ANALYSIS 1

(ENGLISH)

LECTURE 10

CAUCHY SEQUENCES

EXAMPLE

$$x_n = \left(1 + \frac{1}{n}\right)^n \qquad n \geq 1$$

$$x_n \xrightarrow{n \to \infty} e \longleftarrow$$

def of e as the limit of x_n

CAUCHY SEQUENCES

EXAMPLE $x_n = \left(1 + \frac{1}{n}\right)^n$

$$x_n \xrightarrow{n \to \infty} e$$

HOW DID WE PROVE IT ?

CAUCHY SEQUENCES

EXAMPLE

$$x_n = \left(1 + \frac{1}{n}\right)^n$$

$$x_n \xrightarrow{n \to \infty} e$$

HOW DID WE PROVE IT?

1. (x_n) IS STRICTLY INCREASING
2. (x_n) IS BOUNDED

$\overset{\text{THM}}{\Longrightarrow}$ LIMIT EXISTS

IN GENERAL: GIVEN (x_n)

IF WE CAN GUESS $x_n \xrightarrow{n \to \infty} x$

THEN WE CAN VERIFY IF

x SATISFIES THE DEF'N

OF LIMIT FOR (x_n)

SAME

EXAMPLE

$$x_n = \frac{1}{n^2} \qquad \lim x_n = 0$$

ARCH. PROP. $\forall \varepsilon > 0$, $\exists n_\varepsilon$ S.T.

$\forall n \geq n_\varepsilon$ $0 \leq \frac{1}{n^2} \leq \varepsilon$ $\qquad n_\varepsilon \doteq \left[\frac{1}{\sqrt{\varepsilon}}\right] + 1$

IN GENERAL: GIVEN (x_n)

IF WE CAN GUESS $x_n \xrightarrow{n \to \infty} x$

SAME

THEN WE CAN VERIFY IF x SATISFIES THE DEF'N OF LIMIT FOR (x_n)

BUT WHAT IF WE CANNOT GUESS WHAT THE LIMIT MAY BE?

BUT WHAT IF WE CANNOT GUESS WHAT THE LIMIT MAY BE?

EXAMPLE

$x_0 = 3$

$x_1 = 3.1$

$x_2 = 3.14$

$x_3 = 3.141$

$x_4 = 3.1415$

$x_5 = 3.14159$

$$x_n \xrightarrow[n\to\infty]{} \pi$$

$$|x_n - \pi| \le 10^{-n}$$

$$= 0.\underbrace{00000\,0\ldots0}_{n\ 0\text{'s}}\ldots$$

x_n = TAKE DECIMAL REPR'N OF π AND TRUNCATE DECIMAL PART AT n-th DIGIT.

DEF (x_n) IS A CAUCHY SEQUENCE

IF $\forall \varepsilon > 0$ THERE IS $n_\varepsilon \in \mathbb{N}$

$\forall n, m \geq n_\varepsilon$ THEN
$\quad \in \mathbb{N}$

$$|x_n - x_m| \leq \varepsilon$$

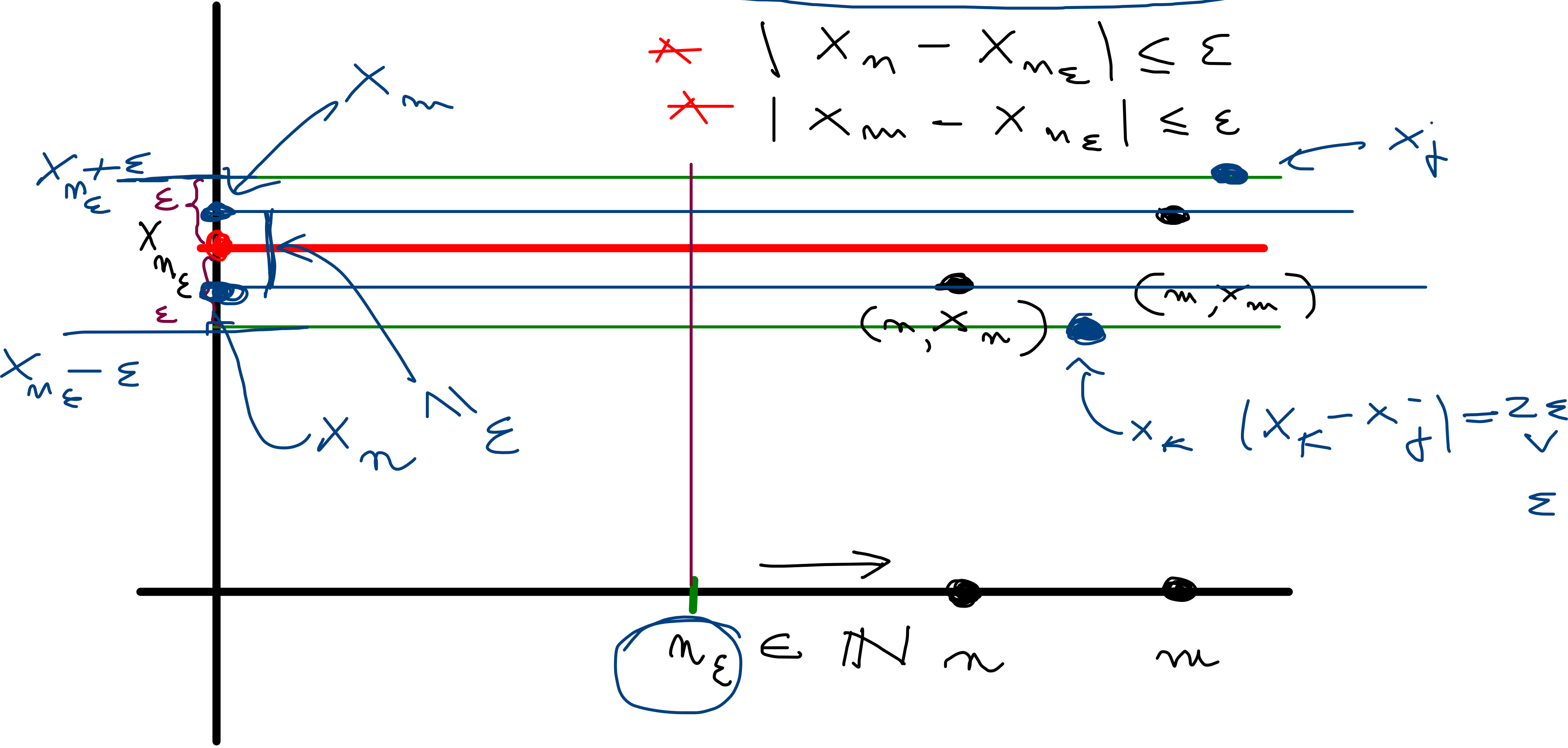
INTUITIVELY: "SELF CONTAINED" DEF'N

Fix $\varepsilon > 0$

$$* \quad |x_n - x_m| \le \varepsilon$$

$$* \quad |x_n - x_{n_\varepsilon}| \le \varepsilon$$

$$* \quad |x_m - x_{n_\varepsilon}| \le \varepsilon$$



EXAMPLE

$x \in \mathbb{R}^*_+$

$x_0 = [x]$, $x_n \doteq$ DECIMAL REPRES. OF x UP TO THE n-TH DECIMAL DIGIT

$$\underset{x_0}{\overset{\shortparallel}{3}}\ ,\ \underset{x_1}{\overset{\shortparallel}{3.1}}\ ,\ \underset{x_2}{\overset{\shortparallel}{3.14}}\ ,\ \underset{x_3}{\overset{\shortparallel}{3.141}}$$

$$x_n \xrightarrow[]{n \to \infty} x$$

CONVERGING SEQ.

