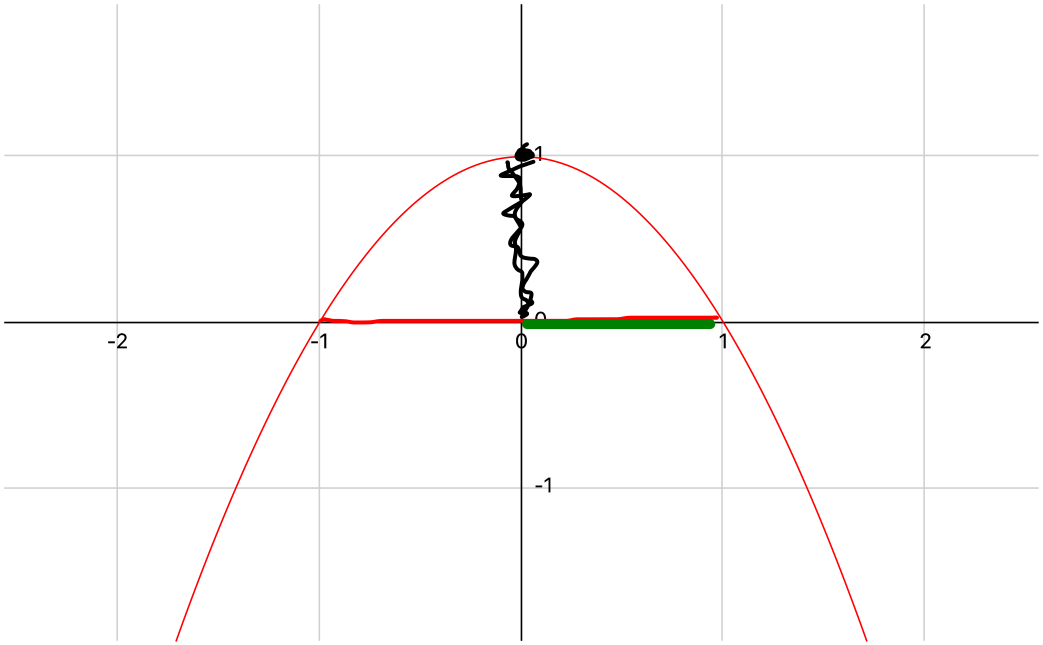
GIVEN $G \subseteq E \rightsquigarrow f|_G : G \to \mathbb{R}$

$$f|_G(x) := f(x)$$

$$R(f|_G) = \{ y \in \mathbb{R} \,/\, \exists x \in G \wedge f|_G(x) = y \}$$

$$R(f|_G) \subseteq R(f)$$

$f(x) = 1 - x^2$



$f|_{[-1,1]} \overset{h}{=}$:

$[-1, 1] \longrightarrow \mathbb{R}$

$R(h) = [0, 1]$

$f|_{[0,1]}$ IS DECREASING

GOOD OLD FRIENDS

$$f : \underset{\mathcal{D}(f)}{\underset{\shortparallel}{E}} \longrightarrow \mathbb{R}$$

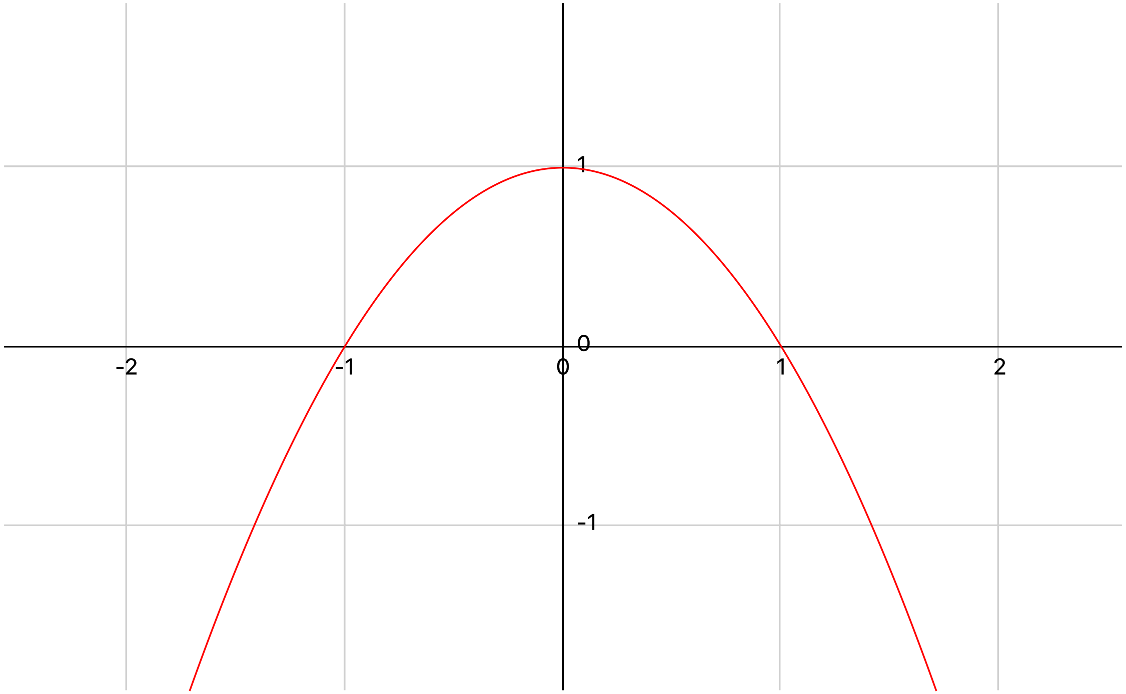
[NOTATION]

