Random Matrix Theory (RMT) Homework 1

March 15, 2025

1. Use the uncorrelated case $Y_t \;=\; X_t' A_P X_t$ to prove the analogous result for the correlated case $Y_t \;=\; S_t' A_P S_t$ with $S_t = \Psi^{1/2} X_t$: Prove that

$$E[Y_t] \;=\; \mathrm{tr}(\Psi A_P)$$

and

$$\mathrm{Var}[P^{-1} Y_t] \;\to\; 0$$

2. Solve the Master equation

$$\lim_{T,P\to\infty, P/T\to c} \hat{m}(-z) \;=\; m(-z;c) \tag{1}$$

exists in probability and $m(-z;c)$ is the unique positive solution to the nonlinear master equation

$$m(-z;c) = \frac{1}{1 \,-\, c \,+\, c\,z\,m(-z;c)}\, m_\Psi\left(\frac{-z}{1 \,-\, c \,+\, c\,z\,m(-z;c)},\right) \tag{2}$$

when (a) $\Psi = I$ (this is the Marcenko-Pastur Theorem) and (b) when Ψ has just two eigenvalues λ_1, λ_2. What else matters in addition to λ_1, λ_2?

3. Prove the Sherman-Morrison formula:

Lemma 1 (Sherman-Morrison Formula). Suppose $A \in \mathbb{R}^{n\times n}$ is an invertible square matrix and $u,\ v \in \mathbb{R}^P$ are column vectors. Then $A + uv'$ is invertible if $1 + v'A^{-1}u \neq 0$. In this case,

$$(A + uv')^{-1} = A^{-1} - \frac{A^{-1}uv'A^{-1}}{1 + v'A^{-1}u} \tag{3}$$

and

$$(A + uv')^{-1} u = A^{-1} u \, \frac{1}{1 + v' A^{-1} u} \tag{4}$$

4. Prove that $\|(zI + \hat{\Psi}_T)^{-1}\| \ \leq \ z^{-1}$.