$$\int \frac{x^5}{(x+1)(x^3+1)} \qquad\qquad \int \frac{P(x)}{Q(x)}\,dx$$

Ici, $\deg P > \deg Q$. On commence par faire la division euclidienne pour se ramener au cas où $\deg P < \deg Q$. On veut $\dfrac{x^5}{X^4+X^3+X+1}$:

$$\begin{array}{l|l}
X^5 & X^4+X^3+X+1 \\ \cline{2-2}
\underline{X^5+X^4+X^2+X} & X-1 \\
-X^4-X^2-X & \\
\underline{-X^4-X^3-X+1} & \\
X^3-X^2+1 &
\end{array}$$

$$\Rightarrow \quad X^5 = (X-1)(X^4+X^3+X+1) + (X^3-X^2+1)$$

$$\Rightarrow \quad \frac{X^5}{X^4+X^3+X+1} = (X-1) + \frac{X^3-X^2+1}{X^4+X^3+X+1}$$

$$\frac{X^5}{(X+1)(X^3+1)} = (X-1) + \frac{X^3-X^2+1}{(X+1)(X^3+1)}$$

factorisation complète du dénominateur :

$X+1$ ✓

X^3+1 : $X=-1$ racine

$$\begin{array}{rrr|l} X^3 & & +1 & X+1 \\ \cline{4-4} X^3 & +X^2 & & X^2-X+1 \\ \cline{1-3} & -X^2 & +1 & \\ & -X^2 & -X & \\ \cline{2-3} & & X+1 & \\ & & X+1 & \\ \cline{3-3} & & 0 & \end{array}$$

$$X^3+1 = (X+1)\underbrace{(X^2-X+1)}_{\text{irréductible}}$$

car $\Delta = 1-4 = -3 < 0$

Donc $\dfrac{x^5}{\underbrace{(x+1)(x^3+1)}_{(x+1)^2(x^2-x+1)}} = (x-1) + \dfrac{x^3-x^2+1}{(x+1)^2(x^2-x+1)} \Longrightarrow$

$$\int \frac{x^5}{(x+1)(x^3+1)}\,dx = \int (x-1)\,dx + \int \underbrace{\frac{x^3-x^2+1}{(x+1)^2(x^2-x+1)}}_{\text{fraction propre } (\deg P < \deg Q)}\,dx$$

- $\displaystyle\int (x-1)\,dx = \frac{x^2}{2} - x + C$

- décomposition en éléments simples de $\dfrac{x^3-x^2+1}{(x+1)^2(x^2-x+1)}$:

$$\frac{x^3-x^2+1}{(x+1)^2(x^2-x+1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2-x+1}$$

$$= \frac{A(x+1)(x^2-x+1) + B(x^2-x+1) + (Cx+D)(x+1)^2}{(x+1)^2(x^2-x+1)}$$

$$\Rightarrow x^3-x^2+1 = A(x^3\cancel{-x^2}\cancel{+x}\cancel{+x^2}\cancel{-x}+1) + B(x^2-x+1) + (Cx+D)(x^2+2x+1)$$

$$= A(x^3+1) + B(x^2-x+1) + C(x^3+2x^2+x) + D(x^2+2x+1)$$

$$= (A+C)x^3 + (B+2C+D)x^2 + (-B+C+2D)x + (A+B+D)$$

On résoud :

$$\begin{cases} A + C = 1 & (1) \\ B + 2C + D = -1 & (2) \\ -B + C + 2D = 0 & (3) \\ A + B + D = 1 & (4) \end{cases}$$

$$(2)+(3): \quad 3C + 3D = -1 \qquad 3(C+D) = -1$$

$$\begin{cases} A + C = 1 \\ B = C + 2D \\ C + D = -\frac{1}{3} \\ A + B + D = 1 \end{cases} \iff \begin{cases} A + C = 1 \\ B = C + 2D \\ C + D = -\frac{1}{3} \\ A + C + 3D = 1 \end{cases} \iff \begin{cases} A + C = 1 \\ B = C + 2D \\ C + D = -\frac{1}{3} \\ 1 + 3D = 1 \end{cases}$$

$$\iff \begin{cases} A + C = 1 \\ B = C \\ C = -\frac{1}{3} \\ D = 0 \end{cases} \iff \begin{cases} A - \frac{1}{3} = 1 \\ B = -\frac{1}{3} \\ C = -\frac{1}{3} \\ D = 0 \end{cases} \iff \begin{cases} A = \frac{4}{3} \\ B = -\frac{1}{3} \\ C = -\frac{1}{3} \\ D = 0 \end{cases}$$

$$\Rightarrow \frac{x^3-x^2+1}{(x+1)^2(x^2-x+1)} = \frac{1}{3}\left(\underbrace{\frac{4}{x+1} - \frac{1}{(x+1)^2} - \frac{x}{x^2-x+1}}_{\text{éléments simples}}\right)$$

$$\Rightarrow \int \frac{x^3-x^2+1}{(x+1)^2(x^2-x+1)}\,dx = \frac{4}{3}\underbrace{\int\frac{dx}{x+1}}_{=\ln|x+1|+C} - \frac{1}{3}\int\frac{dx}{(x+1)^2} - \frac{1}{3}\int\frac{x}{x^2-x+1}\,dx$$

$$\int\frac{dx}{(x+1)^2} = -\frac{1}{(x+1)} + C \qquad \left(\frac{d}{dx}\,\frac{1}{x+1} = -\frac{1}{(x+1)^2}\right)$$

$$\int\frac{x}{x^2-x+1}\,dx = \frac{1}{2}\int\frac{2x-1}{x^2-x+1}\,dx + \frac{1}{2}\int\frac{dx}{x^2-x+1}$$

$$= \int\frac{u'}{u}$$

$$= \frac{1}{2} \ln (x^2 - x + 1) + \frac{1}{\sqrt{3}} \operatorname{Arctg}\left(\frac{2x-1}{\sqrt{3}}\right) + C$$

Finalement,

$$\int \frac{x^5}{(x+1)(x^3+1)}\, dx = \frac{x^2}{2} - x + \frac{4}{3} \ln|x+1| + \frac{1}{3} \frac{1}{x+1}$$

$$- \frac{1}{6} \ln (x^2 - x + 1) + \frac{1}{3\sqrt{3}} \operatorname{Arctg}\left(\frac{2x-1}{\sqrt{3}}\right) + C$$