GIVEN $G \subseteq E \rightsquigarrow f|_G : G \to \mathbb{R}$ $\quad f|_G(x) := f(x)$

$$\mathcal{R}(f|_G) = \{ y \in \mathbb{R} \mid \exists\, x \in G \text{ s.t. } f|_G(x) = y \}$$

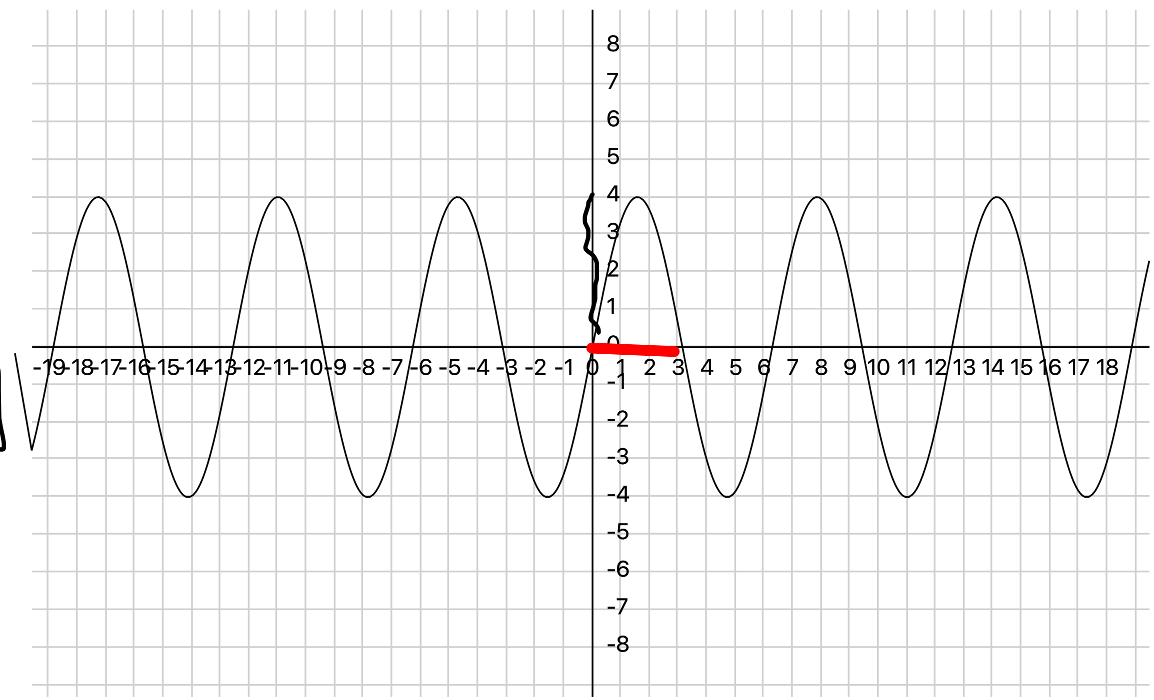
$$\inf_G f \doteq \inf \mathcal{R}(f|_G)$$

$$\sup_G f \doteq \sup \mathcal{R}(f|_G)$$



$$f(x) = 1 - x^2$$

$$f(x) = 4 \sin x$$



$$\mathrm{Im}\left(f|_{[0,\pi]}\right) = [0,4]$$

$$\inf_{[0,\pi]} f = \min_{[0,\pi]} f = 0$$

LIMITS

EXAMPLE

$$f : E \longrightarrow \mathbb{R}$$

$$f(x) = \frac{\sin(x-1)}{(x-1)}$$

LIMITS

EXAMPLE

$$f : E \longrightarrow \mathbb{R}$$

$$f(x) = \frac{\sin(x-1)}{(x-1)}$$

$$E = \mathbb{R} \setminus \{1\}$$

LIMITS

EXAMPLE

$f: E \longrightarrow \mathbb{R}$

$$f(x) = \frac{\sin(x-1)}{(x-1)}$$

$E = \mathbb{R} \setminus \{1\}$

Q: CAN WE $\underbrace{\text{EXTEND}}_{f \text{ to}}$ $\tilde{f}: \mathbb{R} \to \mathbb{R}$?

