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[2] GALE, D., 1987, *Math. Intelligencer*, **19**, 50.
[3] POLYA, G., *How to Solve It: a new aspect of mathematical method* (Harmondsworth: Penguin).

Partitions of the interval in the definition of Riemann's integral

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The definition of Riemann's integral involves a redundant requirement.

1. Introduction

The purpose of this note is to suggest that the definition of Riemann's integral involves an unnecessary requirement. Let us recall the history.

In 1854, Riemann gave a precise definition of the integral $\int_a^b f(x)\,\mathrm{d}x$, which set a firm foundation for integration theory. For nearly one and a half centuries, there were numerous investigations by several generations of mathematicians, but there are some things still worth discussing. The definitions of a partition of an interval and of Riemann's integral of a function are as follows.

Definition 1. A partition P of the interval $[a,b]$ is a finite set of numbers: $P=\{a=x_0<x_1<\ldots<x_n=b\}$. The norm of the partition P is given by $\|P\|=\max\{x_k-x_{k-1}:1\leqslant k\leqslant n\}$.

Definition 2. A function $f(x)$ defined on the interval $[a,b]$ is said to be Riemann integrable if there is a number I such that for arbitrarily small $\varepsilon>0$, there is a $\delta>0$, such that for any partition P with $\|P\|<\delta$, and any choice of the numbers $c_k\in[x_{k-1,}x_k]$ $(k=1,2,\ldots,n)$, $|\sum_{k=1}^n f(c_k)(x_k-x_{k-1})-I|<\varepsilon$.

There are two 'any's in Definition 2. It is easily understood that 'any choice of the numbers $c_k\in[x_k,x_{k-1}]$' is necessary by a simple example: $f(x)=1$ for rational x and $f(x)=0$ for irrational x. For this function, $\sum_{k=1}^n f(c_k)(x_k-x_{k-1})$ is 0 if the c_k's are all irrational and is 1 if the c_k's are all rational. Naturally we may ask the question why 'any partition P with $\|P\|<\delta$' is necessary. We need an example of a function $f(x)$ defined on $[a,b]$ such that the sum $\sum_{k=1}^n f(c_k)(x_k-x_{k-1})$ approaches two different values according to two different kinds of partitions P, Q, for instance P is an equal partition while Q is not.

Before stating our result, we recall a necessary and sufficient condition for Riemann integrability due to Darboux in 1875.

Theorem 1. Let $f(x)$ be a bounded function defined on the interval $[a,b]$. For any partition P of $[a,b]$, $P=\{a=x_0<x_1<\ldots<x_n=b\}$, let

$$\omega(f,[x_{k-1},x_k])=\sup\{f(x):x\in[x_{k-1},x_k]\}-\inf\{f(x):x\in[x_{k-1},x_k]\}.$$

Then $f(x)$ is Riemann integrable if and only if for arbitrarily small $\varepsilon > 0$, there is a $\delta > 0$ such that $|\sum_{k=1}^{n} \omega(f, [x_{k-1}, x_k])(x_k - x_{k-1})| < \varepsilon$ whenever $\|P\| < \delta$.

The following two properties of $\omega(f, [a, b])$ are obvious.

(1) $\omega(f, [a, b]) \leqslant \omega(f, [c, d])$ if $[a, b] \subset [c, d]$;
(2) $\omega(f, [a, b]) \leqslant \omega(f, [a, c]) + \omega(f, [c, b])$ if $a < c < b$.

In the sequel, if the function $f(x)$ is the only given function in the context, we will simply write $\omega([a, b])$ instead of $\omega(f, [a, b])$.

2. Main Result

Now we give the main result.

Theorem 2. Let $f(x)$ be a function defined on the interval $[a, b]$. If there is a number I such that for arbitrarily small $\varepsilon > 0$, there is a natural number N, such that whenever $n \geqslant N$ and c_k are arbitrarily chosen from the interval

$$\left[a + \frac{(k-1)(b-a)}{n}, a + \frac{k(b-a)}{n}\right],$$

we have

$$\left|\sum_{k=1}^{n} f(c_k)\frac{b-a}{n} - I\right| < \varepsilon,$$

then $f(x)$ is Riemann integrable.

Remark 1. In the definition above, no partition P of $[a, b]$ is explicitly mentioned. Actually the numbers

$$\left\{a + \frac{k(b-a)}{n} : k = 0, 1, 2, \ldots, n\right\}$$

form an equal partition.

