NTK and Asset Pricing Homework

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1. Suppose that

$$f(x;\theta) \;=\; P^{-1/2} W_2' \sigma(d^{-1/2} W_1 x),\; x \in \mathbb{R}^d,\; W_2 \in \mathbb{R}^P,\; W_1 \in \mathbb{R}^{P \times d}. \quad (1)$$

   Suppose W_1, W_2 are sampled i.i.d., W_k elements from $N(0, v_k^2)$, and σ is a smooth activation function.

   Compute the NTK and show that, in the limit as $P \to \infty$, it converges to a limit that is independent of the actual realization of W_1, W_2.

2. Now, simulate $f_*(x) = \sin(\gamma w' x)$ where γ is a scale controlling the speed of oscillation. Train a neural network to learn $y = f_*(x) + \varepsilon$, and verify that for P large enough we indeed get the formula from the lecture:

$$\begin{aligned} f(x;\theta_t) \;=\; & \underbrace{f(x;\theta_0)}_{\textit{initial random seed}} \\ +\; & K(x;X;\theta_t)\, K(X;X;\theta_t)^{-1} (I - e^{-\eta t n^{-1} K(X;X;\theta_t)})(y - f(X;\theta_0)) \end{aligned} \quad (2)$$

   How does the out-of-sample performance of the trained neural net depend on P?

3. Simulate $G_t \in \mathbb{R}$ i.i.d. with some mean μ. Simulate $X_t \in \mathbb{R}^{N \times d}$ i.i.d. Gaussian, $N = 100,\; d = 10$. Simulate excess stock returns

$$R_{i,t+1} \;=\; f_*(X_{i,t}) G_{t+1} + \varepsilon_{t+1},\; \varepsilon_{t+1} \;\sim N(0,1) \quad (3)$$

   $f_*(x) = \sin(\gamma w' x)$ where γ is a scale controlling the speed of oscillation.

   - Compute the infeasible efficient portfolio π_t and its conditional expected Sharpe ratio $E_t[\pi_t' R_{t+1}]/\mathrm{Var}_t[\pi_t' R_{t+1}]$.

- Simulate this dataset with $T = 500$ and use the first 250 observations as in-sample.
- Then, build linear managed portfolios $F_{t+1}^{lin} = X_t' R_{t+1}$, estimate ridge-penalized efficient portfolio in-sample and report its OOS Sharpe ratio
- Now, sample $\theta \in \mathbb{R}^{d \times P}$ and build random features $S_t = \sin(X_t \theta)$ and the corresponding factors $F_{t+1}^{nonlin} = S_t' R_{t+1}$. Then, proceed as you did for F^{lin}, and investigate the dependence on OOS SR on P.