LIMITS

EXAMPLE

$$f : E \longrightarrow \mathbb{R}$$

$$f(x) = \frac{\sin(x-1)}{(x-1)}$$

$$E = \mathbb{R} \setminus \{1\}$$

f to

Q: CAN WE EXTEND $\tilde{f} : \mathbb{R} \to \mathbb{R}$?

$$\tilde{f}(x) = \begin{cases} \dfrac{\sin(x-1)}{(x-1)} & \forall x \neq 1 \\ C & x = 1 \end{cases}$$

f(x)

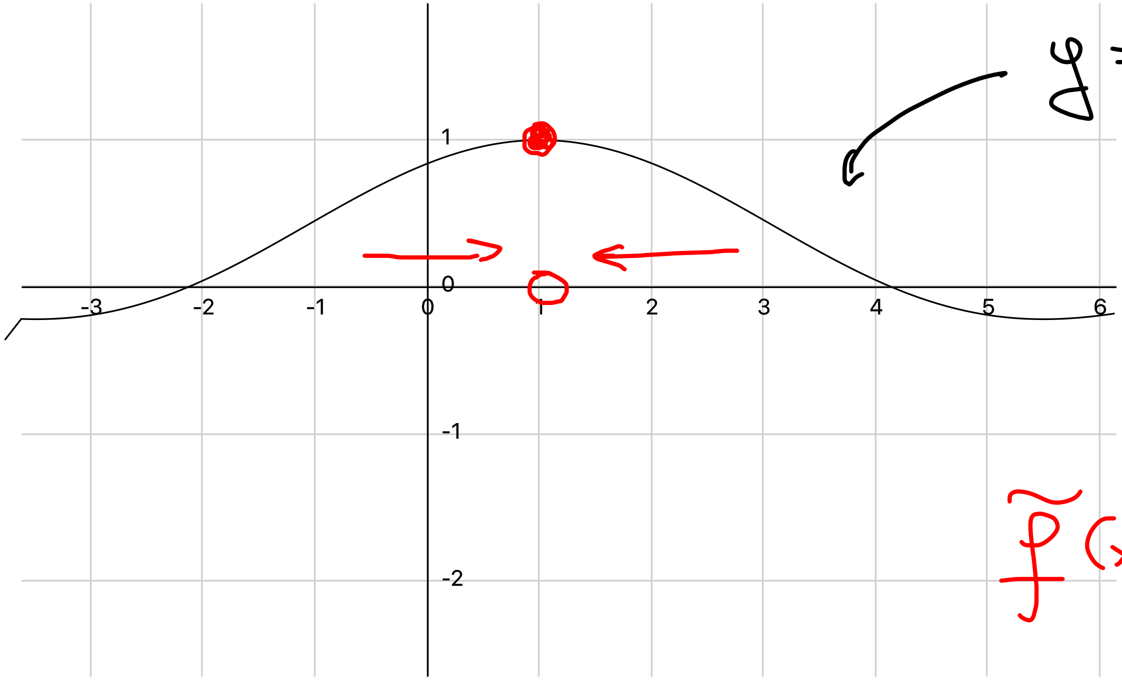
FIXED CONST.

LIMITS

EXAMPLE

$f: E \longrightarrow \mathbb{R}$ $\quad E = \mathbb{R} \setminus \{1\}$

$$f(x) = \frac{\sin(x-1)}{(x-1)}$$



$y = f(x)$

x close to 1

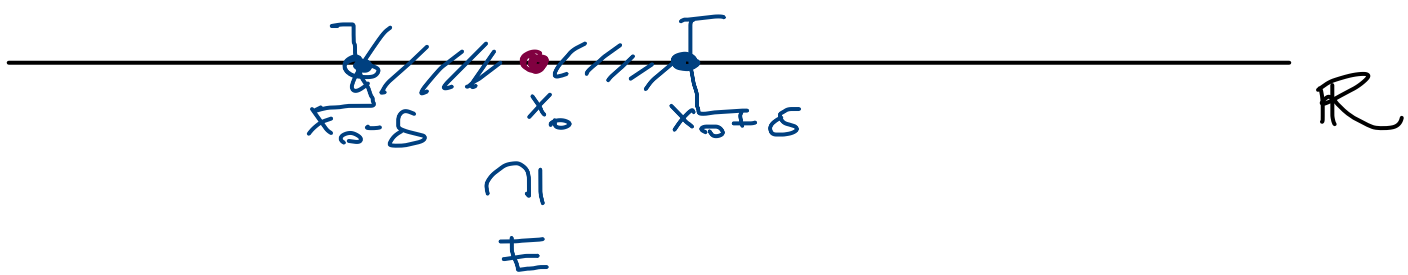
$f(x) \approx 1$

$$\tilde{f}(x) = \begin{cases} f(x) & x \neq 1 \\ 1 & x = 1 \end{cases}$$

DEF $E \subseteq \mathbb{R}$, $x_0 \in \mathbb{R}$ [x_0 not necess. in E]