EXAMPLE

$x \in \mathbb{R}_+^*$

$x_0 = [x]$, $x_n \doteq$ DECIMAL REPRES. OF x UP TO THE n-TH DECIMAL DIGIT

THE SAME $n \in \mathbb{N}$

$x = \pi \implies$

$$x_0 = 3$$
$$x_1 = 3.1$$
$$x_2 = 3.14$$
$$x_3 = 3.141$$
$$x_4 = 3.1415$$
$$x_5 = 3.14159$$

EXAMPLE

(a_n), $a_n \in \{0, 1, 2, 3, 4, \dots, 9\}$

$n \geq 1$

$x_0 = a_0 \in \mathbb{N}$

$x_1 = a_0 + a_1/10 \longrightarrow a_0 . a_1$

$x_2 = a_0 + a_1/10 + a_2/100 \longrightarrow a_0 . a_1 a_2$

$x_3 = a_0 + a_1/10 + a_2/100 + a_3/1000 \longrightarrow a_0 . a_1 a_2 a_3$

$x_n = x_{n-1} + \underline{\underline{a_n}}/10^n$ $\qquad \underbrace{a_0 . a_1 a_2 a_3 a_4 \dots a_{n-1}}_{x_{n-1}} a_n$

EXAMPLE

(a_n), $a_n \in \{0,1,2,3,4,\ldots,9\}$ $n \geq 1$

$x_0 = a_0$

(x_n) CAUCHY SEQUENCE

$x_n = x_{n-1} + a_n/10^n$

$\forall N \in \mathbb{N}$, IF $\underline{n, m \geq N}$

$\Longrightarrow \quad |x_n - x_m| \leq 10^{-N} \longrightarrow 0$

$N \to \infty$

ASSUME $n < m \Longrightarrow x_n, x_m$ have the first n digits in their decimal part

$$|x_n - x_m| = \underbrace{0.00000\ldots0}_{0's}, a_{n+1} a_{n+2} a_{n+3} \ldots a_m$$

EXAMPLE (a_n), $a_n \in \{0,1,2,3,4,\dots,9\}$

$$x_0 = a_0$$

$$x_n = x_{n-1} + a_n/10^n$$

$\forall N \in \mathbb{N}$, IF $n, m \geq N$

$$\Longrightarrow |x_n - x_m| < 10^{-N}$$

(x_n) CAUCHY

THEOREM (x_n). THEN TFAE

THEOREM (x_n). THEN $\boxed{\text{TFAE}}$

"THE FOLLOWING CONDITIONS ARE LOGICALLY EQUIVALENT."

① (x_n) IS A CAUCHY SEQUENCE

② (x_n) IS CONVERGENT

"(x_n) is conv. IF & ONLY IF (x_n) is CAUCHY"

① $\Longrightarrow$ ②

THEOREM (x_n). THEN $\boxed{\text{TFAE}}$

"THE FOLLOWING CONDITIONS ARE LOGICALLY EQUIVALENT."

1. (x_n) IS A CAUCHY SEQUENCE
2. (x_n) IS CONVERGENT

IMPORTANT:

THEOREM (x_n). THEN $\boxed{\text{TFAE}}$ "THE FOLLOWING CONDITIONS ARE LOGICALLY EQUIVALENT"

① (x_n) IS A CAUCHY SEQUENCE

② (x_n) IS CONVERGENT

IMPORTANT: (x_n) CAUCHY

$$\Downarrow$$

$$x_n \xrightarrow[n \to \infty]{} x \in \mathbb{R}$$

THEOREM (x_n). THEN $\boxed{\text{TFAE}}$

① (x_n) IS A CAUCHY SEQUENCE

② (x_n) IS CONVERGENT

IMPORTANT: (x_n) CAUCHY

$$\Downarrow$$

$$x_n \longrightarrow x \in \mathbb{R}$$

WE MAY BE UNABLE TO COMPUTE IT

BUT IT EXISTS!!

EXAMPLE

$$x_n \doteq \sum_{i=0}^{n} \frac{1}{2^i} \rightsquigarrow x_{n+1} = x_n + \frac{1}{2^{n+1}}$$

(x_n) STRICTLY INCR.

(x_n) is CAUCHY :

Fix $\varepsilon > 0$

$$|x_n - x_m| = \left| \sum_{i=0}^{n} \frac{1}{2^i} - \sum_{i=0}^{m} \frac{1}{2^i} \right| \leq \frac{1}{2^n}$$

$\underline{\underline{n < m}}$

$$\sum_{i=n+1}^{m} \frac{1}{2^i} = \frac{1}{2^{n+1}} \cdot \sum_{j=0}^{m-(n+1)} \frac{1}{2^j}$$

$$2 \cdot \frac{1}{2^{n+1}} \geq$$

$$= \frac{1}{2^n}$$

$$2 \geq \frac{1 - \frac{1}{2}^{m-(n+1)+1}}{1 - \frac{1}{2}}$$

$$\frac{1}{2^n} \xrightarrow{n \to \infty} 0$$

[FIXED $\varepsilon > 0$] $\Longrightarrow \exists\, n_\varepsilon$ S.T.

$$\forall n \geq n_\varepsilon \quad 0 \leq \frac{1}{2^n} \leq \varepsilon$$

$$\forall n, m \geq n_\varepsilon \Longrightarrow |x_n - x_m| \leq \frac{1}{2^n}$$

$n < m$

$$\frac{1}{2^n} \leq \varepsilon$$

$$\varepsilon \geq \frac{1}{2^{n_\varepsilon}}$$

(x_n) IS CAUCHY $\Longrightarrow$ CONVERGING.