Proof of Theorem 2. The proof consists of two steps.

(1) We first prove that for arbitrarily small $\varepsilon > 0$, there is a natural number N such that $n \geqslant N$ implies

$$\left|\sum_{k=1}^{n} \omega\left(f, \left[a + \frac{(k-1)(b-a)}{n}, a + \frac{k(b-a)}{n}\right]\right)\frac{b-a}{n}\right| < \varepsilon.$$

For notational simplicity, let

$$t_k = a + \frac{k(b-a)}{n} \quad \text{for} \quad k = 0, 1, \ldots, n$$

Then for any $\varepsilon > 0$, there are numbers $c_k, d_k \in [t_{k-1}, t_k]$ such that

$$0 \leqslant \sup\{f(x) : t_{k-1} \leqslant x \leqslant t_k\} - f(c_k) < \frac{\varepsilon}{4(b-a)}$$

$$0 \leqslant f(d_k) - \inf\{f(x) : t_{k-1} \leqslant x \leqslant t_k\} < \frac{\varepsilon}{4(b-a)}$$

For this ε, there are two natural numbers N_1, N_2 such that $n \geqslant N_1$ implies

$$\left|\sum_{k=1}^{n} f(c_k)\frac{b-a}{n} - I\right| < \frac{\varepsilon}{4}$$

and $n \geqslant N_2$ implies

$$\left|\sum_{k=1}^{n} f(d_k)\frac{b-a}{n} - I\right| < \frac{\varepsilon}{4}$$

Therefore letting $N = \max\{N_1, N_2\}$, we have $n \geqslant N$ implies the following two inequalities

$$\left|\sum_{k=1}^{n} \sup\{f(x) : t_{k-1} \leqslant x \leqslant t_k\}\frac{b-a}{n} - \sum_{k=1}^{n} f(c_k)\frac{b-a}{n}\right| < \left(\sum_{k=1}^{n}\frac{\varepsilon}{4(b-a)}\right)\frac{b-a}{n} = \frac{\varepsilon}{4}$$

$$\left|\sum_{k=1}^{n} f(d_k)\frac{b-a}{n} - \sum_{k=1}^{n} \inf\{f(x) : t_{k-1} \leqslant x \leqslant t_k\}\frac{b-a}{n}\right| < \frac{\varepsilon}{4}$$

Hence

$$\left|\sum_{k=1}^{n} \sup\{f(x) : t_{k-1} \leqslant x \leqslant t_k\}\frac{b-a}{n} - I\right| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}$$

$$\left|\sum_{k=1}^{n} \inf\{f(x) : t_{k=1} \leqslant x \leqslant t_k\}\frac{b-a}{n} - I\right| < \frac{\varepsilon}{2}$$

Now we have

$$\left|\sum_{k-1}^{n} \omega(f, [t_{k-1}, t_k])\frac{b-a}{n}\right| < \varepsilon$$

(2) Let $\varepsilon > 0$ be given. By the discussion (1), pick a natural number N such that $n \geqslant N$ implies

$$\left|\sum_{k=1}^{n} \omega(f, [t_{k-1}, t_k])\frac{b-a}{n}\right| < \frac{\varepsilon}{3}$$

We specially pick $n = N$, and have

$$\left|\sum_{k=1}^{N} \omega([t_{k-1}, t_k])(t_{k-1} - t_k)\right| < \frac{\varepsilon}{3}$$

Here we write $\omega([t_{k-1}, t_k])$ instead of $\omega(f, [t_{k-1}, t_k])$.

Suppose $P = \{a = x_0 < x_1 < \ldots < x_m = b\}$ is a partition of the interval $[a, b]$ with $\|P\| < \delta$, where $\delta = (b-a)/(3N)$. Since

$$t_k - t_{k-1} = \frac{b-a}{N} \quad \text{for all} \quad k = 1, \ldots, N$$

we know that for any interval $[t_{k-1}, t_k]$, there is at least one number x_{i_k} in the partition P such that $x_{i_k} \in [t_{k-1}, t_k]$. Hence the two groups of numbers $\{t_k : 0 \leqslant k \leqslant N\}$ and $\{x_i : 0 \leqslant i \leqslant m\}$ are related as follows:

$$\begin{aligned} a = t_0 = x_0 < x_1 < \ldots < x_{i_1} < t_1 \leqslant x_{i_1+1} < x_{i_1+2} < \ldots < x_{i_2} < t_2 \\ \leqslant x_{i_2+1} < \ldots < x_{i_k} < t_k \leqslant x_{i_k+1} < \ldots \\ < x_{i_{N-1}} < t_{N-1} \leqslant x_{i_{N-1}+1} < x_{i_{N-1}+2} < \ldots < x_m = t_N = b \end{aligned}$$

