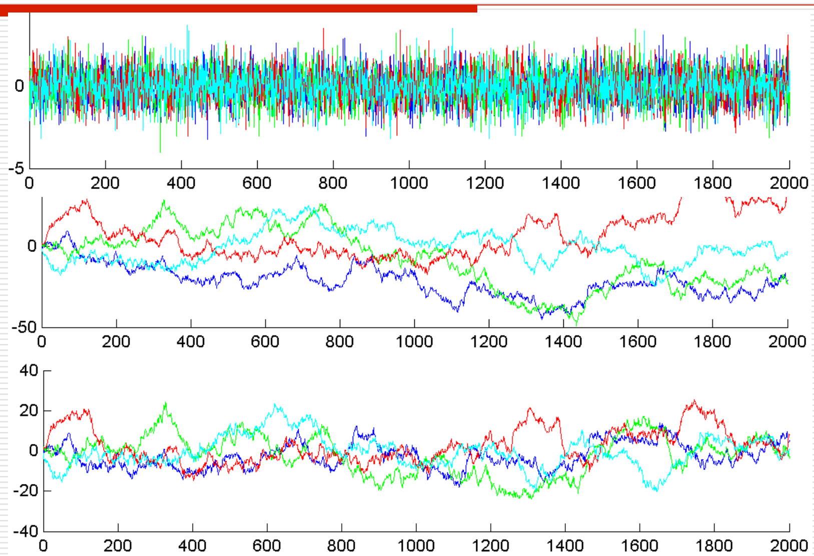
Topics

- Random **process**
  - General concept
  - Classification of random processes
  - Empirical **model identification**
- Random process **examples**
  - White Noise
  - Random bias
  - Shaped white noise
- Characteristic functions
  - AC
  - PSD

Random process – an example



Random processes – general concept

- Random process:
  - **Association** of random variable(s) with a deterministic parameter (e.g. time)
  - Stochastic **collection** – values depend on the outcome of an experiment
  - Time is a **parameter** but the process is ONLY LOOSELY/stochastic "function" of time
  - Discrete or continuous

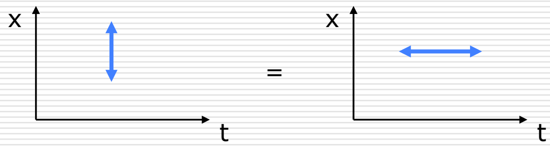
Random variable vs. random process

- Probability
  - Distribution
  - Density
- Characteristics
  - Variable
    - Moments
    - Characteristic function
  - Process
    - Autocorrelation function (AC)
    - Power spectral density (PSD)

TABLE – black board

Classification of random processes

- By ergodicity



$$E(x) = \lim_{t\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\,dt$$

$$E(x^2) = \lim_{t\to\infty} \frac{1}{2T} \int_{-T}^{T} x^2(t)\,dt$$

$$\varphi_{xx}(\tau) = \lim_{t\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t_1)\,x(t_2)\,dt$$

Classification of random processes

- By stationarity
  - Characteristics = const.(t)
  - Can be computed from 'sample' interval

$$\varphi_{xx}(\tau) = E\big(x(t)\,x(t+\tau)\big)$$

$$\varphi_{xx}(0) = E(x^2)$$

$$\varphi_{xx}(-\tau) = \varphi_{xx}(\tau)$$

Empirical model identification

- Autocorrelation techniques:
  - Assumption **1**: Underlying process is **stationary** Gaussian random process
  - Assumption **2**: Data (z) sampled at **constant Δt**

$$m = \frac{1}{N}\sum_{i=1}^{N} z_i$$

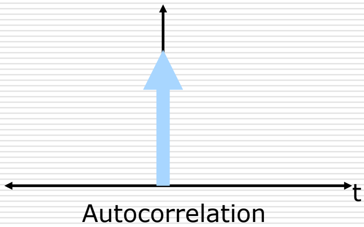
$$\varphi_{zz}\left(l\Delta t\right) = \frac{1}{N-l-1}\sum_{i=1}^{N-l}\left(z_i - m\right)\left(z_{i+l} - m\right)^T; l = 0,1\ldots N-2$$

- Time series analysis:
  - AR, MA, ARMA, ARIMA, …
  - AV, GMWM, …

White noise process

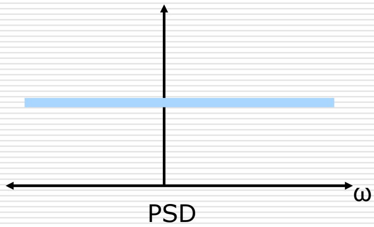
- State propagation: $x_{k+1} = w_k$
- Differential equation: theoretically not differentiable

$$\varphi_{xx}\left(\tau\right) = \varphi_{xx}\left(0\right)\delta\left(\tau\right)$$