EXAMPLE

(a_n), $a_n \in \{0, 1, 2, 3, 4, \ldots, 9\}$

$x_0 = a_0 \qquad x_n = x_{n-1} + a_n / 10^n$

$\forall N \in \mathbb{N}$, IF $n, m \geq N$

$$\Longrightarrow \quad |x_n - x_m| \leq 10^{-N} < \varepsilon$$

FOR $N \gg 0$

(x_n) CAUCHY $\xRightarrow[\text{THM}]{}$ $x_n \xrightarrow{n \to \infty} x \in \mathbb{R}$

DECIMAL REPR OF x

$a_0 . a_1 a_2 a_3 a_4 \ldots$

LIMINF & LIMSUP

FIX (x_n)

VALUES OF (x_n)

$$S_{\underset{\mathbb{N}}{k}} \doteq \{ x_j \mid \underline{\underline{j > k}} \} \subseteq \mathbb{R}$$

LIMINF & LIMSUP

FIX (x_n)

$$S_k \doteq \{ x_j \mid j > k \}$$

$$\ldots \supseteq S_{k-1} \supseteq S_k \supseteq S_{k+1} \supseteq S_{k+2} \supseteq \ldots \supseteq S_{k+n} \supseteq \ldots$$

$k \in \mathbb{N},\ n \in \mathbb{N}$

LIMINF & LIMSUP

FIX (x_n) BOUNDED

$$S_k \doteq \{ x_j \mid j > k \} \subseteq S = \{\text{VALUES OF } (x_n)\}$$

$\{x_0, x_1, \dots, x_k\} = S \setminus S_k \leftarrow$ finite set

$$\dots \supseteq S_{k-1} \supseteq S_k \supseteq S_{k+1} \supseteq S_{k+2} \supseteq \dots \supseteq S_{k+n} \supseteq \dots$$

(x_n) BOUNDED $\Rightarrow \forall k \in \mathbb{N}$

~~IF $\exists\, k \in \mathbb{N}$ S.T.~~ S_k IS BOUNDED $\forall k \in \mathbb{N}$

FROM BELOW / ABOVE, THEN $\inf S_{\cancel{j}\,k} \in \mathbb{R}$ / $\sup S_{\cancel{x}\,k} \in \mathbb{R}$ ~~$\forall j \geq k$~~

LIMINF & LIMSUP

FIX (x_n) BOUNDED $\qquad S_k \doteq \{ x_j \mid j > k \}$

$$\dots \supseteq S_{k-1} \supseteq S_k \supseteq S_{k+1} \supseteq S_{k+2} \supseteq \dots \supseteq S_{k+n} \supseteq \dots$$

$z_k \doteq \inf S_k \in \mathbb{R}$ $\qquad$ z_k IS INCREASING

$y_k \doteq \sup S_k \in \mathbb{R}$ $\qquad$ (**) y_k is DECREASING

since $\quad S_k \supseteq S_{k+1} \implies \inf S_k \leq \inf S_{k+1}$

$S_k \supseteq S_{k+1} \implies \sup S_k \overset{(**)}{\geq} \sup S_{k+1}$

LIMINF & LIMSUP

FIX (x_n) BOUNDED $\qquad S_k \doteq \{ x_j \mid j > k \} \subseteq S$

SET OF VALUES

$\subset [-c, c]$ $\quad c > 0$

$$\dots \supseteq S_{k-1} \supseteq S_k \supseteq S_{k+1} \supseteq S_{k+2} \supseteq \dots \supseteq S_{k+n} \supseteq \dots$$

$$z_k \doteq \inf S_k \in \mathbb{R} \qquad \uparrow \ z_k \text{ IS INCREASING}$$

$\geq -c$, $\quad c \geq$

$$y_k \doteq \sup S_k \in \mathbb{R} \qquad (\star\star)\ y_k \text{ is DECREASING}$$

$\leq c$, $\quad -c \leq$

$$\liminf_{n\to\infty} x_n = \lim_{k\to\infty} z_k \in \mathbb{R}$$

$$\limsup_{n\to\infty} x_n = \lim_{k\to\infty} y_k \in \mathbb{R}$$

EXAMPLE $\quad x_n = (-1)^n \quad n \in \mathbb{N}$

$y_k = x_{2k} = 1$ $\qquad \{\pm 1\}$

$z_k = x_{2k+1} = (-1)$ $\qquad \inf\{x_j \mid j > k\} = -1$

$$\liminf_{n\to\infty} x_n = \lim z_k = -1$$

$$\limsup_{n\to\infty} x_n = \lim y_k = 1$$

$y_k = \sup\{x_j \mid j > k\} = 1$

$\{\pm 1\}$

IN GENERAL:

(x_n) BOUNDED

B-W THM $\Longrightarrow \exists$ converging subseq's of x_n

$$L \doteq \{ \underline{\underline{x \in \mathbb{R}}} \mid x \text{ IS THE LIMIT OF SOME SUBSEQ. OF } x_n \} \subseteq \mathbb{R}$$

L IS BOUNDED SINCE (x_n) is BOUNDED

$$\mathbb{R} \ni \sup L = \limsup_{n \to \infty} x_n$$

$$\mathbb{R} \ni \inf L = \liminf_{n \to \infty} x_n$$

IF (x_n) is CONVERGING

$$\left[\lim_{n \to \infty} x_n = x \right]$$

IF $x_n \to x$

$\implies x_{n_k} \to x$

$L \doteq \{ y \in \mathbb{R} \mid y$ IS A LIMIT OF A SUBSEQ OF $(x_n) \} = \{ x \}$

$$\implies \sup L = x = \inf L$$

$\implies$ IF (x_n) CONVERG. W/ LIMIT x

$$\iff \liminf x_n = x = \limsup x_n$$

SERIES

ACHILLES & THE TORTOISE [ZENO]:

SERIES

ACHILLES & THE TORTOISE [ZENO]:

ACHILLES $\longrightarrow$ RUNS AT 10 m/s

THE TORTOISE $\longrightarrow$ RUNS AT 0.1 m/s

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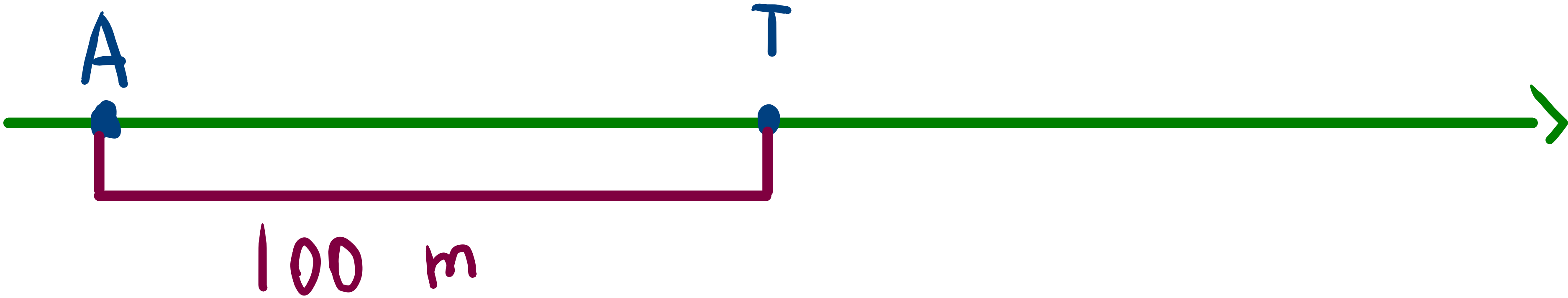

