ANALYSIS 1

(ENGLISH)

LECTURE 11

TODAY

1. SQUEEZE
2. EXAMPLES
3. ROOT, QUOTIENT CRITERIA

IN PARTICULAR:

PROP $\sum_{i=0}^{\infty} x_i$ CONVERGENT

$$\Downarrow \qquad \cancel{\Uparrow} \leftarrow \left[\sum_{i=1}^{\infty} \frac{1}{i}\right]$$

NON CONVERGING

$$x_i \xrightarrow{i \to \infty} 0$$

$$\lim_{n \to \infty} S_n = +\infty$$

$$S_n = \sum_{i=1}^{n} \frac{1}{i}$$

$$S_{n+1} = S_n + \frac{1}{n+1} > S_n$$

EXAMPLE $\sum_{i=1}^{\infty} \frac{1}{i}$ HARMONIC SERIES

DOES IT CONVERGE? $\left[\frac{1}{i} \xrightarrow{i\to\infty} 0\right]$

IF IT CONVERGES THEN

③ $\forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}$ S.T. $\forall n, m \geq n_\varepsilon$

$n < m$ $\quad |S_n - S_m| = \left|\sum_{i=n+1}^{m} \frac{1}{i}\right| \leq \varepsilon$

Fix $\varepsilon = \frac{1}{2}$, $\exists n_{1/2}$ $\Downarrow$ TAKE $m = 2n$ $\quad n \geq n_{1/2}$

$$\frac{1}{2} \geq \sum_{i=n+1}^{2n} \frac{1}{i} \geq \sum_{i=n+1}^{2n} \frac{1}{2n} = \frac{1}{2}$$

$\frac{1}{i} \geq \frac{1}{2n}$

PROP (x_n) S.T. $\exists n_0 \in \mathbb{N} \mid x_n \geq 0 \quad \forall n \geq n_0$

$\Longrightarrow \sum_{i=0}^{\infty} x_i$ CONVERGES

IF AND ONLY IF (S_n) BOUNDED.

PROP (x_n) S.T. $x_n \geq 0 \ \forall n \geq n_0$

$\Rightarrow \sum_{i=0}^{\infty} x_i$ CONVERGES

IF AND ONLY IF (S_n) BOUNDED.

PROOF $\forall \ \underline{\underline{n \geq n_0}}$

$$S_{n+1} - S_n = x_{n+1} \geq 0$$

$$\overset{\shortparallel}{\sum_{i=0}^{n+1} x_i} - \sum_{i=0}^{n} x_i$$

STARTING AT

$(*)$ n_0 $(S_n)_{n \geq n_0}$

INCREASING

(S_n) CONV $\overset{(*)}{\Longleftrightarrow}$ (S_n) BOUNDED

EXAMPLE

$$x_n = \frac{1}{2^n} \qquad n \geq 0$$

$$\text{IF } \sum_{i=0}^{\infty} x_i \text{ CONV} \implies x_i \xrightarrow{i\to\infty} 0$$

$$x_n = 2^{-n} \xrightarrow{i\to\infty} 0$$

$$\forall n \geq 0,\ x_n \geq 0 \stackrel{\text{PROP}}{\implies}$$

$$\sum_{i=0}^{\infty} 2^{-n} \text{ CONV} \iff (S_n) \text{ BOUNDED}$$

$$S_n = \sum_{i=0}^{n} \left(\frac{1}{2}\right)^n = \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}} \quad \forall n \quad \leq \frac{1}{1-\frac{1}{2}} = 2$$

$$\implies (S_n) \text{ IS BOUNDED}$$

S_n CONVERGES

$\left(\overset{\text{DEF}}{\iff} \sum_{i=0}^{\infty} 2^{-i} \text{ CONVERGES} \right)$

$$S_n = \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}}$$

$$\lim S_n = \lim_{n \to \infty} 2 \cdot \left(1 - \left(\frac{1}{2}\right)^{n+1}\right)$$

$n \to \infty \Rightarrow 0$

$$\sum_{i=0}^{\infty} 2^{-i} = 2$$

SQUEEZE THM (DIFFERENT THAN SQUEEZE FOR SEQ.)

PROP $(x_n), (y_n)$

$\exists\, n_0 \in \mathbb{N}$ s.t. $0 \leq x_n \leq y_n \quad \forall n \geq n_0$

SQUEEZE THM (DIFFERENT THAN SQUEEZE FOR SEQ.)

