Kernels Homework 2.

April 15, 2025

1. Show that the spectral decomposition

$$Ax \;=\; UDU'x,\; x \in \mathbb{R}^n\,.$$

implies that, if ϕ_j are the eigenvectors, then

$$Ax \;=\; \sum_{j=1}^{n} \lambda_j \phi_j \langle \phi_j, x\rangle\,.$$

2. Verify that

$$\begin{aligned} \tilde m(-z;c) \;&=\; z^{-1}(1-c+cz\hat m(-z)) \\ &=\; T^{-1}(T-P)z^{-1} + T^{-1}\operatorname{tr}((zI+S'S/T)^{-1})\,. \end{aligned} \tag{1}$$

where

$$\hat m(z) \;=\; P^{-1}\operatorname{tr}((zI+SS'/T)^{-1})$$

3. Let T_K be the Gaussian integral operator studied in the lectures. Show that

$$\operatorname{tr}(T_K(zI+T_K)^{-1} \;=\; \sum_{k=0}^{\infty} \lambda_k (z+\lambda_k)^{-1},\; \lambda_k = (1-\rho)\rho^k$$

4. Verify numerically that

$$Z_*(z;c) \;=\; \frac{z}{T^{-1}\operatorname{tr}((zI+K(X;X))^{-1})} \tag{2}$$

approximately satisfies

$$Z_* = z \;+\; Z_* T^{-1}\operatorname{tr}(T_K(T_K+Z_*I)^{-1})$$

5. Verify the projection theorem numerically: if

$$f(x) \;=\; \sum_{j=1}^{\infty} c_j \psi_j(x) \tag{3}$$

then

$$\hat{f}(x) \;=\; \frac{1}{n} K(x, X)(zI + n^{-1} K(X, X))^{-1} y,\; y = f(X) + \varepsilon \tag{4}$$

satisfies

$$\hat{f}(x) \;\approx\; \sum_{j=1}^{\infty} c_j \frac{1}{1 + Z_* / \lambda_j} \psi_j(x) \tag{5}$$

with Z_* from (2).