E CONTAINS A POINTED NEIGHBORHOOD OF x_0 IF $\exists\, \delta > 0$

S.T. $\underbrace{]x_0 - \delta, x_0 + \delta[}_{\text{open interval}} \setminus \{x_0\} \subseteq E$

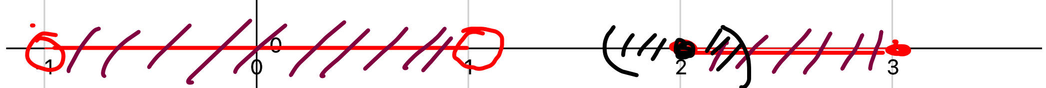


DEF $E \subseteq \mathbb{R}$, $x_0 \in \mathbb{R}$

E CONTAINS A POINTED NEIGHBORHOOD OF x_0 IF $\exists\, \delta > 0$

S.T. $]x_0-\delta, x_0+\delta[\setminus \{x_0\}$

$E = (-1,1) \cup [2,3]$



$\frac{3}{2} \in E$

$1 < 1 + \frac{n-1}{n} < 2$ $\notin E$

$\{x \in E \mid$ E CONTAINS A POINTED NBHD OF $x\} = (-1,1) \cup (2,3)$

EXAMPLE

$$f : E \longrightarrow \mathbb{R}$$

$$f(x) = \frac{\sin(x-1)}{(x-1)}$$

$$E = \mathbb{R} \setminus \{1\}$$

$\forall x \in \mathbb{R}$, E CONTAINS A PTD NBHD of x

$x = 0$ $\quad \mathbb{R}\setminus\{1\} \supseteq (-\delta, \delta) \setminus \{0\} \subseteq \mathbb{R}\setminus\{1\}$ FOR $\delta = \frac{1}{2}$

$x = 1$ $\quad (-\delta + 1, 1+\delta) \setminus \{1\} \subseteq \mathbb{R}\setminus\{1\} \ \forall \delta > 0$

EXAMPLE

$E = \mathbb{Q} \subseteq \mathbb{R}$, $x_0 \in \mathbb{R}$.

E DOES NOT CONTAIN ANY POINTED NEIGHBORHOOD OF x_0

$\delta > 0$ $\quad (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$

$\quad\quad\quad\quad y \notin \mathbb{Q} = E$

DENSITY OF $\mathbb{R} \setminus \mathbb{Q}$

$\exists\, y \in \mathbb{R} \setminus \mathbb{Q}$ S.T. $x_0 - \delta < y < x_0$

DEF $f : E \longrightarrow \mathbb{R}$, $E \subseteq \mathbb{R}$

TAKE $x_0 \in \mathbb{R}$. ASSUME E CONTAINS A POINTED NEIGHBORHOOD OF x_0.

DEF $f : E \longrightarrow \mathbb{R}$, $E \subseteq \mathbb{R}$

TAKE $x_0 \in \mathbb{R}$. ASSUME E CONTAINS A POINTED NEIGHBORHOOD OF x_0.

THEN $\lim_{x \to x_0} f(x) = \ell \in \mathbb{R}$ IF

$\forall \varepsilon > 0$, $\exists \delta_\varepsilon > 0$ S.T.

IF $\underline{0 < |x - x_0| < \delta_\varepsilon} \Longrightarrow |f(x) - \ell| < \varepsilon$

$x \in E$

$(x_0 - \delta, x_0 + \delta) \setminus \{x_0\} \subseteq E$

NEED TO EXIST

DEF $f : E \longrightarrow \mathbb{R}$, $E \subseteq \mathbb{R}$

TAKE $x_0 \in \mathbb{R}$. ASSUME E CONTAINS A POINTED NEIGHBORHOOD OF x_0.

THEN $\lim_{x \to x_0} f(x) = \ell \in \mathbb{R}$ IF

"THE LIMIT OF f FOR $x \to x_0$ IS ℓ"

$\forall \varepsilon > 0$, $\exists \delta_\varepsilon > 0$ S.T.