PROP

$(x_n), (y_n)$

$$0 \leq x_n \leq y_n \quad \forall n \geq n_0$$

THEN ① IF $\sum_{i=0}^{\infty} y_i$ CONVERGES

$\Longrightarrow \sum_{i=0}^{\infty} x_i$ CONVERGES

② IF $\sum_{i=0}^{\infty} x_i$ DOES NOT CONV $\left[\sum_{i=0}^{\infty} x_i = +\infty\right]$

$\Longrightarrow \sum_{i=0}^{\infty} y_i = +\infty$

EXAMPLE

$$x_n = \frac{1}{\sqrt{n}}$$

DOES $\displaystyle\sum_{i=1}^{\infty} \frac{1}{\sqrt{i}}$ CONVERGE? [POLL]

TRUE

FALSE

$$\forall i \in \mathbb{N}, \quad \sqrt{i} \le i = (\sqrt{i})^2 \quad (\Leftarrow \sqrt{i} \ge 0)$$

THEN $y_i = \dfrac{1}{\sqrt{i}} \ge \dfrac{1}{i} = x_i \quad \forall i \ge 1$

$$\sum \frac{1}{i} = +\infty \implies \sum \frac{1}{\sqrt{i}} = +\infty$$

SQUEEZE THM (DIFFERENT THAN SQUEEZE FOR SEQ.)

PROP $(x_n), (y_n)$

$$0 \leq x_n \leq y_n \quad \forall n \geq n_0$$

THEN ① IF $\sum_{i=0}^{\infty} y_i$ CONVERGES

$\Longrightarrow \sum_{i=0}^{\infty} x_i$ CONVERGES

USE CAUCHY SEQ. CARACT. OF SERIES

② IF $\sum_{i=0}^{\infty} x_i = +\infty$

$\Longrightarrow \sum_{i=0}^{\infty} y_i = +\infty$

IF $\sum y_i$ CONVERGES $\Longrightarrow$ FOR FIXED $\varepsilon > 0$

$\exists\, n_\varepsilon \in \mathbb{N}$ S.T. $\forall\, m > n \geq n_\varepsilon$

$* \left| S_m^{y_i} - S_n^{y_i} \right| \leq \varepsilon$

$\forall\, i \geq n_0$

$$= \sum_{i=n+1}^{m} y_i$$

$y_i \geq x_i \geq 0 \quad \forall\, i \gg 0$

$n \gg 0$

$$\geq \sum_{i=n+1}^{m} x_i$$

$\forall\, m > n \geq n'_\varepsilon$

$$= \max\{n_\varepsilon, n_0\}$$

$* \left| S_m^{x_i} - S_n^{x_i} \right| \leq \varepsilon$

EXAMPLE

$$2^{1/2} = \sqrt{2}$$

$$2^{\frac{1}{n}} = \sqrt[n]{2} \quad \left(\text{SOL'N TO } x^n = 2 \right)$$

$$\mathbb{R}_+^* \ni b^{p/q} = \sqrt[q]{b^p} \quad \left(\text{SOL'N TO } x^q = b^p \right)$$

EXAMPLE

$$2^{1/2} = \sqrt{2}$$

$$2^{\frac{1}{n}} = \sqrt[n]{2} \qquad (\text{SOL'N TO } x^n = 2)$$

$$\mathbb{R}_+^* \ni b^{p/q} = \sqrt[q]{b^p} \qquad (\text{SOL'N TO } x^q = b^p)$$

IN GENERAL

$$\mathbb{R}_+^* \ni b^s, \quad s \in \mathbb{R}$$

EXPONENTIAL WITH BASE b & POWER s $\left[b^s \doteq \exp((\log_e b) \cdot s) \right]$

