ANALYSIS 1

(ENGLISH)

LECTURE 12

TODAY

1. MORE CONVERGE CRITERIA FOR SERIES
2. BASIC DEFINITIONS & PROPERTIES OF FUNCTIONS
3. LIMITS

EXAMPLE

$$\sum_{i=0}^{\infty} \frac{1}{i!} = e$$

① DOES $\frac{1}{i!}$ CONVERGE?

$$0 < \frac{1}{i!} \leq \frac{1}{i} \underset{\text{SQUEEZE}}{\Longrightarrow} \frac{1}{i} \xrightarrow{i \to \infty} 0$$

WE NEED TO SHOW:

EXAMPLE $\sum_{i=0}^{\infty} \frac{1}{i!} = e$

① DOES IT CONVERGE?

$0 \leq \frac{1}{i!} \leq \frac{1}{i} \underset{\text{SQUEEZE}}{\Longrightarrow} \frac{1}{i} \xrightarrow{i \to \infty} 0$

WE SHOWED: (S_n) BOUNDED $\longrightarrow \sum_{i=0}^{\infty} \frac{1}{i!}$

IF $i \geq 2$, THEN $\frac{1}{i!} \leq \frac{1}{2^{i-1}} \Longleftrightarrow \sum^{\infty} \frac{1}{2^{i-1}}$

SO $\sum \frac{1}{i!} < +\infty$ SINCE $\sum \frac{1}{2^i} < +\infty$.

$\sum_{i=0}^{\infty} \frac{1}{i!}$ CONVERGES. HOW TO PROVE $= e$?

$\left(1+\frac{1}{n}\right)^n \xrightarrow{n\to\infty} e$

$$\left(1+\frac{1}{n}\right)^n = \sum_{i=0}^{n} \frac{1}{i!}\, \frac{n}{n}\,\frac{n-1}{n}\,\frac{n-2}{n}\cdots\frac{n-i+1}{n}$$

$1 \quad \frac{n-i+1}{n} \quad \frac{n-i+1}{n} \quad \frac{n-i+1}{n}$

$$\frac{1}{i!}\cdot 1 \left(\frac{n-i+1}{n}\right)^{i-1}$$

$$1+\sum_{i=1}^{n} \frac{1}{i!}\left(\frac{n-i+1}{n}\right)^{i-1} \leq \left(1+\frac{1}{n}\right)^n \leq \sum_{i=0}^{n} \frac{1}{i!}$$

$\sum_{i=0}^{\infty} \frac{1}{i!}$ CONVERGES. HOW TO PROVE $= e$?

$$\left(1+\frac{1}{n}\right)^{n} = \sum_{i=0}^{n} \frac{1}{i!} \quad \underset{\underset{1}{\parallel}}{\frac{n}{n}} \cdot \overset{\overset{\frac{n-i+1}{n}}{\wedge}}{\underset{\underset{1}{\wedge}}{\frac{n-1}{n}}} \cdot \overset{\overset{\frac{n-i+1}{n}}{\wedge}}{\underset{\underset{1}{\wedge}}{\frac{n-2}{n}}} \cdot \ldots \cdot \overset{\overset{\frac{n-i+1}{n}}{\parallel}}{\underset{\underset{1}{\wedge}}{\frac{n-i+1}{n}}}$$

$$\Downarrow$$

$$\sum_{i=0}^{n} \frac{1}{i!} \left(1 - \frac{i-1}{n}\right)^{i-1} \leq \left(1+\frac{1}{n}\right)^{n} \leq \sum_{i=0}^{n} \frac{1}{i!}$$

$$* \quad \underline{1+\sum_{i=1}^{n}\frac{1}{i!}\left(1-\frac{i-1}{n}\right)^{i-1}} \leqslant \underline{\left(1+\frac{1}{n}\right)^{n}} \leqslant \underline{\sum_{i=0}^{n}\frac{1}{i!}}$$

NEGATIVE BERNOULLI INEQ : $x \in (-1, 0)$

$$\left[q > 1 \Rightarrow \forall n \in \mathbb{N},\ q^{n} \geqslant 1 + n(q-1)\right] \Longrightarrow \forall k \in \mathbb{N}\quad (1+x)^{k} \geqslant 1 + kx$$

ORIG. VERS

$$\sum_{i=0}^{n} \frac{1}{i!}\left(1-\frac{i-1}{n}\right)^{i-1} \leq \left(1+\frac{1}{n}\right)^{n} \leq \sum_{i=0}^{n} \frac{1}{i!}$$

NEGATIVE BERNOULLI INEQ : $x \in (-1,0) \implies \forall k \in \mathbb{N}$

$$(1+x)^{k} \geq 1+kx$$

$n \geq i \geq 2$

$\Updownarrow$

$n-1 \geq i-1 \geq 1$

$$\Bigl(1 \overbrace{- \frac{i-1}{n}}^{x \,\in\, (-1,0)}\Bigr)^{\overset{k}{\downarrow}\; i-1} \geq \left(1-\frac{(i-1)^{2}}{n}\right)$$

$$(1-x)^{k} \geq 1+kx$$

$$2+\sum_{i=2}^{n}\frac{1}{i!}\left(1-\frac{i-1}{n}\right)^{i-1} \leq \left(1+\frac{1}{n}\right)^{n} \leq \sum_{i=0}^{n}\frac{1}{i!}$$

NEGATIVE BERNOULLI INEQ: $x \in (-1,0) \implies \forall k \in \mathbb{N}$

$$(1+x)^{k} \geq 1+kx$$

$$\sum_{i=2}^{n}\frac{1}{i!}\left(1-\overbrace{\frac{i-1}{n}}^{x}\right)^{\overset{k}{i-1}} \geq \sum_{i=2}^{n}\frac{1}{i!}\left(1-\frac{(i-1)^2}{n}\right)$$

$$\sum_{i=0}^{n} \frac{1}{i!}\left(1-\frac{i-1}{n}\right)^{i-1} \leqslant \left(1+\frac{1}{n}\right)^{n} \leqslant \sum_{i=0}^{n} \frac{1}{i!}$$

NEGATIVE BERNOULLI INEQ : $x \in (-1, 0) \implies \forall K \in \mathbb{N}$

$$(1+x)^{K} \geqslant 1+Kx$$

$$\sum_{i=0}^{n} \frac{1}{i!}\Big(1 \underbrace{-\frac{i-1}{n}}_{x}\Big)^{\overset{K}{i-1}} \geqslant \sum_{i=0}^{n} \frac{1}{i!}\left(1-\frac{(i-1)^2}{n}\right)$$

$$2+\sum_{i=2}^{n} \frac{1}{i!}\left(1-\frac{(i-1)^2}{n}\right) \leqslant \left(1+\frac{1}{n}\right)^{n} \leqslant \sum_{i=0}^{n} \frac{1}{i!}$$

$$2+\sum_{i=2}^{n}\frac{1}{i!}-\underbrace{\frac{1}{n}\sum_{i=2}^{n}\frac{(i-1)^2}{i!}}\le\left(1+\frac{1}{n}\right)^n\le\sum_{i=0}^{n}\frac{1}{i!}$$

$n \to \infty$ $\quad$ $n \to \infty$ $\quad$ $n \to \infty$

$$\underbrace{\lim_{n\to\infty} 2+\sum_{i=2}^{n}\frac{1}{i!}=\sum_{i=0}^{+\infty}\frac{1}{i!}\le e\le\sum_{i=0}^{\infty}\frac{1}{i!}}_{=}$$

$$\sum_{i=0}^{\infty}\frac{1}{i!} = e$$

IF WE CAN PROVE THAT

$[a \le b \le a]$

$\Uparrow$

$a = b$

TRUE !

$$\lim_{n\to\infty} -\frac{1}{n}\sum_{i=2}^{n}\frac{(i-1)^2}{i!}=0$$

$$\lim_{n\to\infty} \underbrace{x_n}_{\xrightarrow{n\to\infty} 0}\cdot y_n \overset{\Leftarrow}{=} 0$$

IF (y_n) is BOUNDED.

$\Uparrow$

ok $(\ast)$ IF $\displaystyle\sum_{i=2}^{\infty}\frac{(i-1)^2}{i!}<+\infty$

NEGATIVE BERNOULLI INEQ: $x \in (-1,0) \implies \forall k \in \mathbb{N}$

$$(1+x)^k \geq 1+kx$$

<u>PROOF</u> INDUCTION ON $k \in \mathbb{N}$

STARTING STEP: $\underbrace{(1+x)^0}_{=1} \geq 1+0\cdot x$

INDUCTIVE STEP: ASSUME WE KNOW $(1+x)^j \geq 1+jx \quad \forall j<k$.