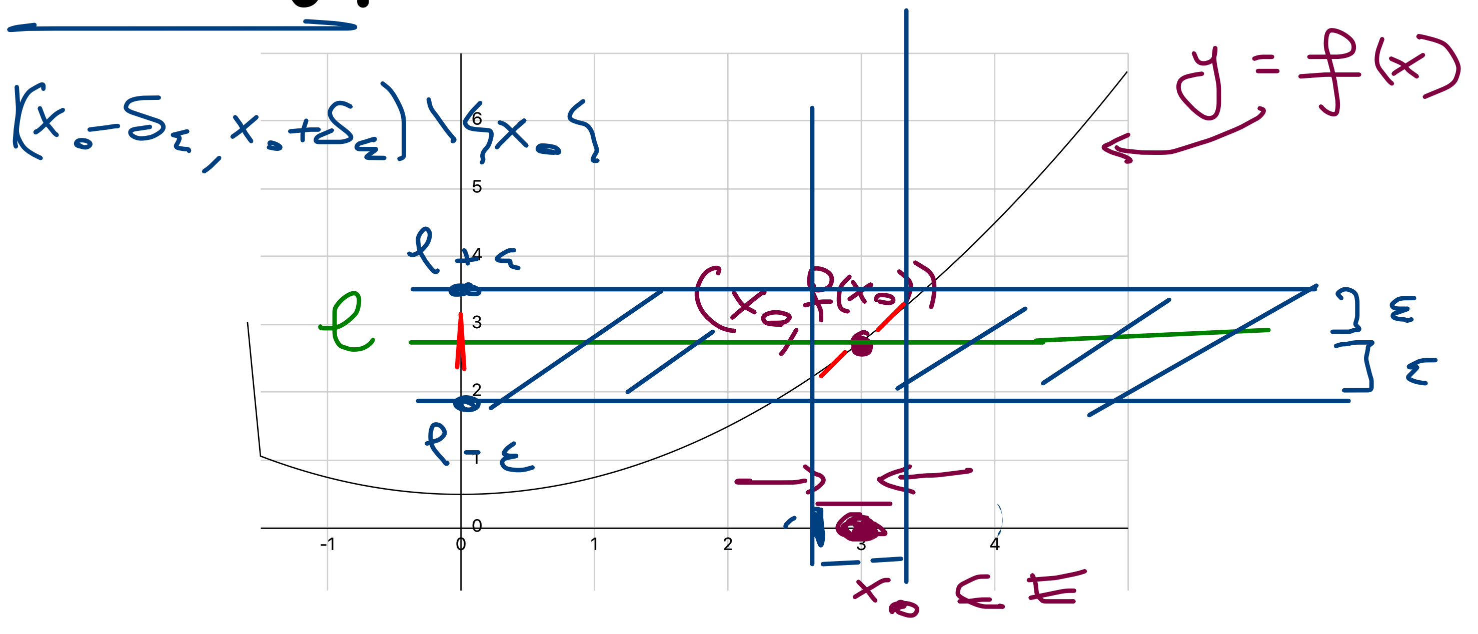
IF $0 < |x - x_0| < \delta_\varepsilon \Longrightarrow |f(x) - \ell| < \varepsilon$

DEF $f : E \longrightarrow \mathbb{R}$, $E \subseteq \mathbb{R}$, $x_0 \in \mathbb{R}$

ASSUME E CONTAINS A POINTED NEIGHB. OF x_0. THEN $\lim_{x \to x_0} f(x) = \ell$ IF

$\forall \varepsilon > 0$, $\exists \delta_\varepsilon > 0$ S.T

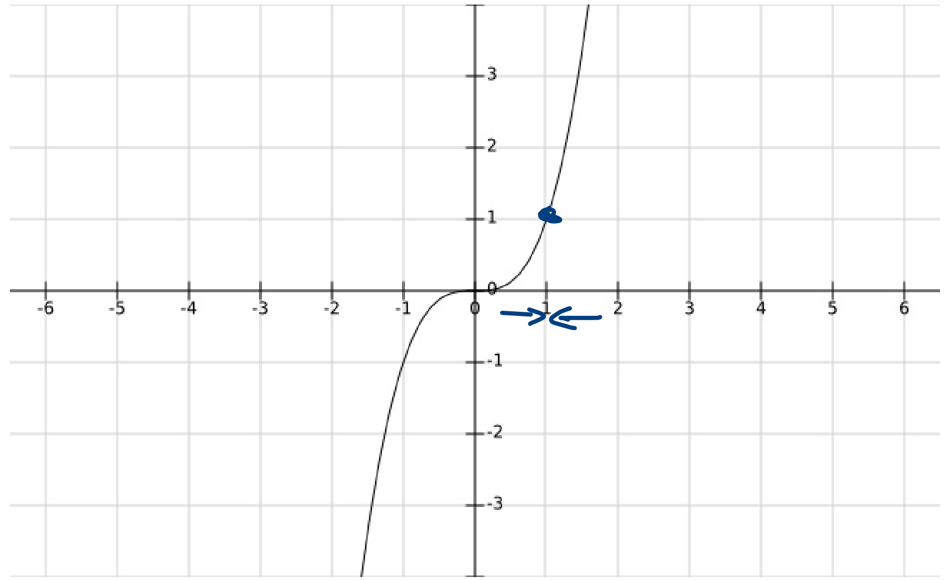
$$0 < |x - x_0| < \delta_\varepsilon \Longrightarrow |f(x) - \ell| < \varepsilon$$