$e^{s \cdot \log b}$

$\log_{10}$

$\log = \log_e = \ln$

IN GENERAL $\mathbb{R}_+^* \ni b^s$, $s \in \mathbb{R}$

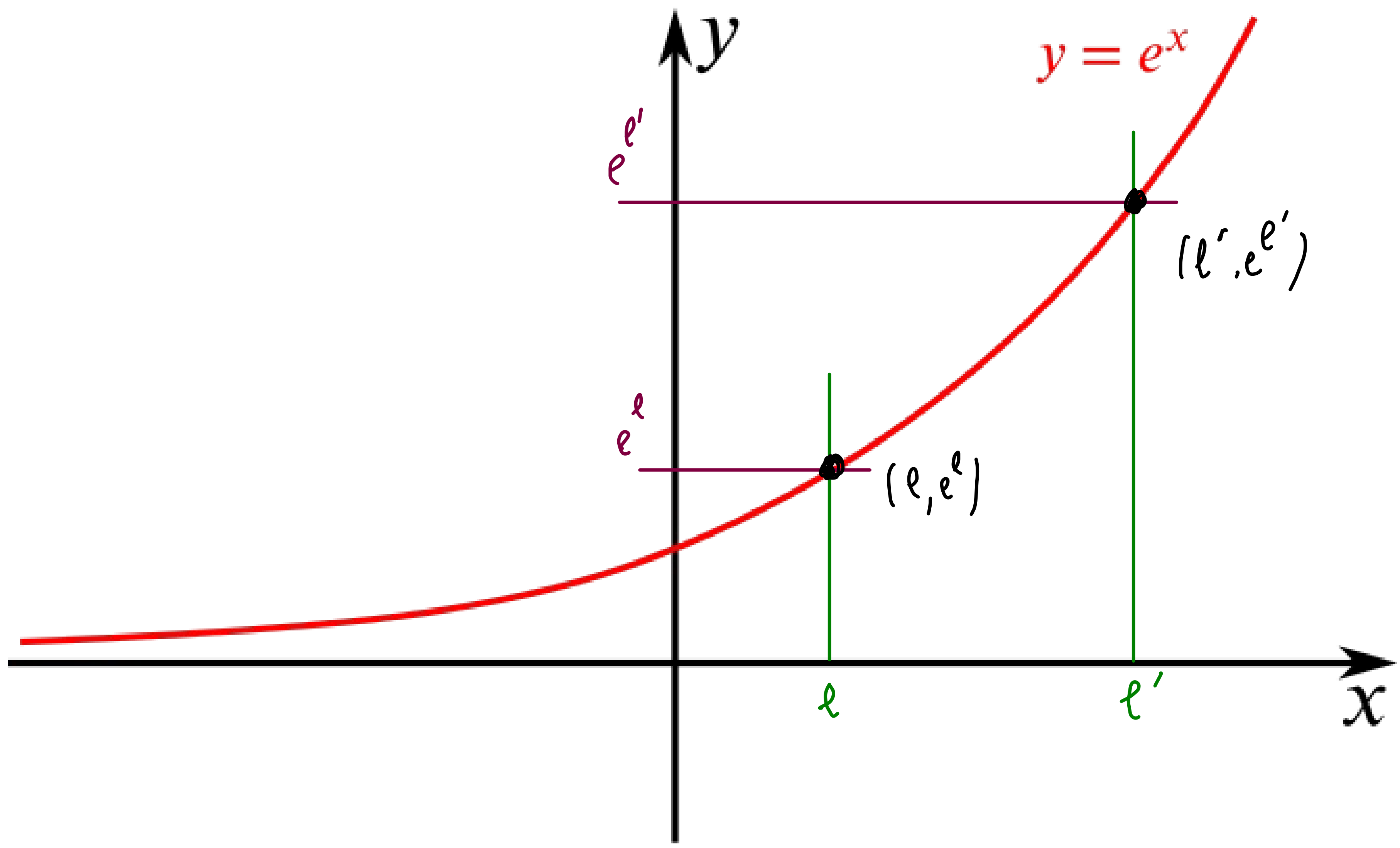
IF $b > 1 \Longrightarrow b^s$ INCREASING WITH RESPECT TO s

$$\boxed{t > s \Longrightarrow b^t > b^s}$$

$$b^t = e^{t \cdot \log b} \qquad b^s = e^{s \cdot \log b}$$

$$\left[\, b > 1 \Longrightarrow \log b > 1 \,\right]$$

ENOUGH TO KNOW THAT e^{ℓ} IS INCREASING FUNCTION OF ℓ ($\ell' > \ell \Longrightarrow e^{\ell'} > e^{\ell}$)



y
y = e^x
e^ℓ'
e^ℓ
(ℓ', e^ℓ')
(ℓ, e^ℓ)
ℓ
ℓ'
x

IN GENERAL $\mathbb{R}_+^* \ni b^s$, $s \in \mathbb{R}$

IF $b > 1 \Longrightarrow b^s$ INCREASING WITH RESPECT TO s

$$\boxed{t > s \Longrightarrow b^t > b^s}$$

EXAMPLE

$$x_n = \frac{1}{n^s}, \quad s \in [0,1)$$

Q: $\sum_{i=1}^{\infty} x_i$ CONVERGES?

[THE ANSWER MAY DEPEND ON s]

$$\frac{1}{n^s} \longrightarrow n^s \underset{n>1}{\leq} n^1$$

$$n^s = e^{s \cdot \log n} \leq \quad n>1,\ s<1$$

$$\frac{1}{n} \leq \frac{1}{n^s} \quad \forall n > 1$$

SQUEEZE THM: $\sum_{i=1}^{\infty} \frac{1}{n} = +\infty \Longrightarrow \sum \frac{1}{n^s} = +\infty$

SQUEEZE THM (DIFFERENT THAN SQUEEZE FOR SEQ.)

PROP $(x_n), (y_n)$

$$0 \leq x_n \leq y_n \quad \forall n \geq n_0$$

THEN ① IF $\sum_{i=0}^{\infty} y_i$ CONVERGES

$\Longrightarrow \sum_{i=0}^{\infty} x_i$ CONVERGES

USE CAUCHY SEQ. CARACT. OF SERIES

② IF $\sum_{i=0}^{\infty} x_i$ DIVERGES

$\Longrightarrow \sum_{i=0}^{\infty} y_i$ DIVERGES

EXAMPLE $\sum_{i=0}^{\infty} \frac{1}{i!} = e$

① DOES IT CONVERGE?