INDUCT. HYP. $\rightarrow$ $* \ (1+(k-1)x) \leq (1+x)^{k-1}$

$$\underline{(1+x)^k} = (1+x)(1+x)^{k-1} \overset{*}{\geq} (1+x)(1+(k-1)x) = 1+kx+(k-1)x^2 \geq \underline{1+kx}$$

IN

EXAMPLE

$q \in \mathbb{R}$ $\qquad$ $x_n \doteq q^n$

FOR WHAT VALUES OF $q \in \mathbb{R}$ DOES $\sum_{i=0}^{\infty} q^i$ CONVERGE?

EXAMPLE

$q \in \mathbb{R}$ $\qquad x_n \doteq q^n$

FOR WHAT VALUES OF $q \in \mathbb{R}$ DOES $\sum_{i=0}^{\infty} q^i$ CONVERGE? [POLL]

1. ~~FOR ALL $q \in \mathbb{R}$~~ $\quad q = 2$, x_n UNBOUNDED
2. ~~FOR NO $q \in \mathbb{R}$~~ $\quad q = \frac{1}{2} \rightsquigarrow \sum_{i=0}^{\infty} \frac{1}{2^i} < \infty$
3. ~~WHEN $|q| > 1$~~ $\quad q = 2$
4. WHEN $|q| < 1$ ✓

CAUCHY CRITERION

PROP LET (x_n) BE A SEQ.

ASSUME $\lim_{n\to\infty} \sqrt[n]{|x_n|} = \ell \in \mathbb{R}_+ \cup \{+\infty\}$

$$\underset{y_n}{\shortparallel}$$

CAUCHY CRITERION

PROP (x_n)

ASSUME $\lim_{n\to\infty} \sqrt[n]{|x_n|} = \ell \in \mathbb{R}_+ \cup \{+\infty\}$

THEN

① IF $\ell > 1$ OR $\ell = +\infty$ $\Longrightarrow$ $\sum_{n=0}^{\infty} x_n$ IS NON-CONV.
$\Uparrow$ (x_n) IS UNBOUNDED

② IF $\ell < 1$ $\Longrightarrow$ $\sum_{n=0}^{\infty} x_n$ CONV. ABSOLUTELY
$\ell \in [0,1)$
$\sum_{n=0}^{\infty} |x_n| < \infty \Longleftrightarrow$ ABSOLUTELY

EXAMPLE

$q \in \mathbb{R}$ $\quad x_n \doteq q^n$

FOR WHAT VALUES OF $q \in \mathbb{R}$ DOES $\sum_{i=0}^{\infty} q^i$ CONVERGE?

CAUCHY CRIT: $\sqrt[n]{|x_n|} = \sqrt[n]{|q|^n} = |q|$

$\Longrightarrow$ IF $|q| > 1 \Longrightarrow (x_n)$ is UNBOUNDED

$\Longrightarrow \sum q^n$ non CONV.

$|q| < 1 \Longrightarrow \sum_{i=0}^{\infty} q^i$ CONV. ABS.

EXAMPLE

IF $\lim_{n\to\infty} \sqrt[n]{|x_n|} = 1$ THEN ANYTHING CAN HAPPEN

- $x_n = (-1)^n \Longrightarrow \sum_{i=0}^{\infty} x_n$ NON CONV.

  TR. SUMS $S_n = \begin{cases} 1 & n \text{ even} \\ 0 & n \text{ odd.} \end{cases}$

  $\sqrt[n]{n} \xrightarrow{n\to\infty} 1$

- $x_n = n \Longrightarrow \sum_{i=0}^{\infty} x_n = +\infty$

- $x_n = \frac{1}{n^2} \Longrightarrow \sum_{i=0}^{\infty} x_n$ CONV.

  $\sqrt[n]{\frac{1}{n}} \to 1$

- $x_n = \frac{(-1)^n}{n} \Longrightarrow \sum_{i=0}^{\infty} x_n$ CONV. (LEIBNIZ CRIT.)

  CONV. ~~ABSOLUTELY~~

PROP LET (x_n) BE A SEQ.

ASSUME THAT $\lim\limits_{n\to\infty} \sqrt[n]{|x_n|} = \rho$

$\rho \in \mathbb{R}_+ \cup \{+\infty\}$

IF $\rho > 1$ OR $\rho = +\infty$ $\Longrightarrow$ (x_n) IS UNBOUNDED

IF $\rho < 1$ $\Longrightarrow$ $\left\{\sum x_n = \ell \in \mathbb{R}\right\}$ $\Longrightarrow$ $\lim x_n = 0$.

D'ALEMBERT'S CRITERION

PROP LET (x_n) BE A SEQ.

ASSUME

$$\lim_{k\to\infty} \frac{|x_{k+1}|}{|x_k|} = \ell \in \mathbb{R}_+ \cup \{+\infty\}$$

$$\underbrace{\qquad}_{y_k}$$

D'ALEMBERT'S CRITERION

PROP (x_n)

ASSUME $\displaystyle\lim_{k\to\infty} \frac{|x_{k+1}|}{|x_k|} = \ell \in \mathbb{R}_+ \cup \{+\infty\}$

① IF $\underset{\substack{\cap \\ [0,1)}}{\ell} < 1 \Rightarrow \displaystyle\sum_{k=0}^{\infty} x_k$ CONV. ABSOLUTELY

② IF $\ell > 1 \Rightarrow \displaystyle\sum_{k=0}^{\infty} x_k$ DOES NOT CONV.

OR

$\ell = +\infty \Rightarrow (x_n)$ IS UNBOUNDED

EXAMPLE

$$x_k = \frac{k}{2^k}$$

$$\frac{|x_{k+1}|}{|x_k|} = \frac{(k+1)}{2^{\cancel{k+1}}} \cdot \frac{\cancel{2^k}}{k} = \frac{k+1}{k} \cdot \frac{1}{2}$$

$$2^{k+1} = 2 \cdot 2^k$$

$$\lim_{k\to\infty} \frac{|x_{k+1}|}{|x_k|} = \lim \underbrace{\frac{1}{2}}_{\to \frac{1}{2}} \underbrace{\frac{k+1}{k}}_{\xrightarrow{k\to\infty} 1} = \frac{1}{2}$$

$$\ell < 1$$

$$\overset{\text{D'AL.}}{\underset{\text{CRIT.}}{\Longrightarrow}} \sum_{k=0}^{\infty} \frac{k}{2^k} < +\infty$$

EXAMPLE

$q \in \mathbb{R}$ $\qquad x_n = \dfrac{q^n}{n!}$

DOES $\sum_{n=0}^{\infty} x_n$ CONVERGE? [POLL] $\forall q \in \mathbb{R}$

TRUE

FALSE

$$\lim_{k\to\infty} \frac{|x_{k+1}|}{|x_k|} = \frac{|q|^{\cancel{k+1}}}{(k+1)!} \cdot \frac{\cancel{k!}}{\cancel{|q|^k}} = \lim_{k\to\infty} \frac{|q|}{k+1} = 0$$

$|q|^{k+1} = |q|^k \cdot |q|$

$(k+1)! = (k+1) \cdot k!$

DAL. CRIT $\Rightarrow \sum q^n/n!$ CONV. ABS.

EXAMPLE

$x_n = \frac{n^n}{n!}$ DOES $\sum_{i=0}^{\infty} x_n$ CONVERGE?

$$\frac{|x_{n+1}|}{|x_n|} = \frac{(n+1)^{n+\cancel{1}}}{\cancel{(n+1)!}} \cdot \frac{\cancel{n!}}{n^n} = \frac{(n+1)^n}{n^n} = \left(1+\frac{1}{n}\right)^n$$

$$\sum_{i=0}^{\infty} \frac{1}{i!}$$

$$\lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} = \lim \left(1+\frac{1}{n}\right)^n = e > 2$$

$$e > 1$$

(x_n) UNBOUND $\Longrightarrow \sum x_n = +\infty$

EXAMPLE

$$x_n = \frac{(n-1)^2}{n!}$$

DOES $\sum_{i=0}^{\infty} x_n$ CONVERGE?