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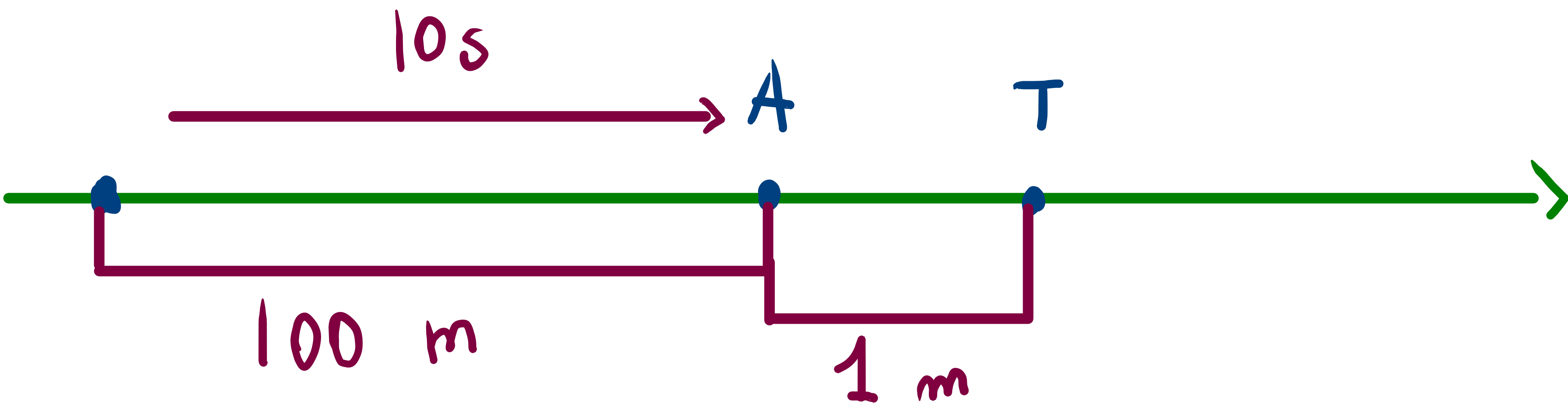

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ACHILLES $\longrightarrow$ RUNS AT 10 m/s

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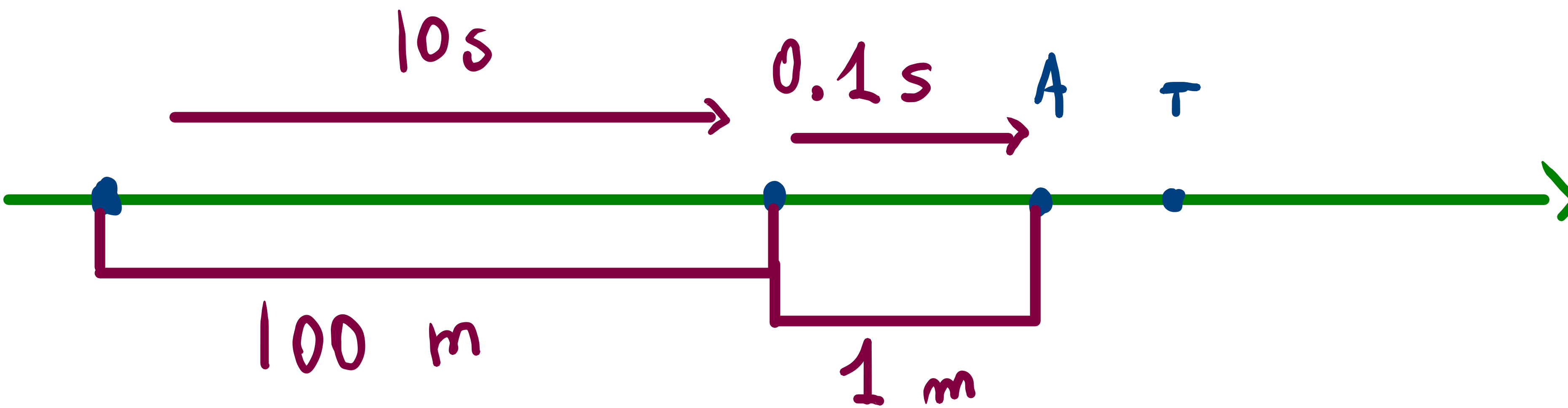

SERIES

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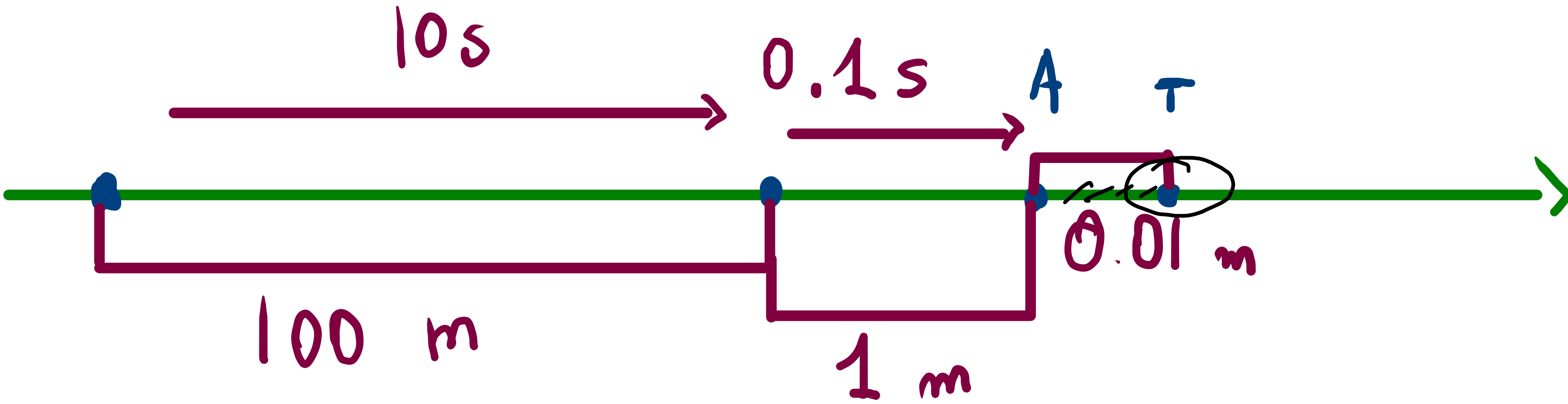