$0 \leqslant \frac{1}{i!} \leqslant \frac{1}{i} \overset{\text{SQUEEZE}}{\Longrightarrow} \frac{1}{i} \xrightarrow{i \to \infty} 0$

WE NEED TO SHOW: (S_n) BOUNDED

IF $i \geqslant 2 \Longrightarrow \frac{1}{i!} \leqslant \frac{1}{2^{i-1}}$ &

$\sum_{i=1}^{\infty} \frac{1}{2^{i-1}}$ CONV. $\overset{\text{SQUEEZE}}{\Longrightarrow}$

EXAMPLE

$$x_n = \frac{1}{n^s}, \quad s > 1$$

$$S_n = \sum_{k=1}^{n} \frac{1}{k^s} \leq \sum_{k=1}^{2n+1} \frac{1}{k^s} = S_{2n+1}$$

$$= \underbrace{1}_{k=1} + \underbrace{\sum_{k=1}^{n} \frac{1}{(2k)^s}}_{\text{even } k\text{'s}} + \underbrace{\sum_{k=1}^{n} \frac{1}{(2k+1)^s}}_{\substack{\text{odd indices} \\ > 1}}$$

EXAMPLE

$$x_n = \frac{1}{n^s}, \quad s > 1$$

$$S_n = \sum_{k=1}^{n} \frac{1}{k^s} \le \sum_{k=1}^{2n+1} \frac{1}{k^s}$$

$$= 1 + \sum_{k=1}^{n} \frac{1}{(2k)^s} + \sum_{k=1}^{n} \frac{1}{(2k+1)^s}$$

$$= \frac{1}{2^s} \cdot \frac{1}{k^s}$$

$$\sum_{k=1}^{n} \frac{1}{(2k)^s} = \frac{1}{2^s} \cdot \sum_{k=1}^{n} \frac{1}{k^s} = \frac{1}{2^s} \cdot S_n$$

$$\sum_{k=1}^{n} \frac{1}{(2k+1)^s} \le \sum_{k=1}^{n} \frac{1}{(2k)^s} = \frac{1}{2^s} \cdot S_n$$

$$a > b > 1 \Longrightarrow a^s > b^s$$

$$s > 0$$

$$0 \leq S_n \leq S_{2n+1}$$

$S_n = \sum$ positive term

$S_{2n+1} = 1 + \sum$ even indices $+ \sum$ odd indices

$$\geq 1 + \frac{1}{2^s} S_n + \frac{1}{2^s} S_n$$

$$= 1 + \frac{1}{2^{s-1}} S_n$$

$$\geq 0 \quad S_n \leq 1 + \frac{1}{2^{s-1}} S_n$$

$$\Updownarrow$$

$$0 \leq S_n \left(1 - \frac{1}{2^{s-1}}\right) \leq 1$$

$$s > 1$$

$$\Updownarrow$$

$$(s-1) > 0$$

$$0 \le S_n \le \frac{1}{\left(1 - \frac{1}{2^{s-1}}\right)}$$

INDEP of n

$\Rightarrow (S_n)$ is BOUNDED

STR. INCR.

$\Rightarrow \sum_{i=1}^{\infty} \frac{1}{i^s}$ is CONV.

$\forall s > 1$

$$\sum_{i=1}^{\infty} \frac{1}{i^s} \quad \begin{cases} 0 \le s \le 1 & = +\infty \\ s > 1 & \text{CONV.} \end{cases}$$

DEF $\sum_{i=0}^{\infty} x_i$ CONVERGES ABSOLUTELY

IF $\sum_{i=0}^{\infty} |x_i|$ CONVERGES

ABS. VALUE OF THE x_i

ALL TERMS ARE ≥ 0

DEF $\sum_{i=0}^{\infty} x_i$ CONVERGES ABSOLUTELY

IF $\sum_{i=0}^{\infty} |x_i|$ CONVERGES

EXAMPLE $\sum_{i=1}^{\infty} \frac{(-1)^i}{i}$ DOES NOT CONV. ABS

$$\sum_{i=1}^{\infty} \left|\frac{(-1)^i}{i}\right| = \sum_{i=1}^{\infty} \frac{1}{i} = +\infty$$

HARM. SERIES

$$\sum_{i=1}^{\infty} \frac{(-1)^i}{i^2} \longrightarrow \sum_{i=1}^{\infty} \left|\frac{(-1)^i}{i^2}\right| = \sum_{i=1}^{\infty} \frac{1}{i^2}$$

ABSOLUTELY CONVERGENT.