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \frac{n^2}{(n+1)!} \cdot \frac{\cancel{n!}}{(n-1)^2} = \lim_{n\to\infty} \left(\frac{n}{n-1}\right)^2 \cdot \frac{1}{n+1} = 0$$

$\left(\frac{1}{2}\right)^2 \le \frac{n}{n-1}$ BOUNDED 2^2

$\frac{1}{n+1} \xrightarrow{n\to\infty} 0$

$0 = \ell < 1$

$$\sum_{i=2}^{\infty} \frac{(i-1)^2}{i!} \qquad , \qquad \sum_{i=0}^{\infty} \frac{(i-1)^2}{i!} \quad \text{CONV.}$$

BOUNDED

$$1 - \frac{1}{n} \to 0 \qquad \cdot \sum_{i=2}^{n} \frac{(i-1)^2}{i!} \xrightarrow{n\to\infty} 0$$

FUNCTIONS

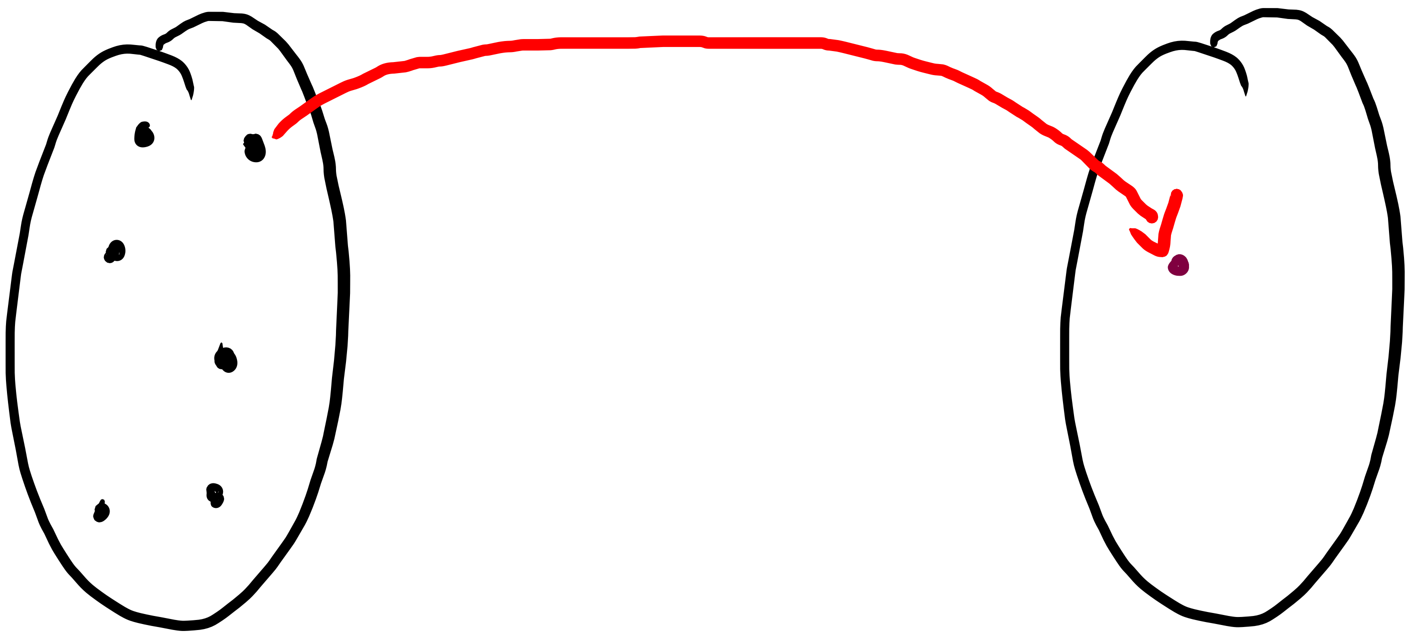
FUNCTION :

S = STARTING SET [DOMAIN]

T = ARRIVAL SET

FUNCTIONS

FUNCTION :



S = STARTING SET [DOMAIN]

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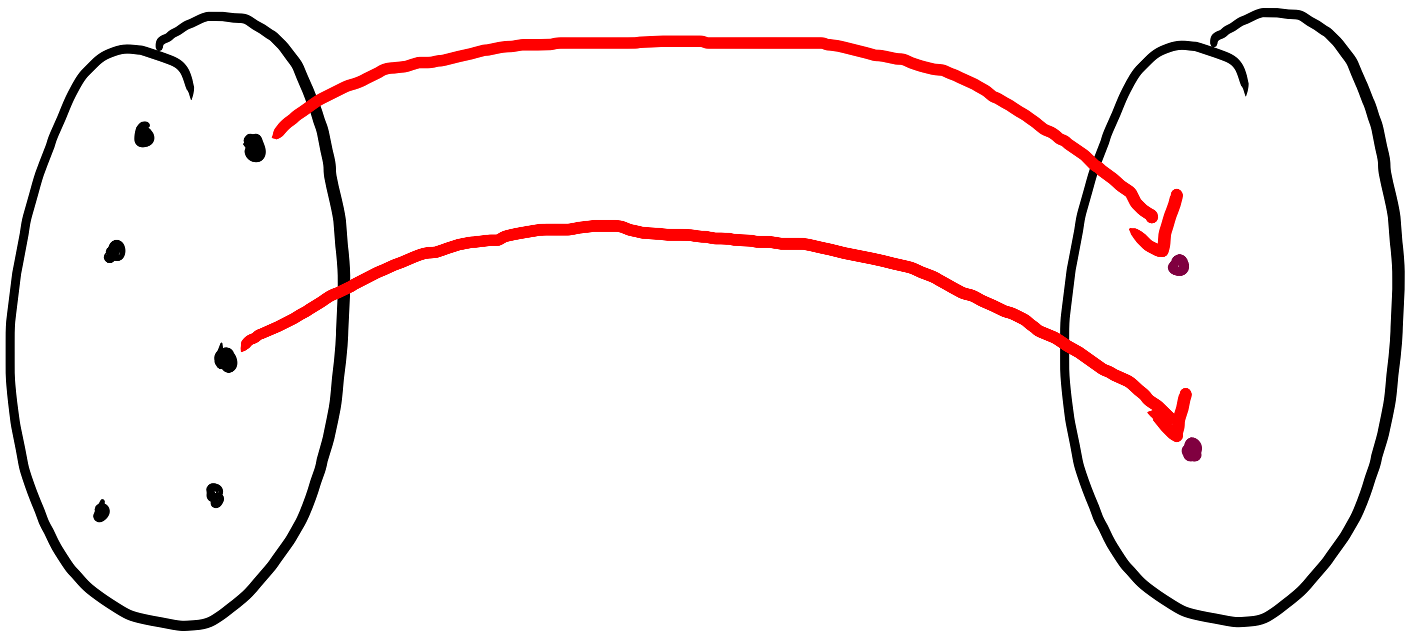
FUNCTIONS

FUNCTION :



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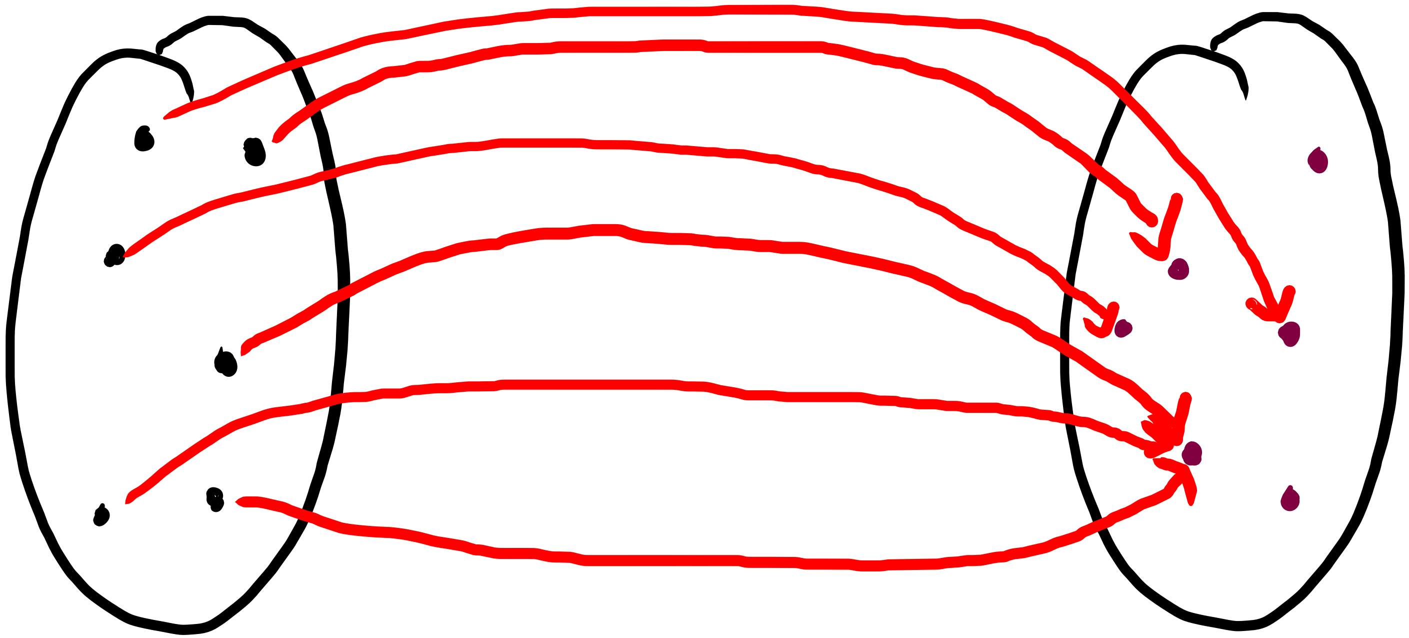
FUNCTIONS