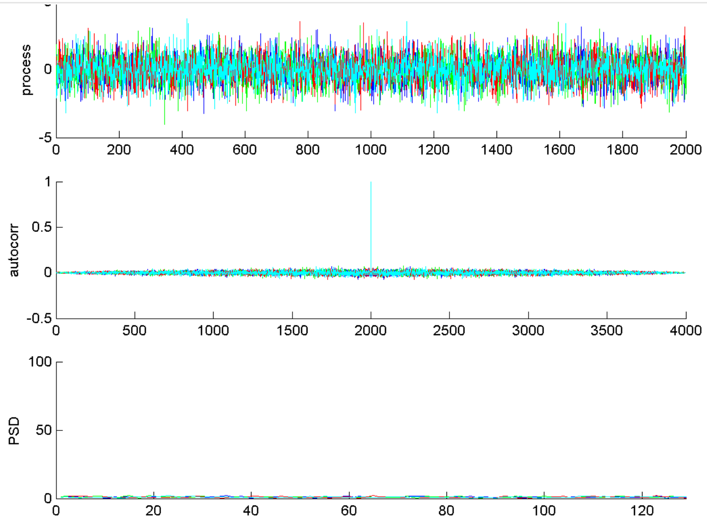
Autocorrelation

$$\Phi_{xx}\left(\omega\right) = \sigma^2$$



PSD

White noise - example

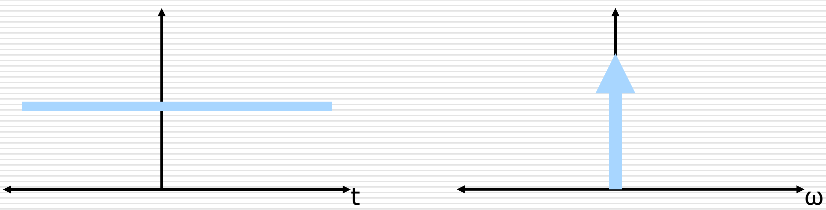


Random bias/constant

- State propagation: $x_{k+1} = x_k = m$
- Differential equation: $\dot{x}(t) = 0$

$\varphi_{xx}(\tau) = m^2$

$\Phi_{xx}(\omega) = 2\pi m^2 \delta(\omega)$



Autocorrelation

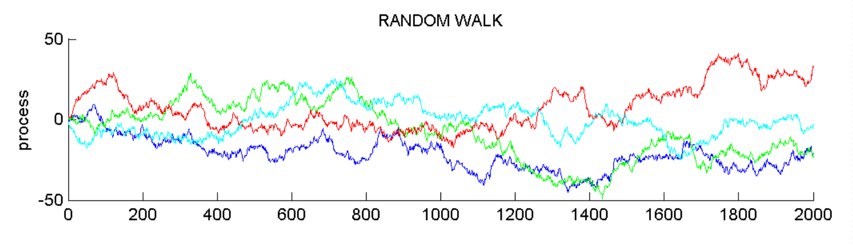
PSD

Shaped white noise

- Amplitude:
  - Gaussian probability density -> results of number of independent random variables.
- Correlation in time (frequency)
  - **White noise**: not correlated in time
  - **Colored noise**: white noise put through a small linear system can duplicate virtually any form of time-correlated noise.
- Linear transformation of white noise = **Shaping Filter**

Random walk

- State propagation: $x_{k+1} = x_k + w_k$
- Differential equation: $\dot{x}(t) = w(t)$
- Not a stationary process!

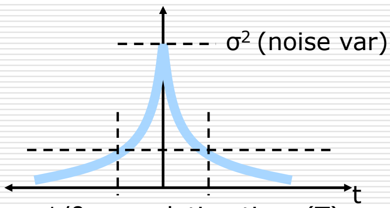


1st order Gauss-Markov process

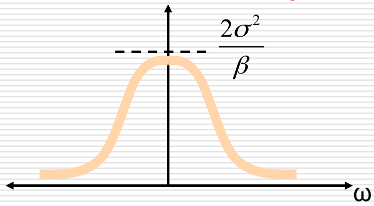
- State propagation: $x_{k+1} = e^{-\beta \Delta t} x_k + w_k$
- Differential equation: $\dot{x}(t) = -\beta x(t) + w(t)$

$$\varphi_{xx}(\tau) = \sigma^2 e^{-\beta|\tau|}$$

$$\Phi_{xx}(\omega) = \frac{2\beta\sigma^2}{\omega^2 + \beta^2}$$



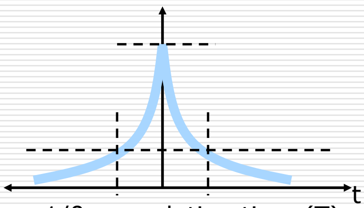
1/β=correlation time (T)



