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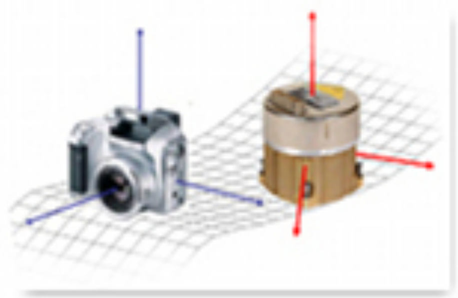


Sensor Orientation – Random Processes

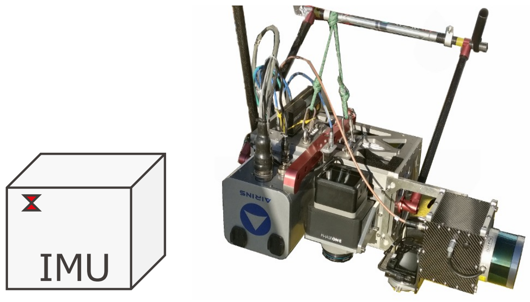
Jan SKALOUD

EPFL, spring semester 2024

École polytechnique fédérale de Lausanne

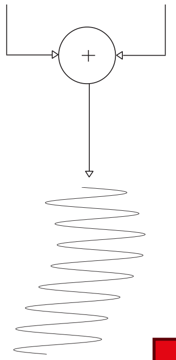
Welcome to the SO course!

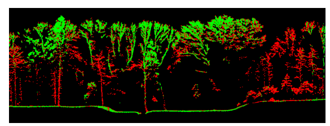
This translates into three rough big areas



$$\begin{bmatrix} \delta\dot{p} \\ \delta\dot{v} \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & g \\ 0 & \frac{1}{R} & 0 \end{bmatrix} \begin{bmatrix} \delta p \\ \delta v \\ \varphi \end{bmatrix} + \begin{bmatrix} 0 \\ \varepsilon_a \\ \varepsilon_g \end{bmatrix}$$

1. Fundamentals
   - How to characterize sensor noise
   - How to transform from the sensed signals to navigation frame?
2. Position, velocity, attitude (navigation)
   - How to formulate navigation equation in different frames?
   - How to resolve them numerically?
3. Sensor fusion
   - How to formulate models for sensor fusion?
   - How to implement it in optimization and use it for mapping?





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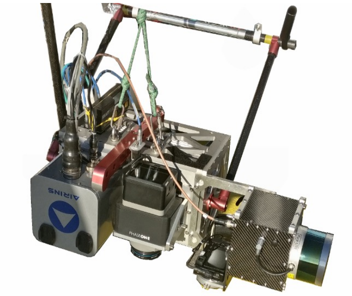
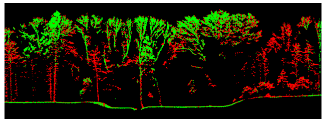
Sensor orientation – main topics