For each interval $[x_i, x_{i+1}]$, there are two possible cases:

(i) There is a k such that $[x_i, x_{i+1}] \subset [t_k, t_{k+1}]$. Then $\omega([x_i, x_{i+1}] \leqslant \omega([t_k, t_{k+1}])$.
(ii) The interval $[x_i, x_{i+1}]$ does not fall completely in an interval $[t_k, t_{k+1}]$. Hence i must be an i_k such that $t_{k-1} < x_{i_k} < t_k \leqslant x_{i_k+1} < t_{k+1}$. The following inequalities are immediate

$$\omega([x_{i_k}, x_{i_k+1}] \leqslant \omega([t_{k-1}, t_{k+1}]) \leqslant \omega([t_{k-1}, t_k]) + \omega([t_k, t_{k+1}])$$

Now we divide the addends in the expression $\sum_{i=0}^{m} \omega([x_{i,} x_{i+1}])(x_{i+1} - x_i)$ into two parts according to the case (i) or (ii).

In the first part, since the total length of all the adjacent intervals $[x_i, x_{i+1}]$ falling into the interval $[t_k, t_{k+1}]$ is $x_{i_{k+1}} - x_{i_k+1} < t_{k+1} - t_k$ and each $\omega([x_i, x_{i+1}]) \leqslant \omega([t_k, t_{k+1}])$, we know that the sum of all addends in this part is less than

$$\sum_{k=0}^{N} \omega([t_k, t_{k+1}])(t_{k+1} - t_k)$$

On the other hand, for the intervals in case (ii),

$$x_{i_k+1} - x_{i_k} \leqslant \|P\| \leqslant \frac{b-a}{3N} < \frac{b-a}{N} = t_k - t_{k-1} = t_{k+1} - t_k$$

Hence

$$\omega([x_{i_k+1}, x_{i_k}])(x_{i_k+1} - x_{i_k}) \leqslant \omega([t_{k-1}, t_k])(t_k - t_{k-1}) + \omega([t_k, t_{k+1}])(t_{k+1} - t_k)$$

The sum of all addends this part cannot exceed $2\sum_{k=0}^{N} \omega([t_k, t_{k+1}])(t_{k+1} - t_k)$.

Therefore

$$\sum_{i=0}^{m} \omega([x_i, x_{i+1}])(x_{i+1} - x_i) < 3\sum_{k=0}^{N} \omega([t_k, t_{k+1}])(t_{k+1} - t_k) < 3\left(\frac{\varepsilon}{3}\right) = \varepsilon$$

The proof of Theorem 2 is completed.

3. Discussion

Theorem 2 reveals that there is a redundant condition in Riemann's definition of integral. The requirement 'any partition of the interval $[a, b]$ with $\|P\| < \delta$' can be relaxed. As matter of fact, rewriting

$$\sum_{k=1}^{n} f(c_k)\frac{b-a}{n} \quad \text{as} \quad (b-a)\frac{\sum_{k=1}^{n} f(c_k)}{n}$$

we may give Riemann's integral a statistical definition since

$$\frac{\sum_{k=1}^{n} f(c_k)}{n}$$

is the mean of the sequence of the samples taken from the space

$$S_k = \left\{ f(x) : a + \frac{(k-1)(b-a)}{n} \leqslant x \leqslant a + \frac{k(b-a)}{n} \right\}$$

Definition 3. Let $f(x)$ be a function defined on $[a, b]$. Then $\int_a^b f(x)\,\mathrm{d}x = m(b-a)$ if and only if

$$m = \lim_{n\to\infty} \frac{\sum_{k=1}^{n} f(c_k)}{n}$$

exists, where

$$f(c_k) \in \left\{ f(x) : a + \frac{(k-1)(b-a)}{n} \leq x \leq a + \frac{k(b-a)}{n} \right\}.$$

Theorem 2 asserts that for Riemann's integrable function we need only consider the equal partitions of the the interval $[a, b]$. If we make some minor verbal revision we can prove that whether a function is Riemann integrable only depends one sequence of partitions with norm approaches to 0.