EXAMPLE

$f:\mathbb{R}\to\mathbb{R}$

$f(x)=x^3$



$\lim_{x\to 1} f(x)=?$ $\qquad f(1)=1$

Fix $\varepsilon>0$. NEED TO FIND $\delta_\varepsilon>0$ S.T.

IF $|x-1|<\delta_\varepsilon \Longrightarrow |f(x)-1|<\varepsilon$

Fix $\varepsilon > 0$. NEED TO FIND $\delta_\varepsilon > 0$ S.T.

$$|x - 1| < \delta_\varepsilon \Longrightarrow \underline{\underline{|f(x) - 1| < \varepsilon}}$$

$$|f(x) - 1| = |x^3 - 1| = \underbrace{|x - 1|}_{\overset{\wedge\!\shortparallel}{\delta}} \underbrace{|x^2 + x + 1|}_{\overset{\wedge\!\shortparallel}{h(\delta)}} \leq \varepsilon$$

$\delta < \delta_\varepsilon \Longrightarrow (1 - \delta_\varepsilon, 1 + \delta_\varepsilon) \setminus \{1\}$

$\cup\!\shortmid$

$\forall\ x \in (1 - \delta, 1 + \delta) \setminus \{1\}$

$\delta_\varepsilon h(\delta_\varepsilon) \leq \varepsilon$

$\triangle$ ineq

$$|x^2 + x + 1| \leq |x|^2 + |x| + 1$$

$$\wedge\!\shortparallel$$

$$1 + 1 + \delta + \delta + 1 + \delta^2 \overset{+2\delta + 2}{} = 3 + 3\delta + \delta^2$$

$$\wedge$$

$$3 + \cancel{4}\delta$$

ASSUME $\delta < 1$

$$|f(x) - 1| \le |x-1|\,|x^2+x+1|$$

$< 1 \cdot$

$x \in (1-\delta, 1+\delta)\setminus\{1\}$ $\quad |x-1|\,(3+4\delta)$

< 1

$\delta\,(3+4\delta)$

< 7

7δ

$\delta < 1$

$\varepsilon = 10^6$

$\delta_\varepsilon = 10^6/7 > 1$

IF $\delta < 1$ $\quad |f(x) - 1| \le 7\delta = \varepsilon$

on $(1-\delta, 1+\delta)\setminus\{1\}$

$\delta_\varepsilon = \min\left(\frac{1}{99}, \frac{\varepsilon}{8}\right)$

$7\delta_\varepsilon = \min\left(\frac{7}{99}, \frac{7\varepsilon}{8}\right) < \varepsilon$

< 1

$$7\delta_\varepsilon = 7 \cdot \min\left(\frac{1}{99}, \frac{\varepsilon}{8}\right)$$

$$7\min\left(\frac{1}{99}, \frac{\varepsilon}{8}\right) = \min\left(\frac{7}{99}, \frac{7\varepsilon}{8}\right)$$