This translates into three rough big areas





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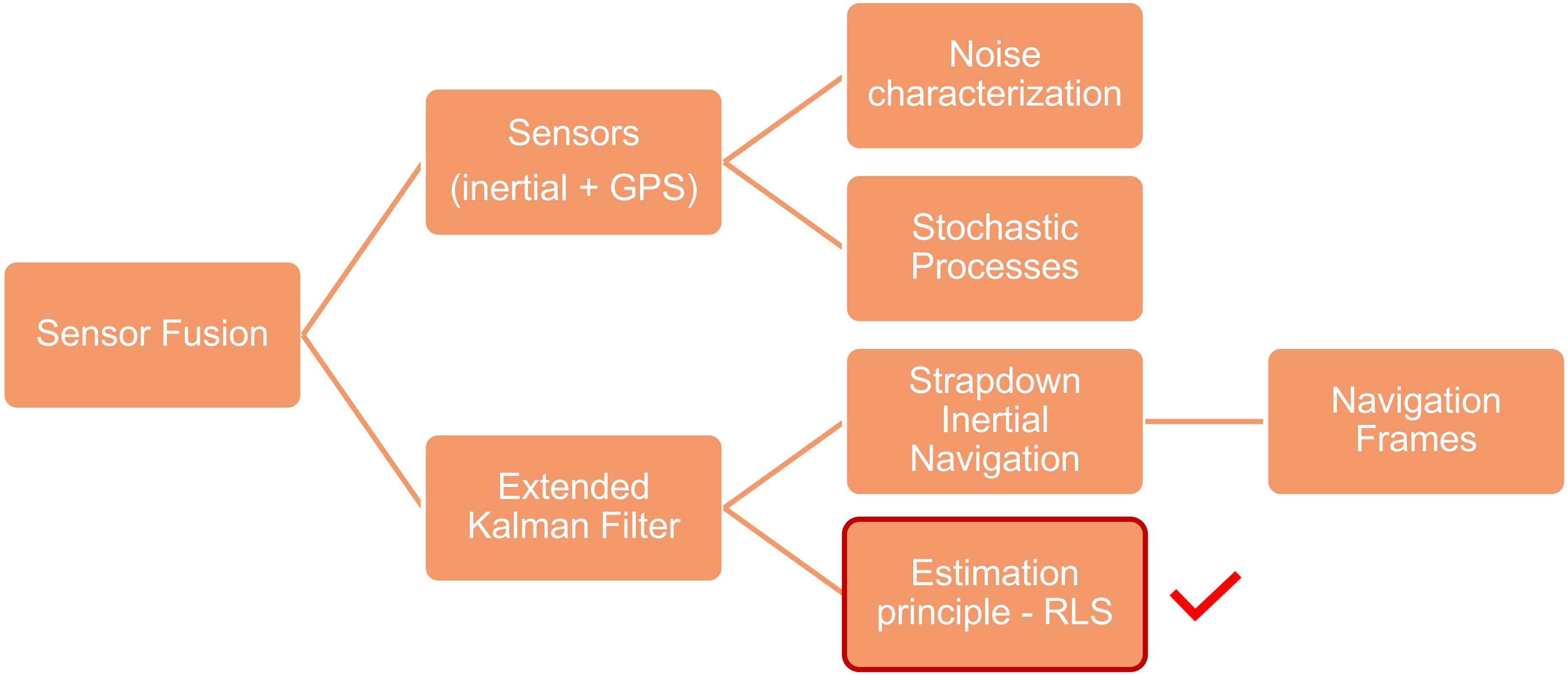
You need the frames

You need the navigation quantities and the noise properties

Cockpit view of SO course’s topics

How to reach *integrated* sensor orientation?



Similarities RLS vs. Kalman Filter (2.2.3)

Recursive Least Square (RLS)

(2.28-2.30) gain, state, covariance:

$$
\begin{aligned}
\mathbf{K}_j &= \mathbf{P}_{j-1}\,\mathbf{H}_j^{\mathrm{T}}\left(\mathbf{H}_j\,\mathbf{P}_{j-1}\,\mathbf{H}_j^{\mathrm{T}}+\mathbf{R}_j\right)^{-1}\\
\hat{\mathbf{x}}_j &= \hat{\mathbf{x}}_{j-1}+\mathbf{K}_j\left(\mathbf{z}_j-\mathbf{H}_j\,\hat{\mathbf{x}}_{j-1}\right)\\
\mathbf{P}_j &= \left(\mathbf{I}-\mathbf{K}_j\,\mathbf{H}_j\right)\mathbf{P}_{j-1}
\end{aligned}
$$

Discrete Kalman Filter (KF)

(2.32-2.34) gain, state, covariance:

$$
\begin{aligned}
\mathbf{K}_k &= \tilde{\mathbf{P}}_k\mathbf{H}_k^{\mathrm{T}}\left(\mathbf{H}_k\tilde{\mathbf{P}}_k\mathbf{H}_k^{\mathrm{T}}+\mathbf{R}_k\right)^{-1}\\
\hat{\mathbf{x}}_k &= \tilde{\mathbf{x}}_k+\mathbf{K}_k\left(\mathbf{z}_k-\mathbf{H}_k\tilde{\mathbf{x}}_k\right)\\
\mathbf{P}_k &= \left(\mathbf{I}-\mathbf{K}_k\mathbf{H}_k\right)\tilde{\mathbf{P}}_k
\end{aligned}
$$