Theorem 3. Let $f(x)$ be a function defined on $[a, b]$. Let $P_i = \{a = x_0(i) < x_1(i) < \ldots < x_{n(i)}(i) = b\}$ be a sequence of partitions of the interval $[a, b]$ and $\lim_{i\to\infty}\|P_i\| = 0$. If I is a real number and for any $\varepsilon > 0$, there is a natural number N such that $i \geqslant N$ implies $|\sum_{k=1}^{n(i)} f(c_k)(x_k(i) - x_{k-1}(i)) - I| < \varepsilon$, where c_k are taken arbitrarily from the interval $[x_{k-1}(i), x_k(i)]$, then $\int_a^b f(x)\,\mathrm{d}x = I$.

Riemann's integral has a long history. It is incredible that such a very basic property has been neglected for a long time. It even escaped the notice of many great mathematical analysts. In most of the textbooks in calculus, the Riemann integral of a function is defined as the limit $\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)(x_k - x_{k-1})$ if it exists. This limit process on partitions is not the same limit process on the set of real numbers. The textbooks confuse the two limits as one. We know that the set of real numbers has very rich structures in both algebra and topology, while the set of partitions of an interval has very poor structure. Many properties of the limit process on the set of real numbers cannot be tranferred to the new limit process on partitions. Most of people think a counterexample can be found because in the limit process involving real numbers, it is common sense that the convergence of a subsequence never guarantee the convergence of the sequence. The author was a victim of this wrong belief.

All the knowledge we need in this paper is elementary and self-contained in the classical theory of calculus founded by Cauchy. The readers can find all the preliminaries in an undergraduate course 'Advanced Calculus'. No concepts like topology or measure are needed.

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References

[1] Foulis, D. J., and Munem, M. A., 1989, *After Calculus: Analysis* (San Francisco: Dellen Publishing Company).

[2] LARSON, R., HOSTETLER, R., and EDWARDS, B., 1994, *Calculus*, fifth edition (Lexington: D. C. Heath and Company).
[3] KESTELMAN, H., 1960, *Modern Theory of Integration* (New York: Dover Publications).

On the power series expansions for the sine and cosine

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In this note we give an elementary derivation for the power series expansions for the sine and the cosine function.

The functions sin and cos have derivatives satisfying $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$. It is not hard to come up with pairs of polynomials whose derivatives are related in *almost* the same way: the functions C_n and S_n defined by

$$C_n(x) = 1 - \frac{x^2}{2!} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} \quad \text{and} \quad S_n(x) = x - \frac{x^3}{3!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!},$$

clearly satisfy

$$S_n' = C_n(x) \quad \text{and} \quad C_n'(x) = -S_{n-1}(x). \tag{$*$}$$

Consider the functions f_n and g_n defined by

$$f_n(x) = C_n(x)\cos x + S_n(x)\sin x \quad \text{and} \quad g_n(x) = C_n(x)\sin x - S_n(x)\cos x.$$

Using $(*)$ it is elementary to verify that

$$f_n'(x) = (-1)^n \cos x \frac{x^{2n+1}}{(2n+1)!} \quad \text{and} \quad g_n'(x) = (-1)^n \sin x \frac{x^{2n+1}}{(2n+1)!}.$$

Applying the Mean Value Theorem to each of the functions f_n and g_n we get the following system of equations

$$C_n(x)\cos x + S_n(x)\sin x = 1 + xf_n'(c) \quad \text{and} \quad C_n(x)\sin x - S_n(x)\cos x = 0 + xg_n'(\bar{c}),$$

for c and $\bar{c}$ between x and 0. Clearly $|xf_n'(c)| \leqslant x^{2n+2}/(2n+1)!$ and $|xg_n'(\bar{c})| \leqslant x^{2n+2}/(2n+1)!$. Using that $x^{2n+2}/(2n+1)! \to 0$, we obtain

$$C_n(x)\cos x + S_n(x)\sin x \to 1 \quad \text{and} \quad C_n(x)\sin x - S_n(x)\cos x \to 0$$

as $n \to \infty$. It follows that

$$C_n(x) = \{C_n(x)\cos x + S_n(x)\sin x\}\cos x + \{C_n(x)\sin x - S_n(x)\cos x\}\sin x \to \cos x$$

and