$$\frac{7}{99} \le \frac{7\varepsilon}{8} < \varepsilon$$

$$\frac{7}{99} < \varepsilon \qquad \frac{7\varepsilon}{8} < \varepsilon$$

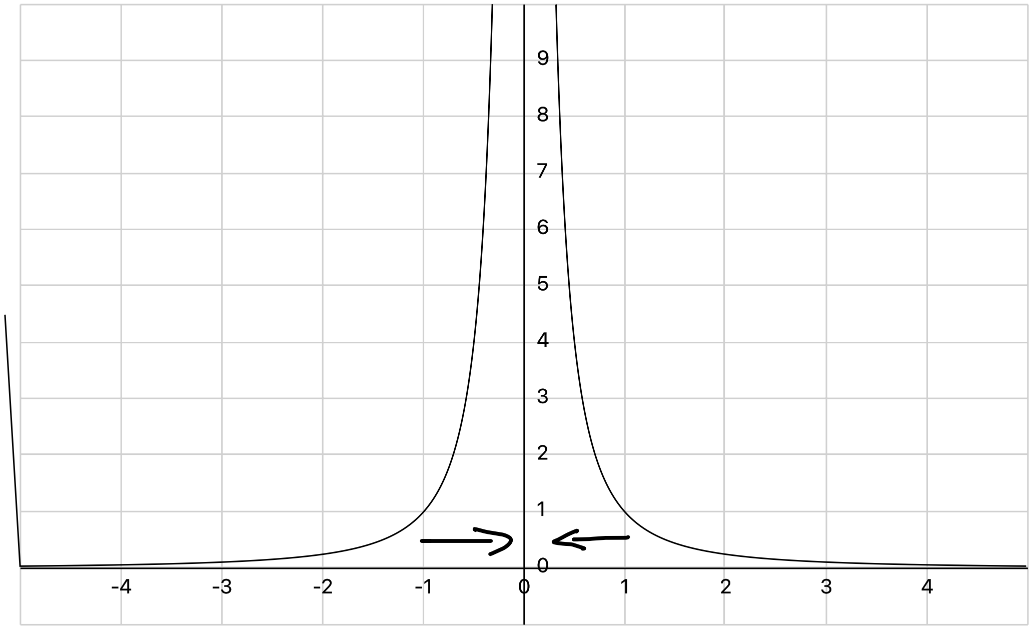
$$\delta_\varepsilon < 1$$

for $x \in (1-\delta_\varepsilon, 1+\delta_\varepsilon) \setminus \{1\}$

$$|f(x) - 1| < 7 \cdot \delta_\varepsilon < \varepsilon$$

EXAMPLE

$f : \mathbb{R}^* \to \mathbb{R}$, $f(x) = \dfrac{1}{x^2}$



$\lim\limits_{x \to 0} f(x)$ EXISTS?

NO

$\dfrac{1}{x^2}$ $\quad x = 0$

IF $\lim\limits_{x \to 0} f(x) = \ell$ $\quad \varepsilon = 1$, $\exists \delta_1 > 0$

S.T. $\forall x \in (-\delta_1, \delta_1) \setminus \{0\} \Longrightarrow |f(x) - \ell| < 1$

$f(x) \in [\ell - 1, \ell + 1]$

$R(f|_{(-\delta_1, \delta_1) \setminus \{0\}})$ is BOUNDED

$\dfrac{1}{n} \longrightarrow 0$ BUT $\left(\dfrac{1}{n}\right)^{-2} = n^2 \longrightarrow +\infty$

EXAMPLE $E = \mathbb{Q} \subseteq \mathbb{R}$, $x_0 \in \mathbb{R}$.

E DOES **NOT** CONTAIN ANY POINTED NEIGHBORHOOD OF x_0

$f : \mathbb{Q} \longrightarrow \mathbb{R}$, $x_0 = 0$

EXAMPLE

$E = \mathbb{Q} \subseteq \mathbb{R}$, $x_0 \in \mathbb{R}$.

E DOES NOT CONTAIN ANY POINTED NEIGHBORHOOD OF x_0

$f : \mathbb{Q} \longrightarrow \mathbb{R}$, $x_0 = 0$

$\lim_{x \to 0} f$ DOES NOT EXIST

THM $f : E \longrightarrow \mathbb{R}$, $E \subseteq \mathbb{R}$, $x_0 \in \mathbb{R}$

ASSUME E CONTAINS A POINTED NEIGHB. OF x_0. TFAE

THM $f : E \longrightarrow \mathbb{R}$, $E \subseteq \mathbb{R}$, $x_0 \in \mathbb{R}$

ASSUME E CONTAINS A POINTED NEIGHB. OF $\overline{x_0}$. TFAE

1. $\lim_{x \to \overline{x_0}} f(x) = \ell \in \mathbb{R}$

$\forall n$
$\Longleftrightarrow x_n \in E \setminus \{\overline{x_0}\}$

2. FOR ALL $\boxed{(x_n) \subseteq E \setminus \{\overline{x_0}\}}$

IF $x_n \xrightarrow{n\to\infty} x_0 \Longrightarrow f(x_n) \xrightarrow{n\to\infty} \ell$

SEQUENCE $y_n \doteq f(x_n)$

<u>EXAMPLE</u> $f(x) = x^2+1 \quad f: \mathbb{R} \to \mathbb{R}$

$$\lim_{x\to 5} (x^2+1) = \left(\lim_{x\to 5} x^2\right) + 1 = 25+1$$

INTUITIVELY

EXAMPLE

$f(x) = x^2 + 1 \qquad f : \mathbb{R} \to \mathbb{R}$

$$\lim_{x \to 5} (x^2 + 1) = \ell = 26$$

$\Updownarrow$ THM

$$\lim_{n \to \infty} (x_n^2 + 1) = \ell \qquad \left[\forall (x_n),\ x_n \xrightarrow{n \to \infty} 5 \right]$$