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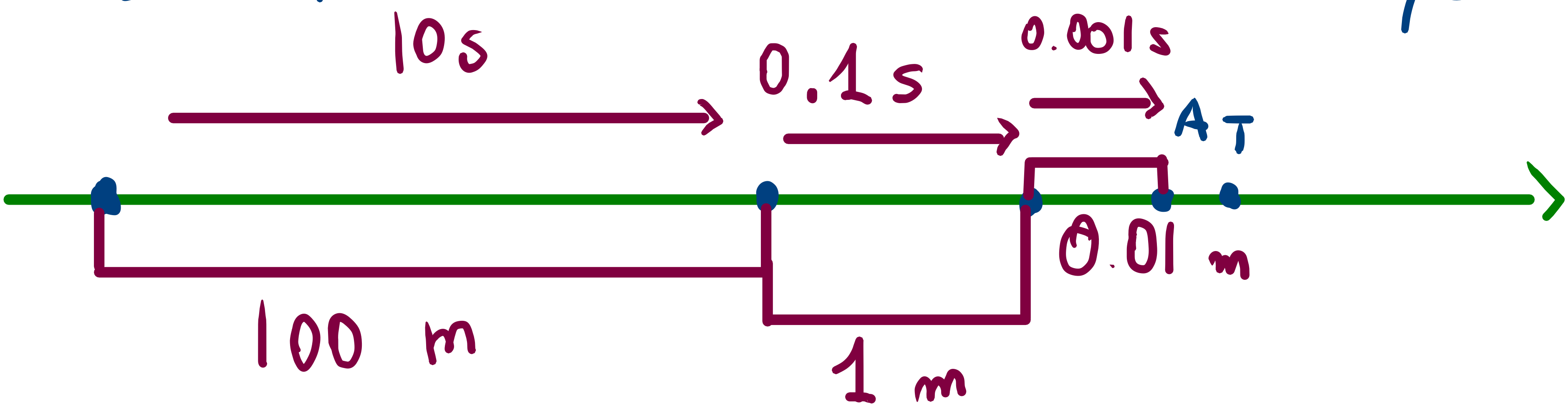

SERIES

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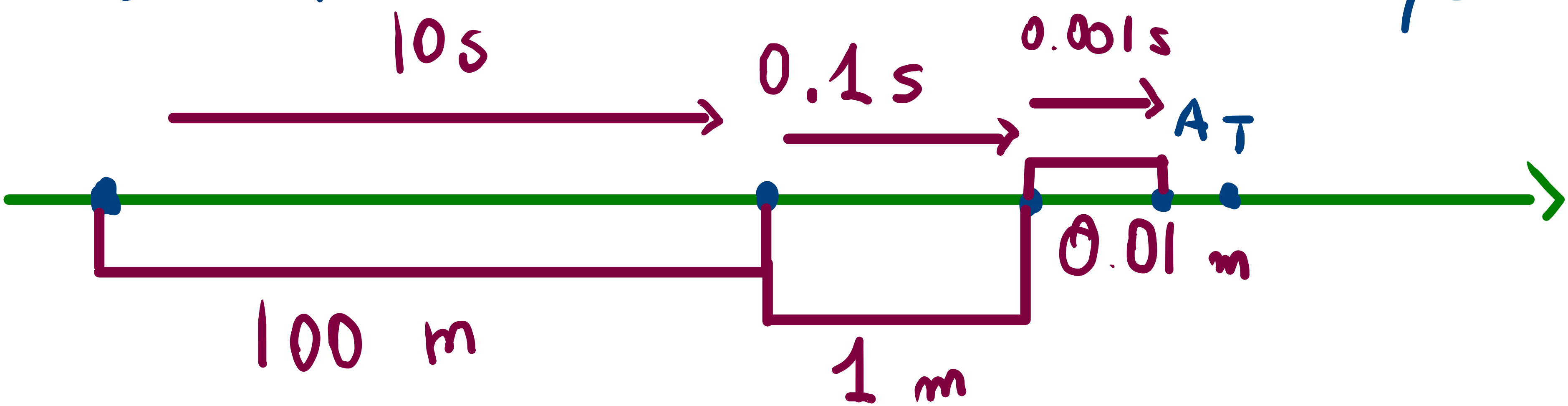


Q [ZENO]: DOES A EVER REACH T?

DEF (x_k) SEQUENCE

DEF (x_k) SEQUENCE

THE SEQUENCE (S_n)

$$S_n \doteq \sum_{i=0}^{n} x_i$$

IS CALLED A

SERIES

DEF (x_n) SEQUENCE

THE SEQUENCE (S_n)

$$S_n \doteq \boxed{\sum_{i=0}^{n}} x_i$$

"THE SUM FOR i THAT GOES FROM 0 TO n"

IS CALLED A SERIES

DEF (x_n) SEQUENCE

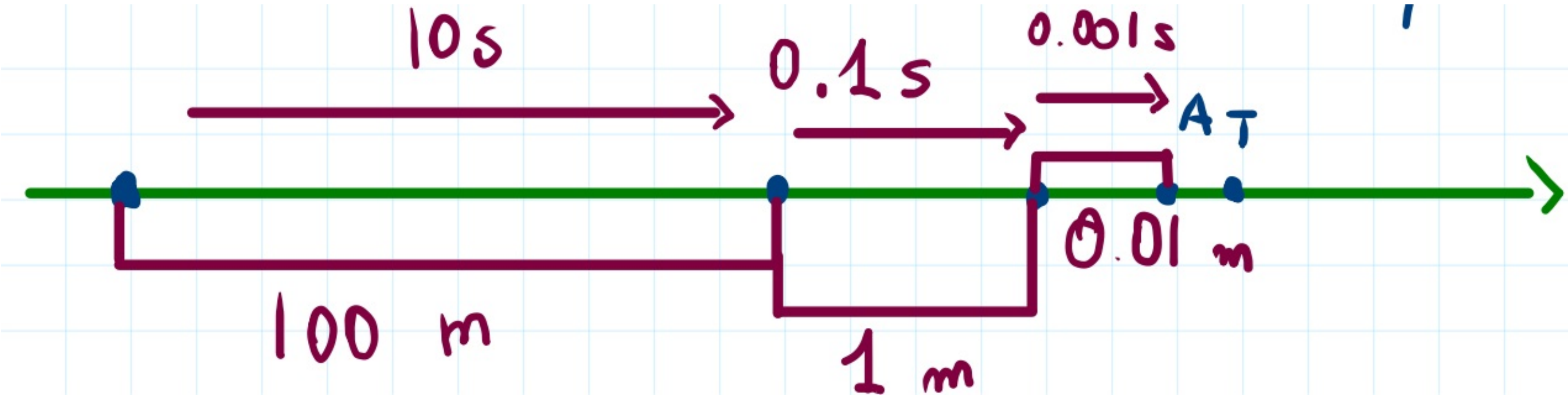
THE SEQUENCE (S_n)