PSD

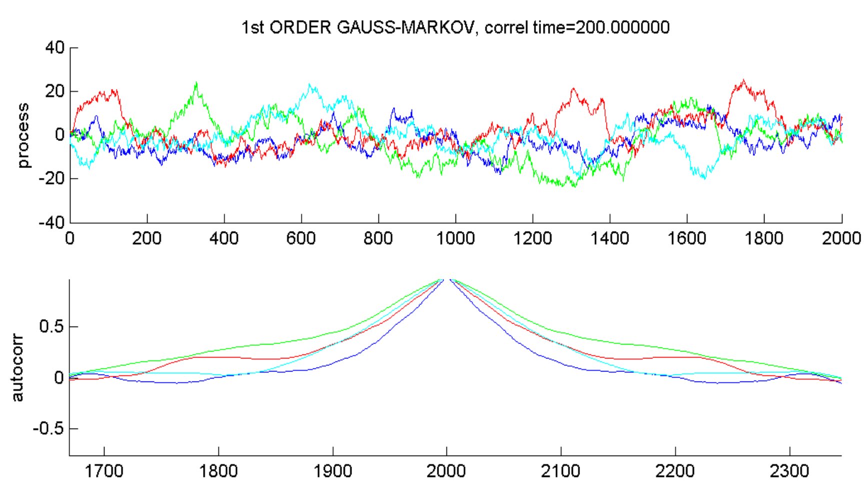
Question:

$$\varphi_{xx}(T) = \varphi_{xx}(1/\beta) = ?$$



1/β=correlation time (T)

1^{st} o. Gauss-Markov process - example



Other random processes

- Random ramp
- Auto-Regressive (AR) process
- Moving-Average (MA) process
- Auto-Regressive Moving Average (ARMA)
- :

Power spectrum (or PSD)

- PSD is the discrete Fourier transform (DFT) of AC:

$$\Phi_{xx}(\omega) = \sum_{\tau=-\infty}^{\infty} \varphi_{xx}(\tau) e^{-j\omega t}$$

- If we have a signal of length N:
  - Sample AC (AC <u>estimate</u>):

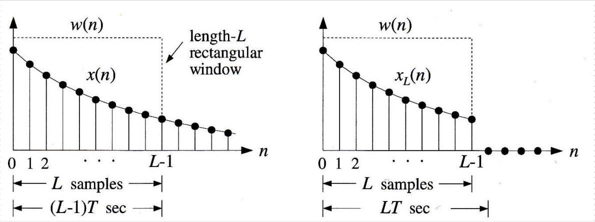
$$\hat{\varphi}_{xx}(\tau) = \frac{1}{N} \sum_{n=0}^{N-1-\tau} x_{n+\tau} x_n ; \tau = 0,1,\ldots,N-1$$

  - Periodogram spectrum (PSD <u>estimate</u>)

$$\hat{\Phi}_{xx}(\omega) = \sum_{\tau=-\infty}^{\infty} \hat{\varphi}_{xx}(\tau) e^{-j\omega t}$$

Note about windowing in DFT

- Time windowing (e.g. rectangular window)



  - Original and time-windowed spectrum:

$$\hat{\Phi}(\omega) = \sum_{n=-\infty}^{\infty} x(nT) e^{-2\pi j\omega nT}$$

$$\hat{\Phi}_L(\omega) = \sum_{n=0}^{L-1} x_L(nT) e^{-2\pi j\omega nT} = \sum_{n=-\infty}^{\infty} x_L(n) e^{-j\omega n}$$

Note about windowing in DFT

- Windowing has two (side) effects:
  - Reduction of the frequency resolution of the computed spectrum