FUNCTION :



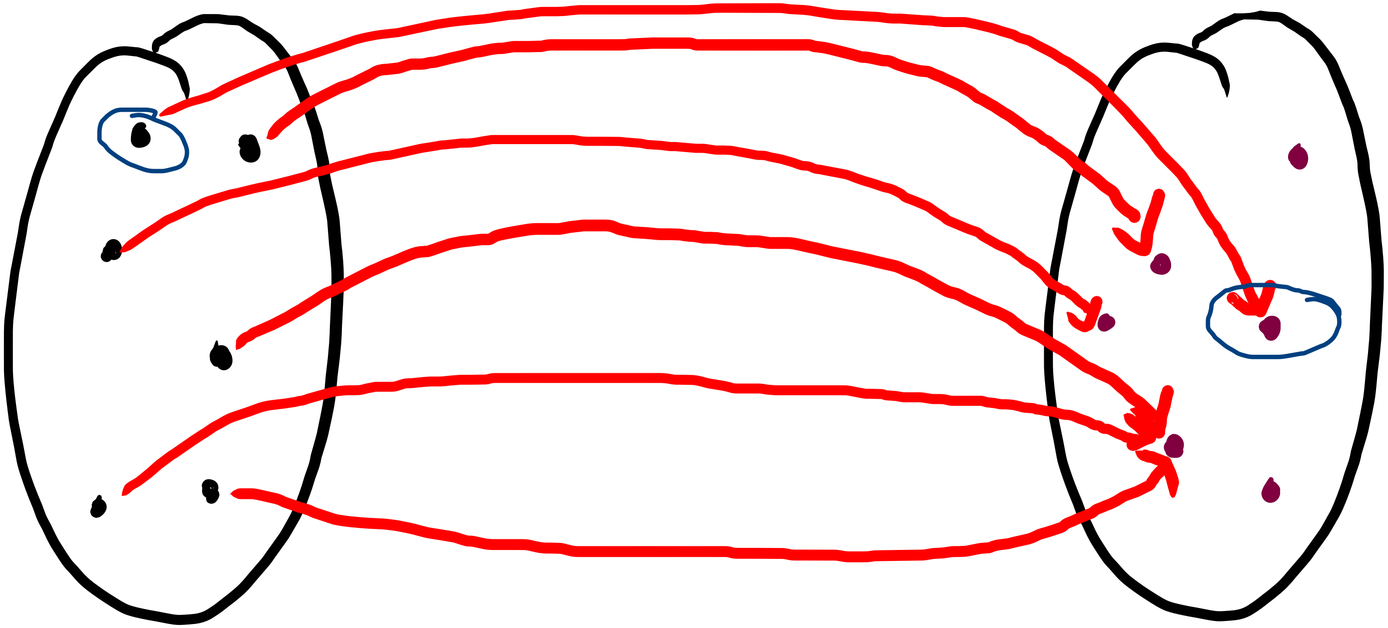
S = STARTING SET [DOMAIN]

T = ARRIVAL SET

FUNCTIONS

FUNCTION: RECIPE f TO ASSOCIATE TO EACH ELEM. OF S A UNIQUE ELEM. OF T

S



S = STARTING SET [DOMAIN]

ARRIVAL SET $= T$

FUNCTIONS

FUNCTION : RECIPE f TO ASSOCIATE TO EACH ELEM. OF S A UNIQUE ELEM. OF T

NOTATION: $f : \overset{\text{DOMAIN}}{S} \xrightarrow{f} T$

$f(s) = t$ ← $s \in S$

$t \in T$ is THE UNIQUE ELT. THAT f ASSOCIATES TO $s \in S$

FUNCTIONS

RECIPE f TO ASSOCIATE

FUNCTION : TO EACH ELEM. OF S

A UNIQUE

ELEM. OF

T

NOTATION: $f : S \longrightarrow T$

EXAMPLE

$f : \mathbb{R} \longrightarrow \mathbb{R}$

MOVEMENT OF A PARTICLE in terms of the time t

$$f(t) = \frac{1}{2} a t^2 + v t + c$$

$a, v, c \in \mathbb{R}$

CONSTANTS

$f : S \longrightarrow T$ FUNCTION

RANGE f $\doteq$ ELEMENTS OF T THAT ARE 'REACHED" BY f

(IMAGE f)

$\cap$

T

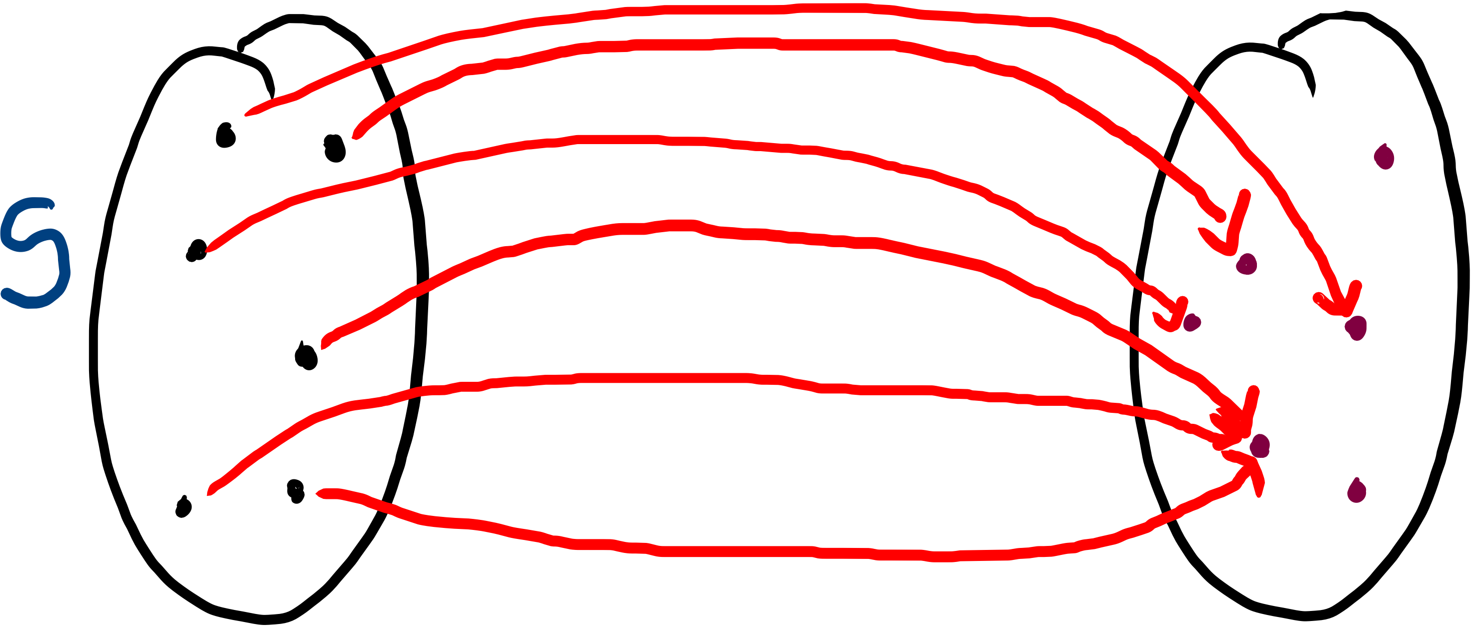
= elt's of T OF THE FORM $f(s)$ for some $s \in S$

$R(f)$

$(\operatorname{Im}(f))$

FUNCTIONS

FUNCTION : RECIPE TO ASSOCIATE TO EACH ELEM. OF S A UNIQUE ELEM. OF T



S = STARTING SET [DOMAIN]