$$S_n \doteq \boxed{\sum_{i=0}^{n}} x_i$$

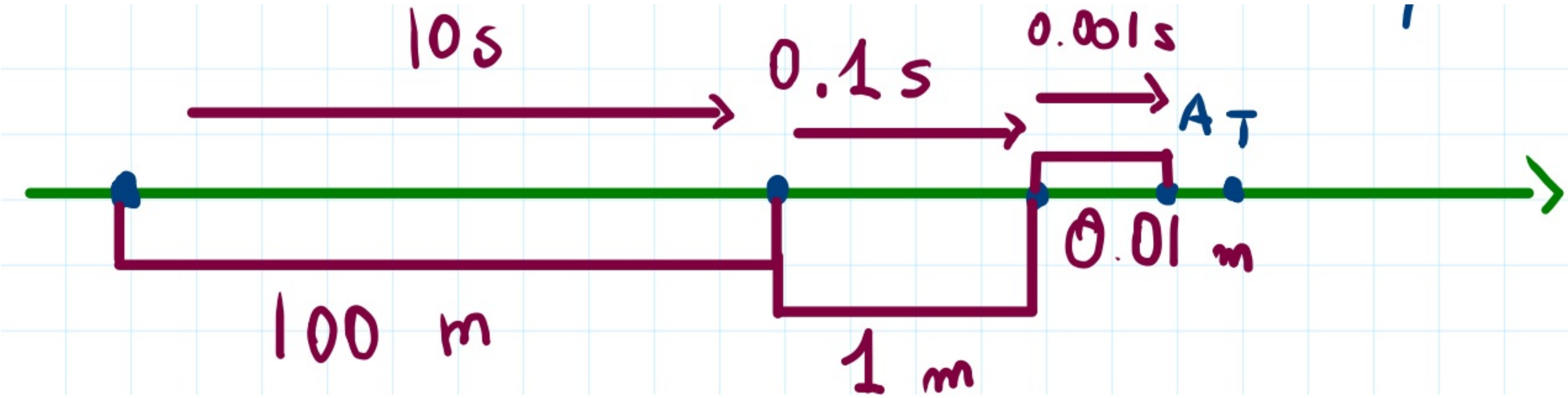
"THE SUM FOR i THAT GOES FROM 0 TO n"

IS CALLED A SERIES

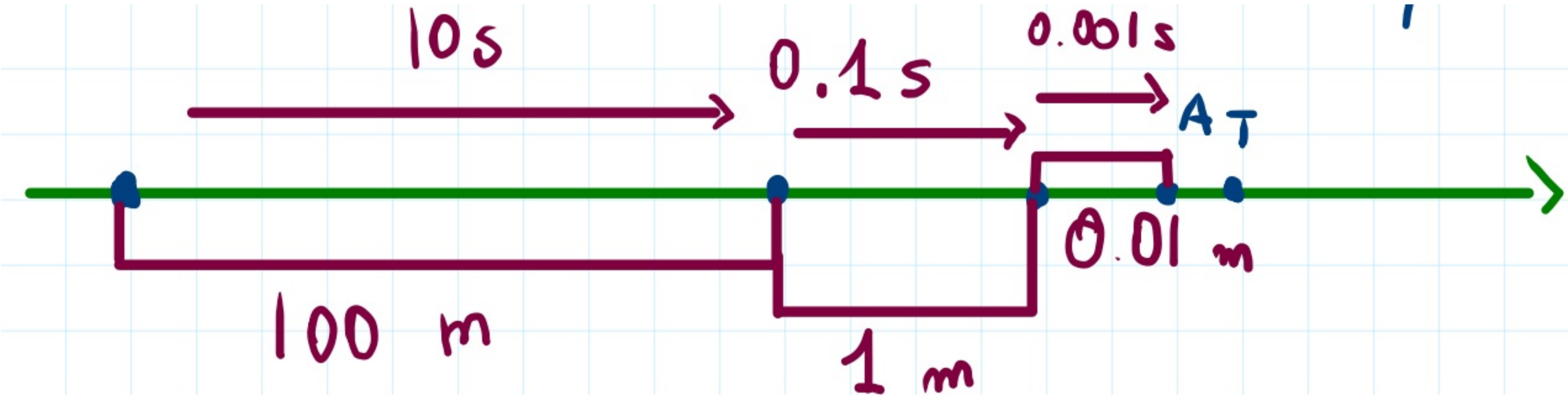
NOTATION: $\displaystyle\sum_{i=0}^{\infty} x_i$