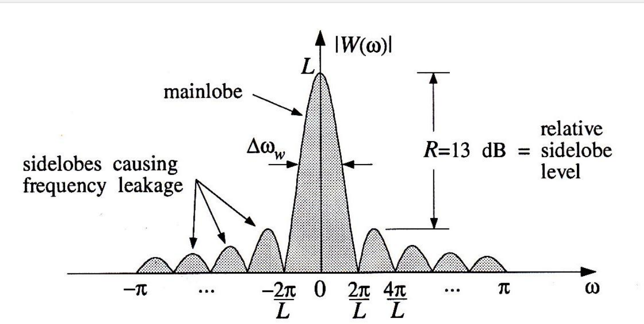
$$\Delta\omega = \frac{1}{T_L}$$

  - Frequency leakage
    - Introduction of high-frequency components into the spectrum
    - Caused by the sharp clipping of the signal at the left & right sides of the rectangular window

$$|W(\omega)| = \left|\frac{\sin(\omega L/2)}{\sin(\omega/2)}\right|$$

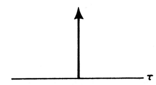
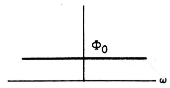
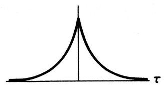
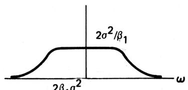
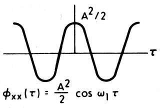
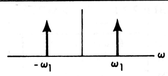
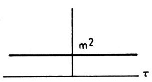
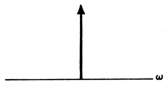
Note about windowing in DFT

- Magnitude spectrum of rectangular window:



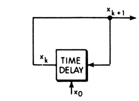
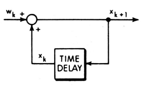
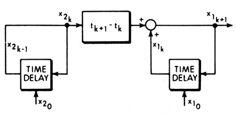
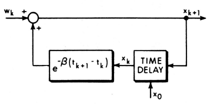
$$|W(\omega)| = \left|\frac{\sin(\omega L/2)}{\sin(\omega/2)}\right|$$

Common Random Processes (distributed)

PROCESS	AUTOCORRELATION FUNCTION	POWER SPECTRAL DENSITY
WHITE NOISE	 $\phi_{xx}(\tau) = \phi_0\,\delta(\tau)$	 $\Phi_{xx} = \Phi_0$
MARKOV PROCESS m = 0	 $\phi_{xx}(\tau) = \sigma^2 e^{-\beta_1 \lvert\tau\rvert}$	 $\Phi_{xx} = \frac{2\beta_1\sigma^2}{\omega^2+\beta_1^2}$
SINUSOID	 $\phi_{xx}(\tau) = \frac{A^2}{2}\cos\omega_1\tau$	 $\Phi_{xx} = \frac{\pi}{2}A^2\left[\delta(\omega-\omega_1)+\delta(\omega+\omega_1)\right]$
RANDOM BIAS	 $\phi_{xx}(\tau) = m^2$	 $\Phi_{xx} = 2\pi m^2\,\delta(\omega)$

After Gelb, A. (1974)

Error Models – Discrete Case (distributed)

NAME	STATE DIFFERENCE EQUATION	BLOCK DIAGRAM	RELATION of DISCRETE q_k to CONTINUOUS q
RANDOM CONSTANT	$x_{k+1} = x_k$		$q_k = 0$
RANDOM WALK	$x_{k+1} = x_k + w_k$		$q_k = q(t_{k+1}-t_k)$
RANDOM RAMP	$x_{1_{k+1}} = x_{1_k} + \overset{\Delta t}{(t_{k+1}-t_k)}\,x_{2_k}$ $x_{2_{k+1}} = x_{2_k}$		$q_k = 0$
EXPONENTIALLY CORRELATED RANDOM VARIABLE	$x_{k+1} = e^{-\beta\overset{\Delta t}{(t_{k+1}-t_k)}}\,x_k + w_k$		$q_k = \frac{q}{2\beta}\left[1-e^{-2\beta(t_{k+1}-t_k)}\right]$

After Gelb, A. (1974)

Allan variance (AV) - D. Allan 1966

- Alternative measure of variability

- = "variations of sample averages of different length τ"

$$\bar{X}_k(\tau) = \frac{1}{\tau}\sum_{j=0}^{\tau-1} X_{k-j} \qquad \sigma^2_{\bar{X}}(\tau) = \frac{1}{2}\mathbf{E}\left[\left(\bar{X}_k(\tau) - \bar{X}_{k-\tau}(\tau)\right)^2\right]$$

- Greenhall (1991) estimator of AV

$$\hat{\sigma}^2_{\bar{x}}(\tau) = \frac{1}{2(N-2\tau+1)}\sum_{k=2\tau}^{N}\left(\bar{x}_k(\tau) - \bar{x}_{k-\tau}(\tau)\right)^2$$

$$x_k : k = 1,\ldots,N$$

Process identification in AV plots