ARRIVAL SET

$f : S \longrightarrow T$ FUNCTION

RANGE $f \doteq$ ELEMENTS OF T THAT ARE 'REACHED" BY f

$$R(f) \doteq \{ f(s) \in T \mid s \in S \}$$

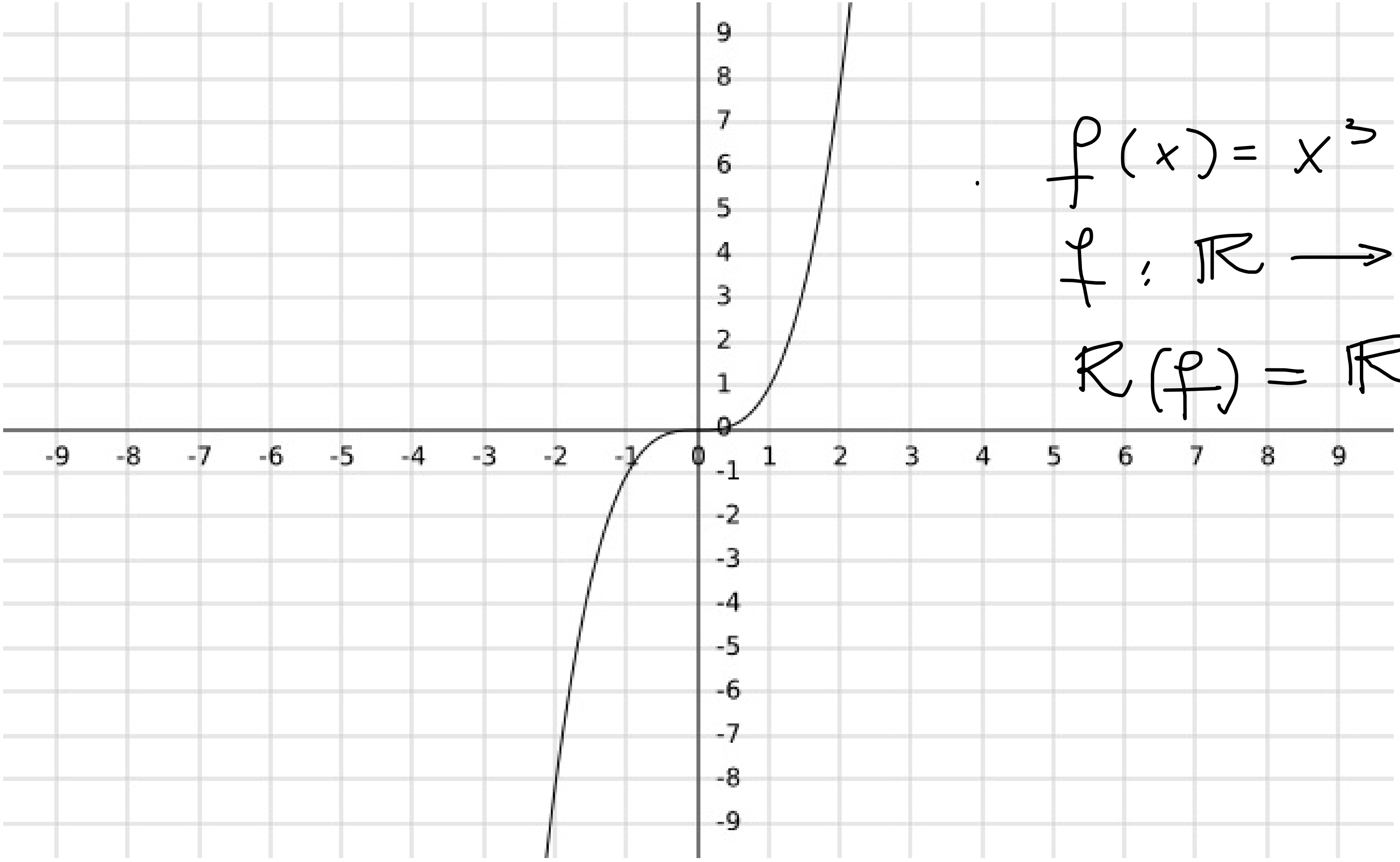
$f : S \longrightarrow T$ FUNCTION

RANGE $f \doteq$ ELEMENTS OF T THAT ARE "REACHED" BY f

$$R(f) \doteq \{ f(s) \in T \mid s \in S \}$$

EXAMPLE $f : \mathbb{R} \longrightarrow \mathbb{R}$ $\quad f(\underset{\substack{\cap \\ \mathbb{R}}}{x}) = x^2$

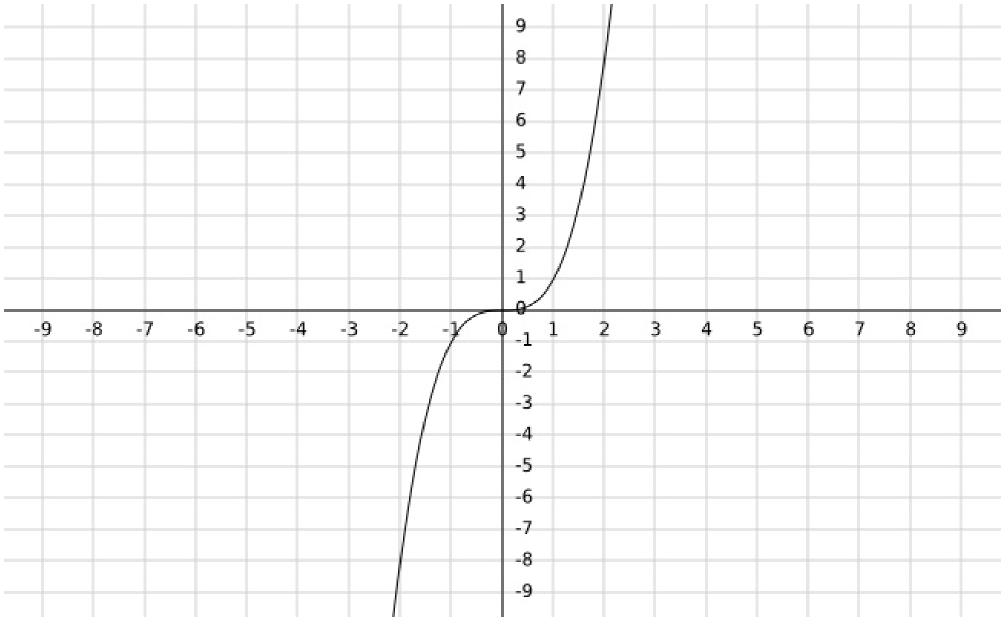
$$R(f) = \{ x^2 \in \mathbb{R} \mid x \in \mathbb{R} \} = \mathbb{R}_+ \subset \mathbb{R}$$



$f(x) = x^3$

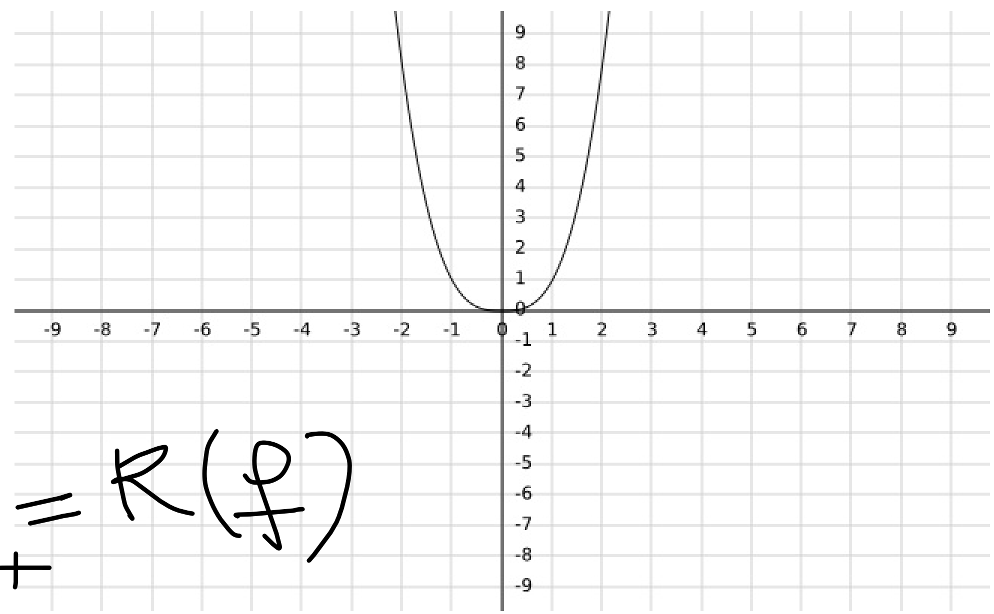
$f : \mathbb{R} \longrightarrow \mathbb{R}$