CONVERGES

PROP (x_n) SEQUENCE

$\sum_{i=0}^{\infty} x_i$ CONVERGES ABSOLUTELY $\left(\Longleftrightarrow \sum_{i=0}^{\infty} |x_i| \text{ CONVERG.} \right)$

$\Longrightarrow \sum_{i=0}^{\infty} x_i$ CONVERGES

PROP (x_n) SEQUENCE

$\sum_{i=0}^{\infty} x_i$ CONVERGES ABSOLUTELY $\left(\Leftrightarrow \sum_{i=0}^{\infty} |x_i| \text{ CONVERG.} \right)$

$\Rightarrow \sum_{i=0}^{\infty} x_i$ CONVERGES

PROOF $\sum_{i=0}^{\infty} x_i$ CONVERGES $\Leftrightarrow (S_n)$ IS CAUCHY

$\sum_{i=0}^{\infty} |x_i|$ CONVERGES $\Rightarrow$ FIX $\varepsilon > 0 \Rightarrow \exists n_\varepsilon \in \mathbb{N}$ S.T. $\forall m > n \geq n_\varepsilon$

$\varepsilon \geq |S_m^{|\cdot|} - S_n^{|\cdot|}| = \sum_{i=n+1}^{m} |x_i|$

$\vee|$ $\triangle$ ineq.

$\varepsilon \geq |S_m - S_n| = |\sum_{i=n+1}^{m} x_i|$

(x_n) SEQ. $\underline{Q}$: $\sum_{i=0}^{\infty} x_n$ CONVERGES?

(x_n) SEQ. **Q**: $\sum_{i=0}^{\infty} x_n$ CONVERGES?

$$\boxed{x_n \xrightarrow{n\to\infty} 0\ ?}$$

(x_n) SEQ. **Q**: $\sum_{i=0}^{\infty} x_n$ CONVERGES?

$x_n \xrightarrow{n\to\infty} 0$?

NO

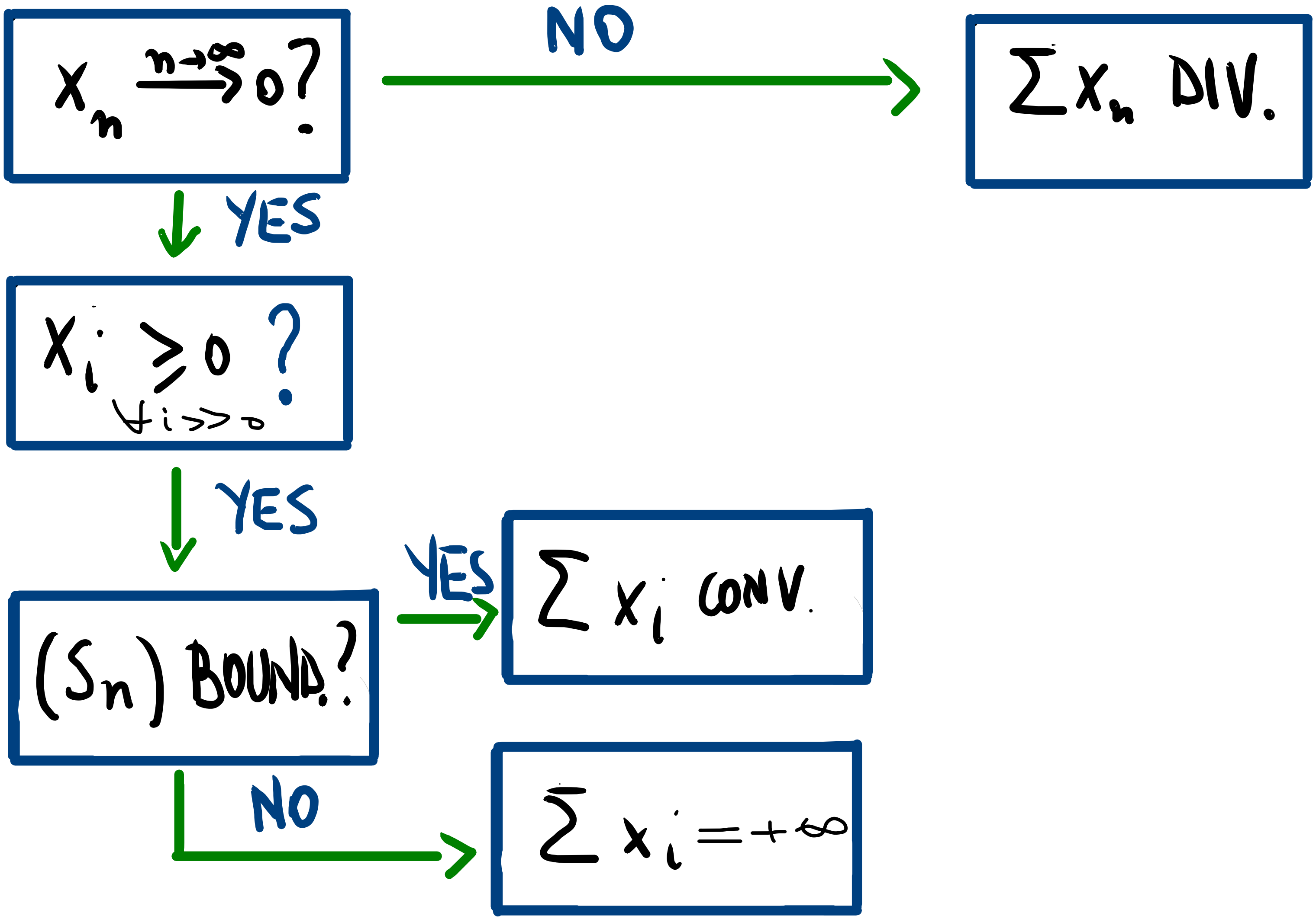
$\sum x_n$ NOT CONV.