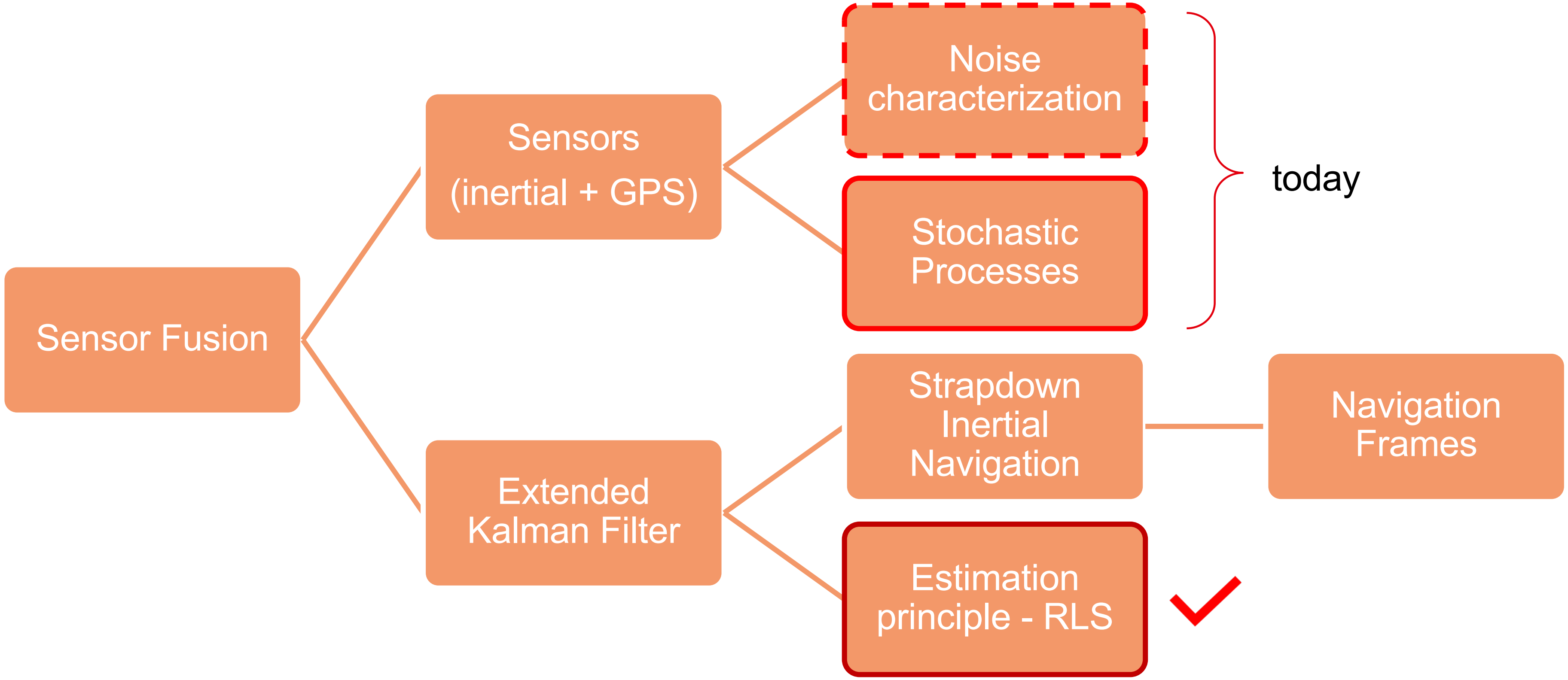
Time update (prediction)

$$
\begin{aligned}
\tilde{\mathbf{x}}_{k+1} &= \mathbf{\Phi}_k\hat{\mathbf{x}}_k\\
\tilde{\mathbf{P}}_{k+1} &= \mathbf{\Phi}_k\mathbf{P}_k\mathbf{\Phi}_k^{\mathrm{T}}+\boxed{\mathbf{Q}_k}
\end{aligned}
$$

system / sensor noise !

Cockpit view of SO course’s topics

How to reach *integrated* sensor orientation?