$R(f) = \mathbb{R}$



$f(x) = x^3$

$R(f) = \mathbb{R}$



$f: \mathbb{R} \to \mathbb{R}$

$f(x) = x^2$

$R(f) = \mathbb{R}_+$

$f: \mathbb{R} \longrightarrow \mathbb{R}_+ = R(f)$

IN GENERAL:

$$
\begin{tikzcd}
f : S \arrow[r] \arrow[d, squiggly] & T \arrow[d, phantom, "\rotatebox[origin=c]{-90}{$\supseteq$}"] \\
f : S \arrow[r] & R(f)
\end{tikzcd}
$$

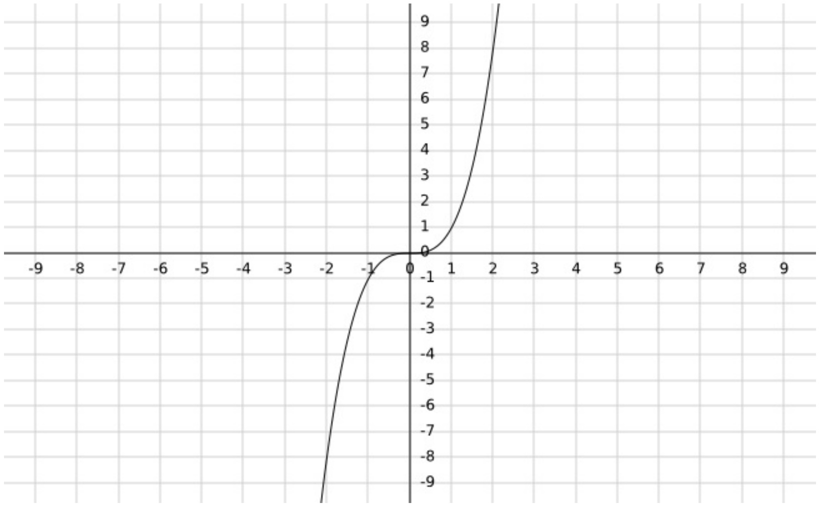
THE SAME FCT.

BASIC PROPERTIES

$$f : S \longrightarrow T \quad \text{FUNCTION}$$

- **INJECTIVE** : $\underset{\overset{\cap}{S}}{s_1} \neq \underset{\overset{\cap}{S}}{s_2} \Rightarrow \underset{\overset{\cap}{T}}{f(s_1)} \neq \underset{\overset{\cap}{T}}{f(s_2)}$

$$\forall s_1, s_2 \in S$$



$f(x) = x^3$

INJECTIVE

$x_1^3 = x_2^3 \Longrightarrow x_1 = x_2$

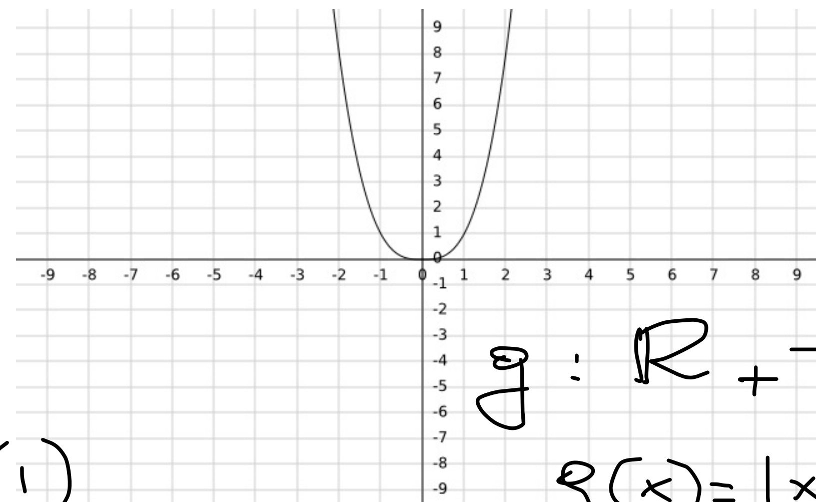
$f : \mathbb{R} \longrightarrow \mathbb{R}_+$

$-1 \neq 1$

$f(x) = |x^3|$

NOT INJ.

$f(-1) = |-1^3| = |1^3| = f(1)$



$g : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$

$g(x) = |x^3|$ INJ.

BASIC PROPERTIES

$$f : S \longrightarrow T \quad \text{FUNCTION}$$

- **INJECTIVE** : $\underset{S}{\overset{\cap}{s_1}} \neq \underset{S}{\overset{\cap}{s_2}} \Rightarrow f(s_1) \neq f(s_2)$

ARRIVAL SET.

- **SURJECTIVE**: $\forall t \in T, \exists s \in S$ s.t.

$$f(s) = t$$

BASIC PROPERTIES

$f : S \longrightarrow T$ FUNCTION

- INJECTIVE : $\underset{S}{\underset{\cap}{s_1}} \neq \underset{S}{\underset{\cap}{s_2}} \Longrightarrow f(s_1) \neq f(s_2)$

- SURJECTIVE : $\forall t \in T, \exists s \in S$ s.t. $f(s) = t$

BASIC PROPERTIES

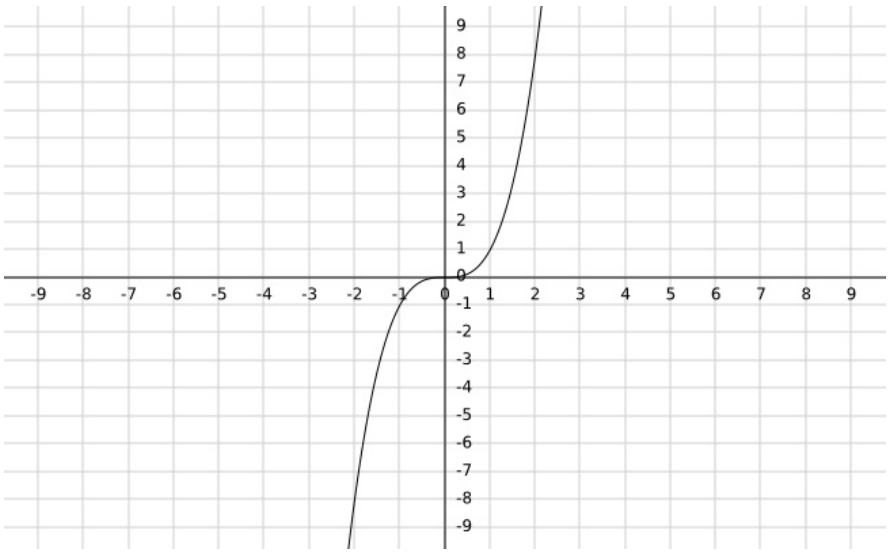
$f : S \longrightarrow T$ FUNCTION

- INJECTIVE : $\underset{\underset{S}{\cap}}{s_1} \neq \underset{\underset{S}{\cap}}{s_2} \Longrightarrow f(s_1) \neq f(s_2)$

- SURJECTIVE : $\forall t \in T, \exists s \in S$ s.t.