YES

$x_i \geq 0$?

$\forall i \geq 0$

(x_n) SEQ. Q: $\sum_{i=0}^{\infty} x_n$ CONVERGES?



(x_n) SEQ. **Q:** $\sum_{i=0}^{\infty} x_n$ CONVERGES?

- $x_n \xrightarrow{n\to\infty} 0$?
  - NO → $\sum x_n$ DIV.
  - YES → $x_i \geq 0$? $\forall i \gg 0$
    - YES → (S_n) BOUND.?
      - YES → $\sum x_i$ CONV.
      - NO → $\sum x_i = +\infty$
    - NO → $\sum_{i=0}^{\infty} x_i$ CONV. ABS.?

      $\sum_{i=0}^{\infty} |x_i|$ CONV.?

      $\Updownarrow$
      - YES → $\sum x_i$ CONV.
      - NO → WE NEED NEW CRITERIA.

EXAMPLE

$$x_n = \frac{(-1)^n}{n}$$

$\sum_{i=1}^{\infty} x_i$ DOES NOT CONV. ABSOLUTELY

[$\sum |x_i|$ IS THE HARM. SERIES]

EXAMPLE

$$x_n = \frac{(-1)^n}{n}$$

$\sum_{i=1}^{\infty} x_i$ DOES NOT CONV. ABSOLUTELY

$S_1 = -1$, $S_2 = -1 + \frac{1}{2} = -\frac{1}{2}$, $S_3 = -1 + \frac{1}{2} - \frac{1}{3}$

$= -\frac{5}{6}$

EXAMPLE

$$x_n = \frac{(-1)^n}{n}$$

$\sum_{i=1}^{\infty} x_i$ DOES NOT CONV. ABSOLUTELY

$$S_1 = -1 < S_2 = -1 + \frac{1}{2} = -\frac{1}{2} > S_3 = -\frac{1}{2} - \frac{1}{3}$$

$$S_{2k-1} < S_{2k} > S_{2k+1} \quad \forall k$$

S_n IS NEVER MONOTONE

$$S_{2k} = S_{2k-1} + \frac{1}{2k} > S_{2k-1}$$

$$S_{2k+1} = S_{2k} - \frac{1}{2k+1} < S_{2k}$$

$\Vert$ (S_n) NOT MONOT.

EXAMPLE

$$x_n = \frac{(-1)^n}{n}$$

$$-\frac{1}{2k+1} + \frac{1}{2k} = \frac{2k+1-2k}{2k(2k+1)} = \frac{1}{2k(2k+1)}$$

$$S_{2n} = -1 + \left(\sum_{i=1}^{n-1} \frac{1}{2i} - \frac{1}{2i+1}\right) + \frac{1}{2n}$$

$$= -1 + \sum_{i=1}^{n-1} \frac{1}{2i(2i+1)} + \frac{1}{2n}$$

$$S_{2n+1} = -1 + \left(\sum_{i=1}^{n} \frac{1}{2i} - \frac{1}{2i+1}\right) = -1 + \sum_{i=1}^{n} \frac{1}{2i(2i+1)}$$

EXAMPLE

$$x_n = \frac{(-1)^n}{n}$$

$\sum \frac{(-1)^i}{i}$ CONVERG.

$$S_{2n} = -1 + \left(\sum_{i=1}^{n-1} \frac{1}{2i} - \frac{1}{2i+1} \right) + \frac{1}{2n}$$

$$= -1 + \sum_{i=1}^{n-1} \frac{1}{2i(2i+1)} + \frac{1}{2n}$$

$$S_{2n+1} = -1 + \left(\sum_{i=1}^{n} \frac{1}{2i} - \frac{1}{2i+1} \right) = -1 + \sum_{i=1}^{n} \frac{1}{2i(2i+1)}$$

$$0 \le \sum_{i=1}^{\infty} \frac{1}{2i(2i+1)} \le \sum_{i=1}^{\infty} \frac{1}{4i^2}$$

CONVERG.