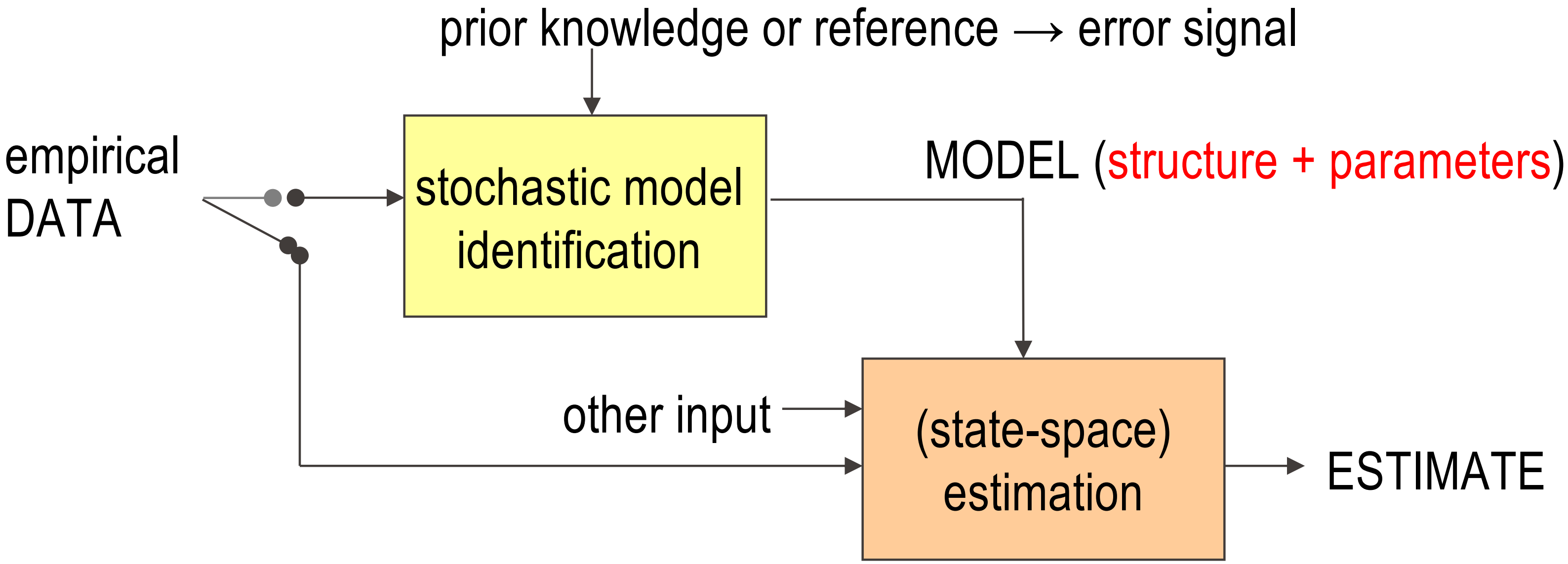


Stochastic Processes – agenda (Ch 5)