$$R(f) = T \Longleftrightarrow f(s) = t$$

" $\{ t \in T \mid \exists s \in S \text{ with } f(s) = t \}$



$$f(x) = x^3$$

$$f : \mathbb{R} \to \mathbb{R}$$

SURJECTIVE: $\left(\sqrt[3]{x}\right)^3 = x$

$$\| $$

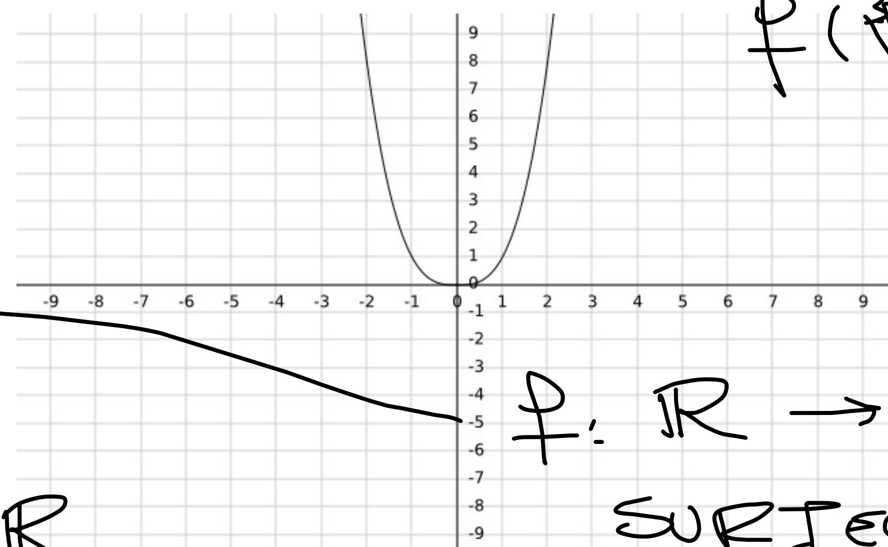
$$f(\sqrt[3]{x})$$

$$f : \mathbb{R} \to \mathbb{R}$$

$$f(x) = |x^3|$$

NOT SURJ-

$R(f) = \mathbb{R}_+ \subsetneq \mathbb{R}$



$$f : \mathbb{R} \to \mathbb{R}_+$$

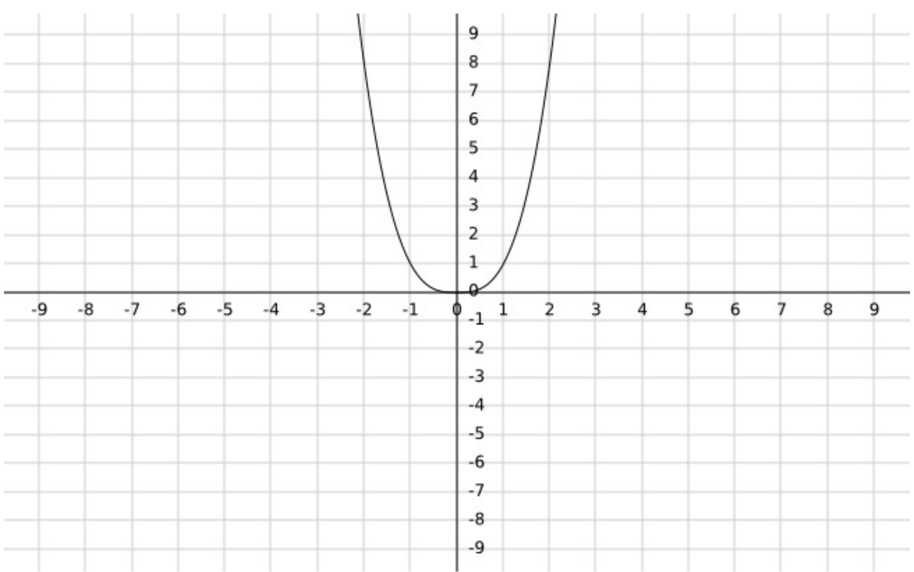
SURJECTIVE

BASIC PROPERTIES

$$f : S \longrightarrow T \quad \text{FUNCTION}$$

- **INJECTIVE** : $\underset{\underset{S}{\cap}}{s_1} \neq \underset{\underset{S}{\cap}}{s_2} \Longrightarrow f(s_1) \neq f(s_2)$
- **SURJECTIVE** : $\forall t \in T, \exists s \in S$ s.t. $f(s) = t$

$$R(f) = T \Longleftrightarrow \forall t \in T, \exists s \in S \text{ s.t. } f(s) = t$$



BASIC PROPERTIES

$f : S \longrightarrow T$ FUNCTION

- INJECTIVE : $\underset{S}{\underset{\cap}{s_1}} \neq \underset{S}{\underset{\cap}{s_2}} \Rightarrow f(s_1) \neq f(s_2)$

- SURJECTIVE : $\forall t \in T, \exists s \in S$ s.t. $f(s) = t$

  $R(f) = T \iff f(s) = t$

- BIJECTIVE :

BASIC PROPERTIES

$$f : S \longrightarrow T \quad \text{FUNCTION}$$

- INJECTIVE : $\underset{S}{s_1} \neq \underset{S}{s_2} \Rightarrow f(s_1) \neq f(s_2)$
  ($s_1, s_2 \in S$)

- SURJECTIVE : $\forall t \in T, \exists s \in S$ s.t. $f(s) = t$

$$R(f) = T \iff f(s) = t$$

- BIJECTIVE : INJECT. + SURJECT.

$$\Updownarrow$$

$$\forall t \in T, \exists! \, s \in S \text{ s.t. } f(s) = t$$

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$f : S \longrightarrow T$ FUNCTION

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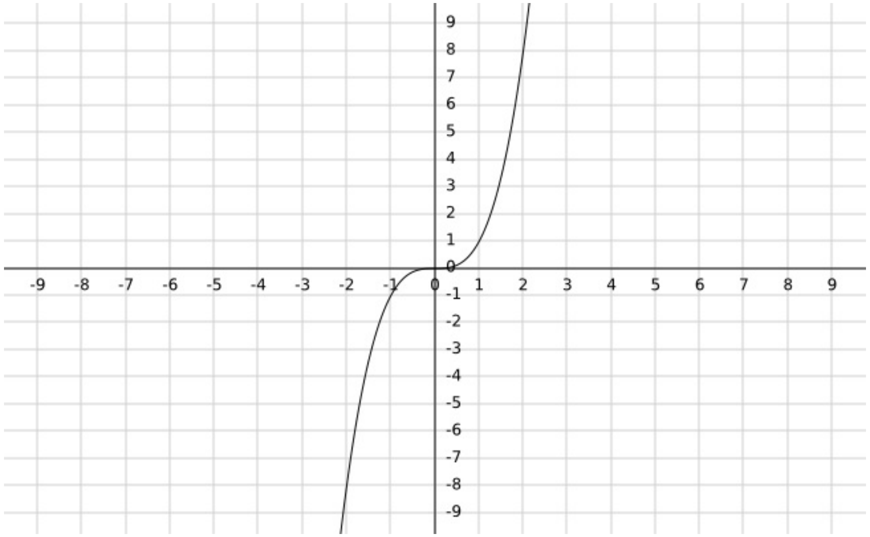
  $\forall t \in T, \exists!\, s \in S$ s.t. $f(s) = t$



$$f(x) = x^3$$

INJ + SURJ.

$\Downarrow$

BIJ.

$f : \mathbb{R} \longrightarrow \mathbb{R} \rightsquigarrow$

$$f(x) = |x^3|$$

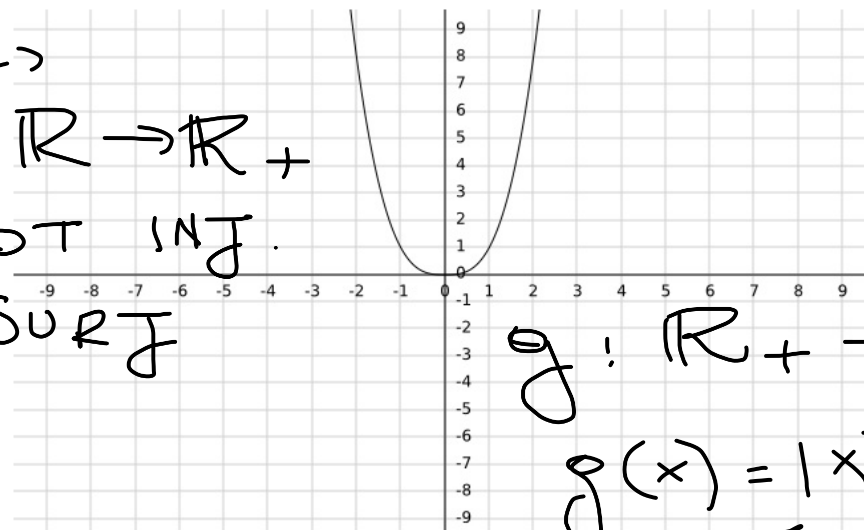
NOT INJ.

NOT SURJ.

$f : \mathbb{R} \to \mathbb{R}_+$

NOT INJ.

SURJ



$g : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$

$g(x) = |x^3|$

INJ. + SURJ.