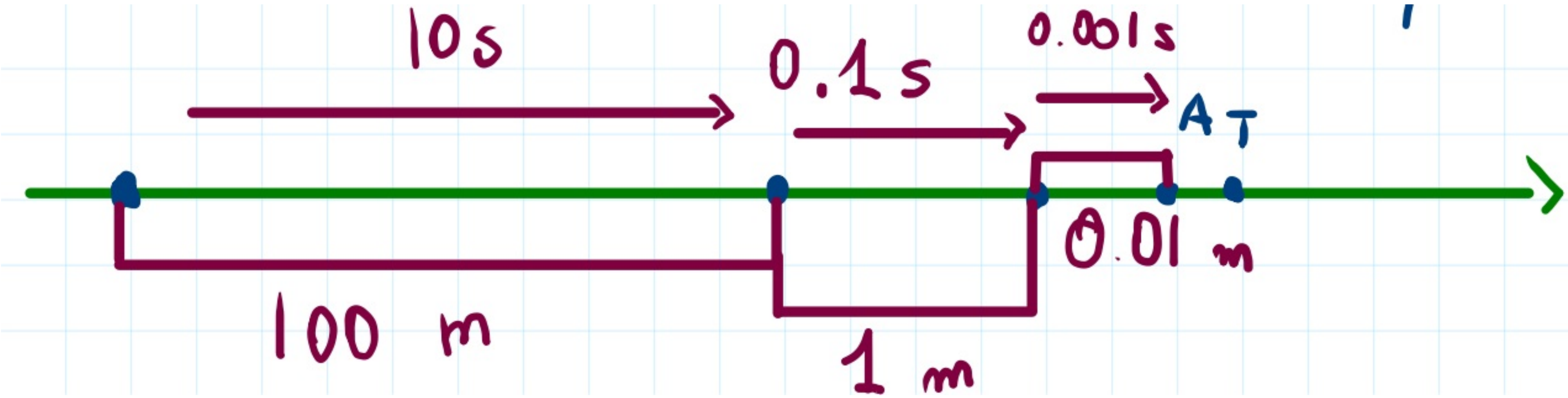
10s
0.1 s
0.001s
A
T
100 m
1 m
0.01 m



$S_0 = 10$

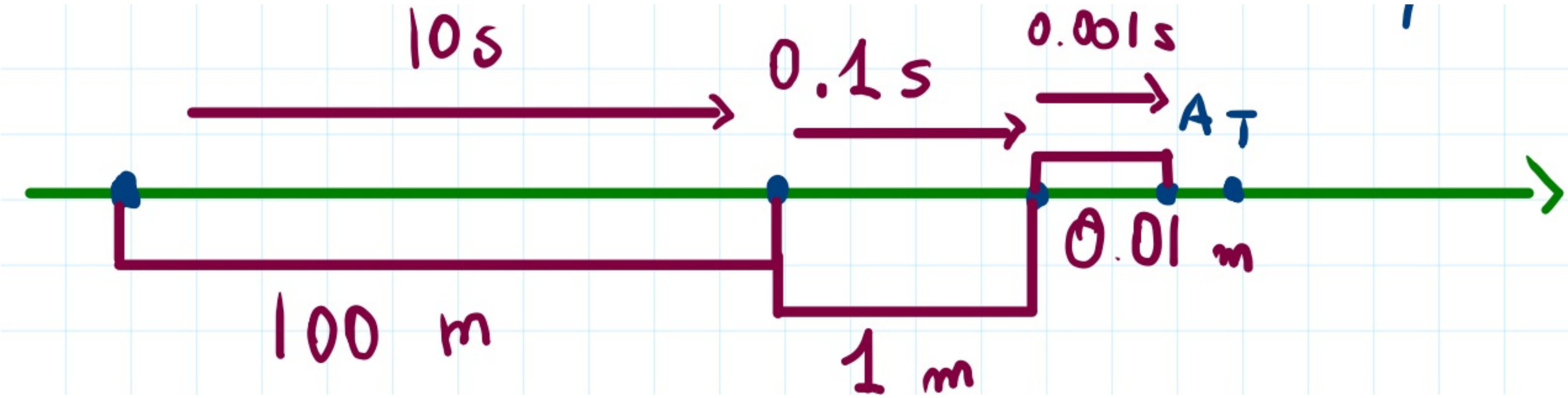


$$S_0 = 10 \quad , \quad S_1 = 10 + 0{,}1 = 10.1$$



$$S_0 = 10 \quad , \quad S_1 = 10 + 0{,}1 = 10 + \frac{10}{100} \, ,$$

$$S_2 = 10 + 0.1 + 0{,}001 = 10 + \frac{10}{100} + \frac{10}{100^2}$$



$$S_0 = 10 \quad , \quad S_1 = 10 + 0{,}1 = 10 + \frac{10}{100} \, ,$$

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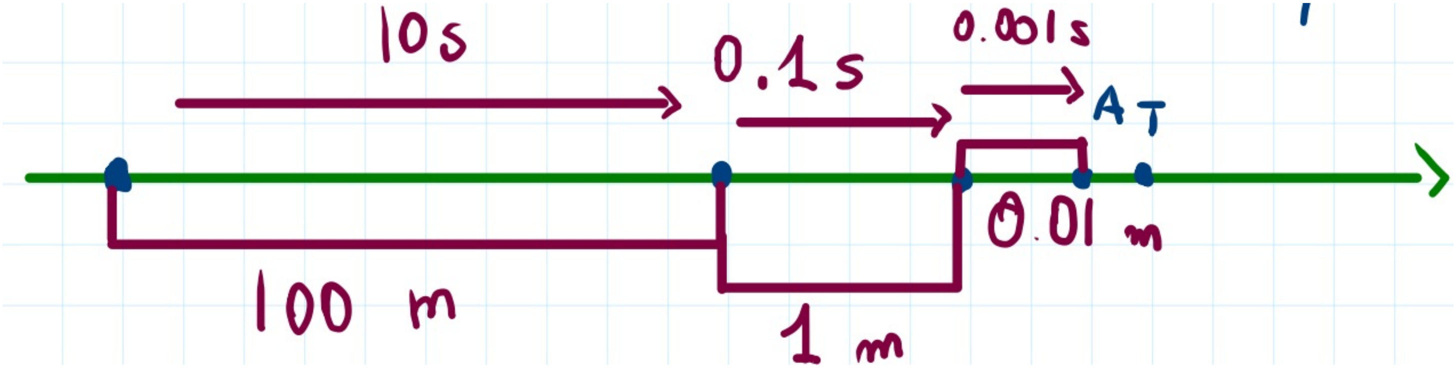

$$S_0 = 10 \quad , \quad S_1 = 10 + 0,1 = 10 + \frac{10}{100} \,,$$

$$S_2 = 10 + 0.1 + 0,001 = 10 + \frac{10}{100} + \frac{10}{100^2}$$

$$S_n = \sum_{i=0}^{n} \frac{10}{100^i} = 10 \cdot \sum_{i=0}^{n} (100)^{-i}$$

SERIES

ZENO'S Q

$$x_k = \frac{10}{100^k} \longrightarrow S_n =$$

IS (S_n) CONVERGENT?

EXAMPLE

① $S_n = \sum_{i=0}^{n} a \cdot q^i$ GEOMETRIC SERIES

GEOM. PROGR. $x_k = a q^k$

② $S_n = \sum_{i=1}^{n} \frac{1}{i}$ HARMONIC SERIES

$x_k = 1/k \quad k \geq 1$

③ $S_n = \sum_{i=1}^{n} (-1) \frac{1}{i}$ LEIBNITZ SERIES

$x_k = (-1)^k / k$

④ $S_n = \sum_{i=1}^{n} \frac{1}{i^2} \longleftarrow x_k = \frac{1}{k^2}$

$$(x_i) \longrightarrow (S_n), \quad S_n \doteq \sum_{i=0}^{n} x_i$$

SEQUENCE

BOUNDED?

MONOTONE?

CONVERGING?

$$x_i \longrightarrow (S_n), \quad S_n \doteq \sum_{i=0}^{n} x_i$$

SEQUENCE

BOUNDED?

CONVERGENT?