$(x_n) \subseteq \mathbb{R} \setminus \{5\}$

$x_n^2 \longrightarrow 5^2$

$x_n^2 + 1 \longrightarrow 5^2 + 1 = 26 \qquad \forall \{x_n\} \subseteq \mathbb{R} \setminus \{5\}$

$x_n \to 5$

EXAMPLE

$f(x) = x^2+1 \qquad f:\mathbb{R}\to\mathbb{R}$

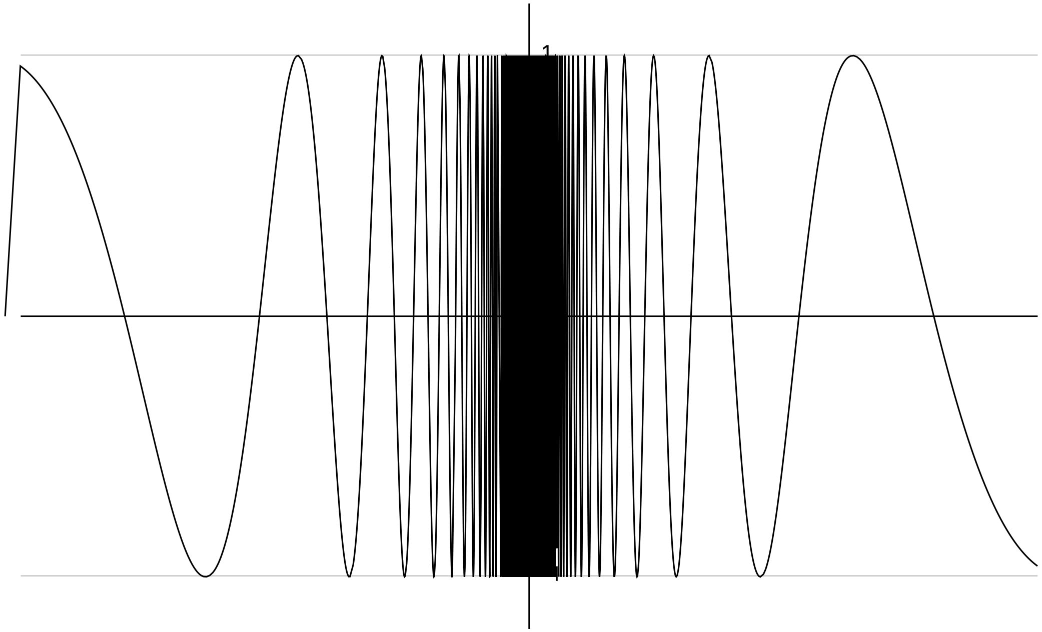
$$\lim_{x\to 5}(x^2+1) = \ell$$

$\Updownarrow$ THM

$$\lim_{n\to\infty}(x_n^2+1) = \ell \qquad \left[\forall (x_n),\ x_n \xrightarrow{n\to\infty} 5\right]$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (x_n) \subseteq \mathbb{R}\setminus\{5\}$$

$$= \left[\lim_{n\to\infty} x_n^2\right]+1$$



$$f : \mathbb{R}^* \to \mathbb{R}$$

$$f(x) = \sin\left(\frac{1}{x}\right)$$

$$\lim_{x \to 0} f(x) = \,?\ [\text{POLL}]$$

THM $f, g : E \longrightarrow \mathbb{R}$, $x_0 \in E$

$$\lim_{x \to x_0} f(x) = F, \quad \lim_{x \to x_0} g(x) = G$$

THEN

1. $$\lim_{x \to x_0} f(x) + g(x) = F + G$$

2. $$\lim_{x \to x_0} f(x) \cdot g(x) = F \cdot G$$

3. IF $G \neq 0$, $g(x) \neq 0 \ \forall x \in E \Rightarrow$

$$\lim_{x \to x_0} f(x)/g(x) = F/G$$

SQUEEZE THM FOR LIMITS

THM $f, g, h : E \to \mathbb{R}$ $\quad x_0 \in \mathbb{R}$

ASSUME THAT ON A PUNCTURED NEIGHB. S OF x_0 "POINTED"

$S = \left]x_0 - \delta, x_0 + \delta\right[\setminus \{x_0\}$

$$f(x) \leq g(x) \leq h(x) \quad \forall x \in S$$

SQUEEZE THM FOR LIMITS

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IF $\lim_{x \to x_0} f(x) = L$
$\lim_{x \to x_0} h(x) = L$ $\Longrightarrow$ $\lim_{x \to x_0} g(x) = L$