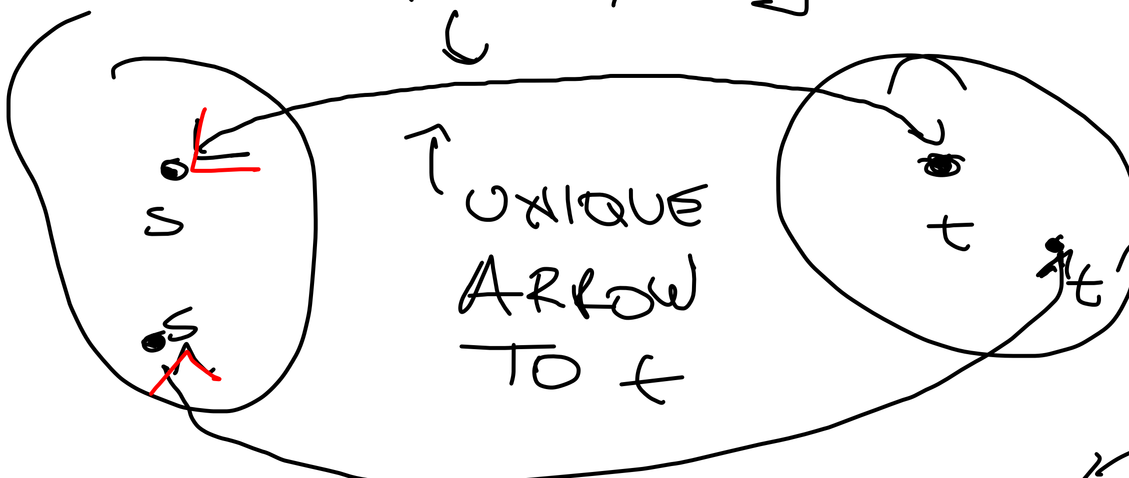
$f : S \longrightarrow T$ BIJECTIVE

EXISTS (INJ + SURJ)

& IT IS UNIQUE

$\forall t \in T, \exists!\ s \in S$ s.t. $f(s) = t$

ALWAYS $\exists$



s.t. $f(s) = t$

$f^{-1} : T \longrightarrow S \qquad f^{-1}(t) = s$

$$f^{-1}(\underset{\underset{t}{\shortparallel}}{f(s)}) = s$$

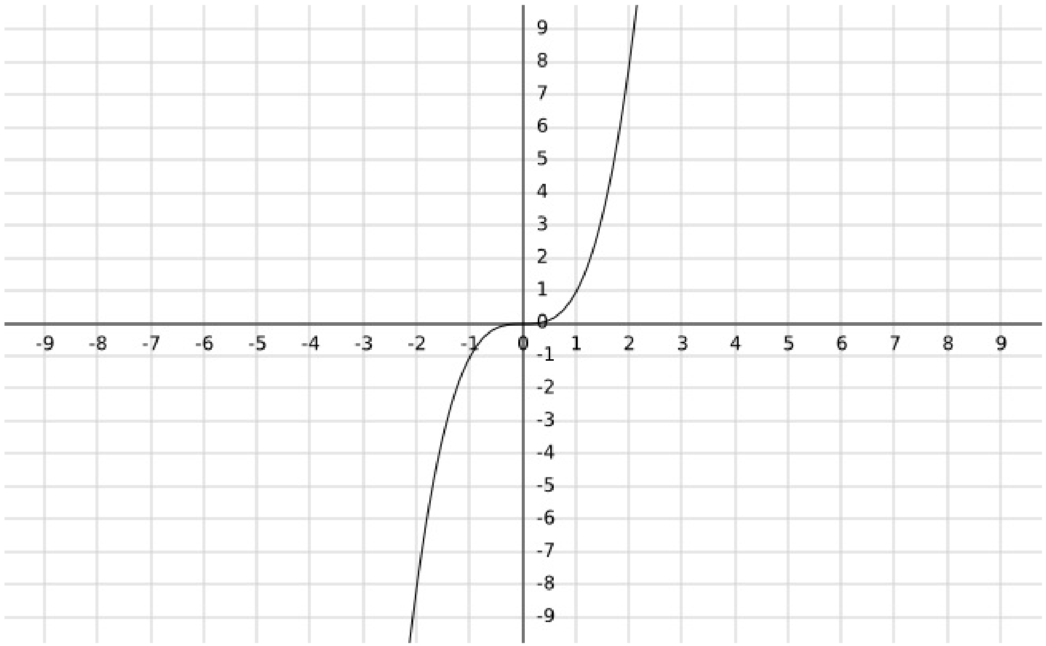
$$f(\underset{\underset{s \text{ s.t } f(s)=t}{\shortparallel}}{f^{-1}(t)}) = t$$

$$\Longrightarrow \quad f^{-1} \circ f = id_S$$

$$f \circ f^{-1} = id_T$$

$$id_S : S \longrightarrow S$$

$$id_S(s) = s$$



$f(x) = x^3$

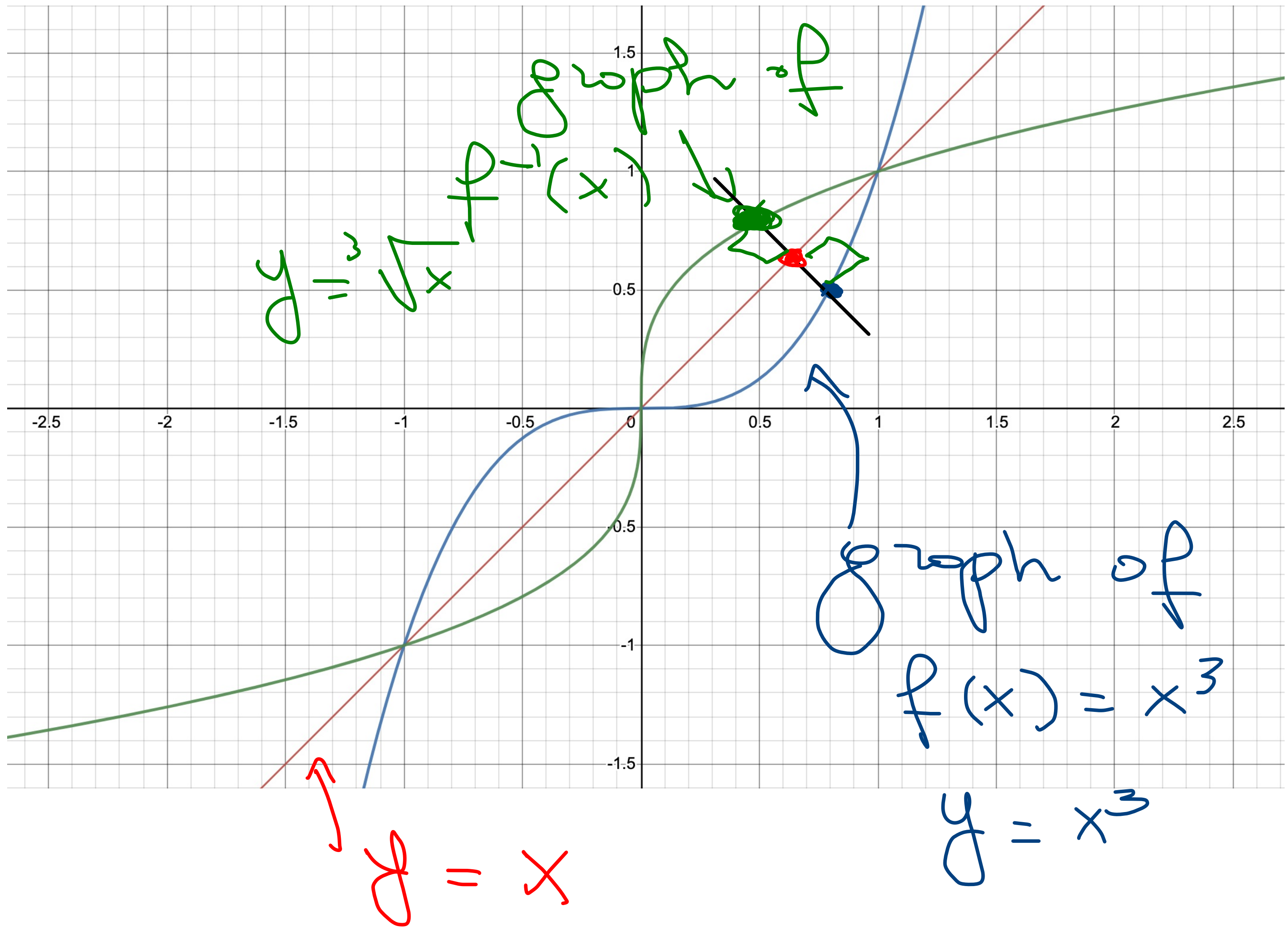
$f : \overset{S}{\mathbb{R}} \to \overset{T}{\mathbb{R}}$

f is BIJ.

INVERSE of f

$f^{-1} : \overset{T}{\mathbb{R}} \to \overset{S}{\mathbb{R}}$

$f^{-1}(x) = \underset{\sqrt[3]{x}}{\underset{\shortparallel}{y}}$ s.t. $f(y) = y^3 = x$


graph of $f^{-1}(x)$
$y = \sqrt[3]{x}$
graph of
$f(x) = x^3$
$y = x^3$
$y = x$
1.5
1
0.5
-0.5
-1
-1.5
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5