TRUNCATED SUMS FOR

IN GENERAL : LEIBNIZ CRITERION

THEOREM (x_n) STR. DECREASING

& $\lim\limits_{n\to\infty} x_n = 0$

$$\left[\; x_n > 0 \quad \forall n \;\right]$$

IN GENERAL: LEIBNIZ CRITERION

THEOREM (x_n) STR. DECREASING

& $\lim_{n\to\infty} x_n = 0$

$$\Longrightarrow \sum_{i=0}^{\infty} (-1)^i x_i \quad \text{CONVERGES}.$$

EXAMPLE

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$$

AUTOMATIC BY LEIBNIZ CRIT.

$x_n = \frac{1}{n}$ STRICTLY DECR.

$$x_n \xrightarrow{n\to\infty} 0$$

$\overset{L.C.}{\Longrightarrow}$ $\sum (-1)^i x_i$ is CONVERGENT.

$$\sum (-1)^i x_i = \sum \frac{(-1)^i}{i}$$

EXAMPLE

$$\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{n}\right)$$

EXAMPLE

$$\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{n}\right)$$

$x_n \doteq \sin\left(\frac{1}{n}\right) \quad n \geq 1$

STR.

IS (x_n) DECREASING? [POLL]

YES

NO

EXAMPLE

$$\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{n}\right)$$

L.C. $\Rightarrow$ CONVERGES

$x_n \doteq \sin\left(\frac{1}{n}\right)$

STR.

IS (x_n) DECREASING? [POLL]

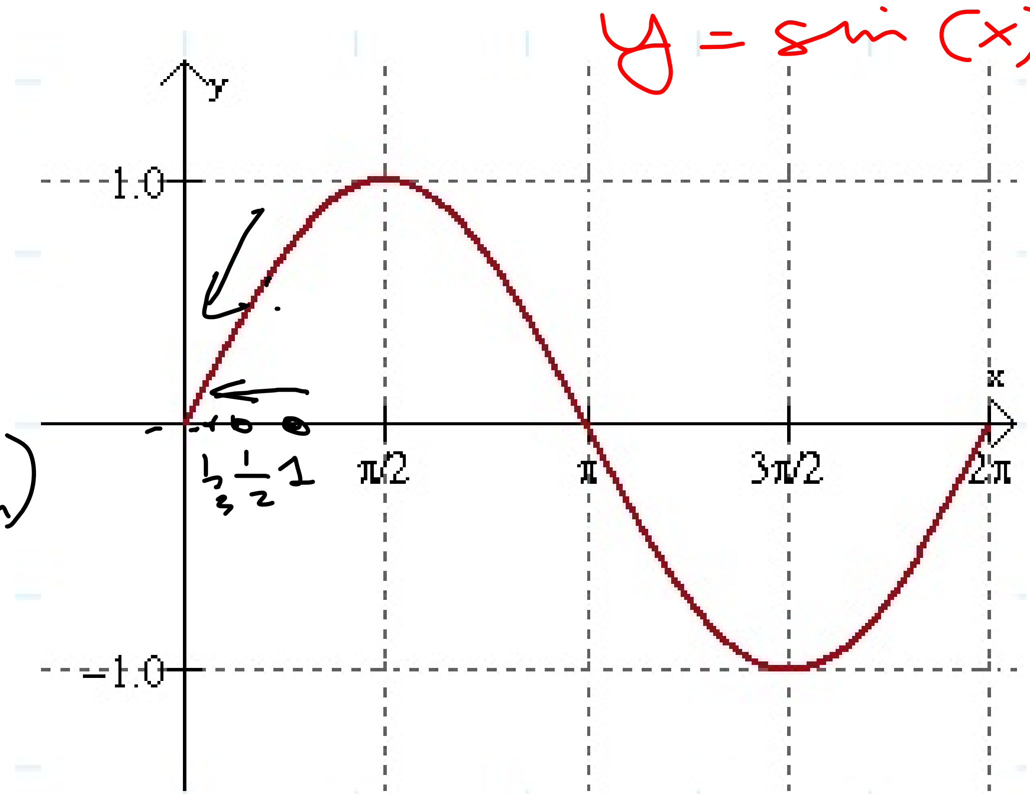
YES

NO

$\sin\left(\frac{1}{n+1}\right) < \sin\left(\frac{1}{n}\right)$

$\downarrow$

0



EXAMPLE

$$\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{n}\right)$$

$$x_n \doteq \sin\left(\frac{1}{n}\right)$$

IS (x_n) DECREASING? YES

LEIBNIZ CRIT. $\Longrightarrow \displaystyle\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{n}\right)$ CONVERGES

EXAMPLE

$$\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{n}\right)$$

$$x_n \doteq \sin\left(\frac{1}{n}\right)$$

IS (x_n) DECREASING? YES

LEIBNIZ CRIT. $\Longrightarrow \sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{n}\right)$ CONVERGES

DOES IT CONVERGE ABSOLUTELY?