DEF A SERIES $\sum_{i=0}^{\infty} x_n = (S_n)$

$$S_n = \sum_{i=0}^{n} x_i$$

IS $\left\{ \begin{array}{l} \text{BOUNDED} \\ \text{CONVERGENT} \end{array} \right.$

$x_i \longrightarrow (S_n)$, $S_n \doteq \sum_{i=0}^{n} x_i$

SEQUENCE

BOUNDED?

CONVERGENT?

DEF A SERIES $\sum_{i=0}^{\infty} x_n$ IS $\begin{cases} \text{BOUNDED} \\ \text{CONVERGENT} \end{cases}$ IF (S_n) IS . BOUNDED / CONVERG.

$$(S_n) \;\; \text{with} \;\; S_n = \sum_{i=0}^{n} x_n$$

PROP $\left(x_i \right)$ TFAE

PROP (x_i) TFAE

① $\sum_{i=0}^{\infty} x_i$ CONVERGENT ((S_n) is CONV.)

② (S_n), $S_n \doteq \sum_{i=0}^{n} x_i$ CAUCHY SEQ.

③ $\forall \varepsilon > 0$, $\exists n_\varepsilon \in \mathbb{N}$ s.t. $\forall m, n \geq n_\varepsilon$

$$|S_m - S_n| \leq \varepsilon$$

$\underset{①}{(S_n)}$ CONV. $\overset{1^{ST}}{\underset{HR}{\iff}}$ $\underset{②}{(S_n)}$ IS CAUCHY

$\Updownarrow$

③ DEFINITION OF (S_n) IS CAUCHY

IN PARTICULAR:

PROP LET (x_i) BE A SEQ.

IF $\sum_{i=0}^{\infty} x_i$ CONVERGENT.

$$\implies x_i \xrightarrow{i\to\infty} 0$$

IN PARTICULAR:

<u>PROP</u> $\sum_{i=0}^{\infty} x_i$ CONVERGENT

$\Longrightarrow$ $x_i \xrightarrow{i \to \infty} 0$

<u>PROOF</u> $\sum_{i=0}^{\infty} x_i$ CONVERGENT

$\Downarrow$

PROP $\sum_{i=0}^{\infty} x_i$ TFAE

① $\sum_{i=0}^{\infty} x_i$ CONVERGENT

② (s_n), $s_n \doteq \sum_{i=0}^{n} x_i$ CAUCHY SEQ.

③ $\forall \varepsilon > 0$, $\exists n_\varepsilon \in \mathbb{N}$ s.t. $\forall m, n \geq n_\varepsilon$
$m > n$

$$|s_n - s_m| = \left|\sum_{i=n+1}^{m} x_i\right| \leq \varepsilon$$

IN PARTICULAR:

PROP $\sum_{i=0}^{\infty} x_i$ CONVERGENT $\Longrightarrow x_i \xrightarrow{i\to\infty} 0$ | $\Longleftrightarrow$ IF $x_i \not\to 0$ $\Rightarrow \sum x_i$ is NON CONVERGENT

PROOF $\sum_{i=0}^{\infty} x_i$ CONVERGENT

$\Downarrow$ TAKE n, $m=n+1$ IN (3)

FIX $\varepsilon>0 \Longrightarrow \exists\, n_\varepsilon \in \mathbb{N}$ S.T. $\forall\, n \geq n_\varepsilon$

$$|s_{n+1} - s_n| \leq \varepsilon$$

$$|x_{n+1}| \leq \varepsilon \Longrightarrow |x_n| \xrightarrow{n\to\infty} 0$$

$$\Updownarrow$$

$$x_n \longrightarrow 0$$

EXAMPLE

$$x_n = (-1)^n$$

Q: DOES $\sum_{n=0}^{\infty} x_n$ CONVERGE? [POLL]

$$S_n = \sum_{i=0}^{n} x_i = \begin{cases} -1 & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

EXAMPLE

$$x_n = (-1)^n$$

Q: DOES $\sum_{n=0}^{\infty} x_n$ CONVERGE? [POLL]

TRUE

FALSE

EXAMPLE $\sum_{i=1}^{\infty} \frac{1}{i}$ HARMONIC SERIES

DOES IT CONVERGE?

(x_i) S.T. $x_i \longrightarrow 0$ BUT $\sum x_i$ IS NOT CONV.

$$S_n = \sum_{i=1}^{n} \frac{1}{i}$$ STR. INCR.

(S_n) CONV. $\Longleftrightarrow$ IT IS BOUNDED

PROP $\sum_{i=0}^{\infty} x_i$ TFAE

① $\sum_{i=0}^{\infty} x_i$ CONVERGENT

② (S_n), $S_n \doteq \sum_{i=0}^{n} x_i$ CAUCHY SEQ.

③ $\forall \varepsilon > 0$, $\exists n_\varepsilon \in \mathbb{N}$ S.T. $m, n \geq n_\varepsilon$

$$|S_m - S_n| \leq \varepsilon$$

$$|S_m - S_n| = \left|\sum_{i=n+1}^{m} x_i\right|$$

EXAMPLE

$$\sum_{i=1}^{\infty} \frac{1}{i} \quad \text{HARMONIC SERIES}$$

DOES IT CONVERGE? $\left[\frac{1}{i} \xrightarrow{i \to \infty} 0 \right]$

IF IT CONVERGES THEN

③ $\forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}$ S.T. $\forall n, m \geq n_\varepsilon$

$$|S_n - S_m| = \left| \sum_{i=n+1}^{m} \frac{1}{i} \right| \leq \varepsilon$$

$m > n$

EXAMPLE $\sum_{i=1}^{\infty} \frac{1}{i}$ HARMONIC SERIES

DOES IT CONVERGE? $\left[\frac{1}{i} \xrightarrow{i\to\infty} 0\right]$

IF IT CONVERGES THEN

③ $\forall \varepsilon > 0$, $\exists n_\varepsilon \in \mathbb{N}$ S.T. $\forall n, m \geq n_\varepsilon$ $\geq n_{1/4}$

$\frac{1}{4}$ $''$

$$|S_n - S_m| = \left|\sum_{i=n}^{m} \frac{1}{i}\right| < \varepsilon$$

$\Downarrow$ TAKE $m = 2n$

$1/4$

$\vee$ $|$ $\therefore$ $\frac{1}{2n}$ $\wedge$

$$|S_n - S_{2n}| = \left|\sum_{i=n+1}^{2n} \frac{1}{i}\right| \geq \sum_{i=n+1}^{2n} \frac{1}{2n} = \frac{n}{2n} = \frac{1}{2}$$

END

IN PARTICULAR:

PROP $\displaystyle\sum_{i=0}^{\infty} x_i$ CONVERGENT

$$\Downarrow \quad \cancel{\Uparrow} \longleftarrow \left[\sum_{i=0}^{\infty} \frac{1}{i} \right]$$

$$x_i \xrightarrow{i \to \infty} 0$$