Does $\sum_{i=1}^{\infty} \sin\left(\frac{1}{n}\right)$ CONVERGES?

EXAMPLE

$$\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{n}\right)$$

$$x_n \doteq \sin\left(\frac{1}{n}\right)$$

IS (x_n) DECREASING? YES

LEIBNIZ CRIT. $\Longrightarrow \sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{n}\right)$ CONVERGES

DOES IT CONVERGE ABSOLUTELY?

$$\sum_{n=1}^{\infty} \left| (-1)^n \sin\left(\frac{1}{n}\right) \right| = \sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$$

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$$

WE WILL SEE THAT :

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)^{\to 0}}{\underset{\to 0}{\frac{1}{n}}} = 1 \qquad \left[\lim_{x\to 0} \frac{\sin(x)}{x} = 1\right]$$

$$\frac{1}{2} \leq \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} \quad \forall n \gg 0$$

$$\underline{\frac{1}{2n} \leq \sin\left(\frac{1}{n}\right)} \qquad \underline{\forall n \geq n_0}$$

$$\sum \frac{1}{2i} = +\infty \overset{\text{SQUEEZE}}{\Longrightarrow} \sum \sin\left(\frac{1}{n}\right) = +\infty$$

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$$

WE WILL SEE THAT:

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1$$

THEN $\forall n \gg 0$

SQUEEZE THM (DIFFERENT THAN SQUEEZE FOR SEQ.)

PROP $(x_n), (y_n)$

$$0 \leq x_n \leq y_n \quad \forall n \geq n_0$$

THEN ① IF $\sum_{i=0}^{\infty} y_i$ CONVERGES

$\Longrightarrow \sum_{i=0}^{\infty} x_i$ CONVERGES

② IF $\sum_{i=0}^{\infty} x_i$ DIVERGES

$\Longrightarrow \sum_{i=0}^{\infty} y_i$ DIVERGES

EXAMPLE

$$\sum_{i=0}^{\infty} \frac{1}{i!} = e$$

(x_n) SEQ. **Q**: $\sum_{i=0}^{\infty} x_n$ CONVERGES?

$\boxed{x_n \xrightarrow{n\to\infty} 0\,?}$ $\xrightarrow{\text{NO}}$ $\boxed{\sum x_n \text{ DIV.}}$

$\downarrow$ YES

EXAMPLE

$$\sum_{i=0}^{\infty} \frac{1}{i!} = e$$

① DOES $x_i := \frac{1}{i!}$ CONVERGES?

$$0 < \frac{1}{i!} \leq \frac{1}{i} \xRightarrow[\text{SEQ.}]{\text{SQUEEZE}} \lim \frac{1}{i!} = 0$$

$$i! = i \cdot (i-1)(i-2) \dots 2 \cdot 1$$

(x_n) SEQ. Q: $\sum_{i=0}^{\infty} x_n$ CONVERGES?

$x_n \xrightarrow{n\to\infty} 0$?

NO → $\sum x_n$ DIV.

YES ↓

$x_i \geq 0$?

YES ↓

(S_n) BOUND.?

YES → $\sum x_i$ CONV.

NO → $\sum x_i$ DIV

EXAMPLE $\sum_{i=0}^{\infty} \frac{1}{i!} = e$

① DOES IT CONVERGE?

$$0 < \frac{1}{i!} \leq \frac{1}{i} \overset{\text{SQUEEZE}}{\Longrightarrow} \frac{1}{i} \xrightarrow{i\to\infty} 0$$

WE NEED TO SHOW:

EXAMPLE $\sum_{i=0}^{\infty} \frac{1}{i!} = e$

① DOES IT CONVERGE?

$0 < \frac{1}{i!} \leq \frac{1}{i} \xRightarrow{\text{SQUEEZE}} \frac{1}{i} \xrightarrow{i \to \infty} 0$

WE NEED TO SHOW: (S_n) BOUNDED

EXAMPLE

$$\sum_{i=0}^{\infty} \frac{1}{i!} = e$$

① DOES IT CONVERGE?

$$0 \leq \frac{1}{i!} \leq \frac{1}{i} \underset{\text{SQUEEZE}}{\Longrightarrow} \frac{1}{i} \xrightarrow{i\to\infty} 0$$

WE NEED TO SHOW: (S_n) BOUNDED

IF $i \geq 2 \Longrightarrow \frac{1}{i!} \leq \frac{1}{1} \cdot \frac{1}{2} \cdot \frac{1}{2^{i-2}} = \frac{1}{2^{i-1}}$

$\frac{1}{k} \leq \frac{1}{2} \quad \forall k \geq 2$

$\sum \frac{1}{2^{i-1}}$ CONV. $\Rightarrow \sum \frac{1}{i!}$ CONV.

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