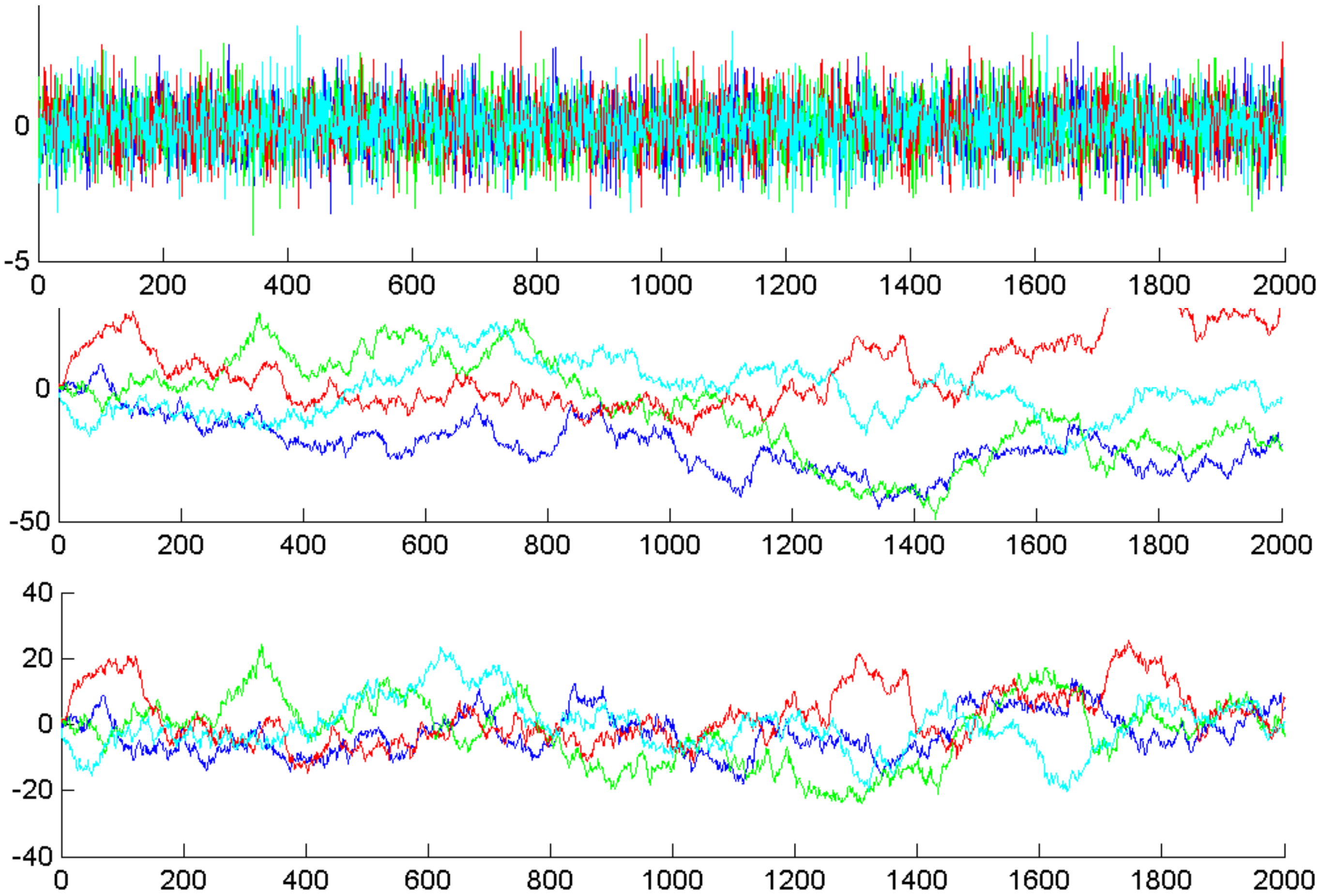
- Introduction
  - Model identification
  - Concept & classification
- Process characteristics
  - Auto-correlation (**AC**)
  - Power Spectral Density (**PSD**)
  - Allan Variance (**AV**) / Wavelet Variance (**WV**)
- The very useful processes
  - White Noise
  - Random bias
  - Random walk
  - Auto-regressive white noise (Gauss-Markov)

Noise model (process) identification

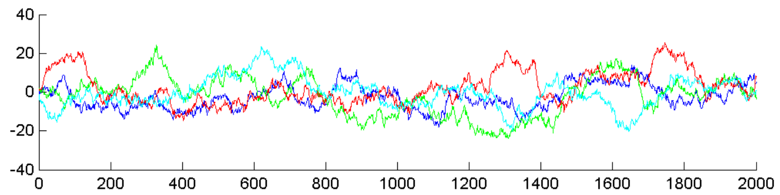
- Stochastic model
  - Prerequisite for sensor fusion (e.g. Kalman filtering)
  - Two elements needed:
    - structure (type of parameters)
    - Values of parameter
- How to identify?



Random process – an example



Random processes – general concept



Association of random variable(s) with a deterministic parameter (e.g. time)

Stochastic **collection** – values depend on the outcome of an experiment

Time is a **parameter,** but the process is only LOOSELY (stochastic) function of time

Can be either *discrete* or continuous

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Connections to random variables

Random variable

Probability density

$$f(x) \quad f(y)$$

Expectation

$$E(x) = \mu_X = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$E(y) = \mu_Y = \int_{-\infty}^{\infty} y\, f(y)\, dy$$

Random process

Probability density

$$f(x_i, t_i), f(x_j, t_j) \longrightarrow f_2(x_i, x_j)$$

Expectation

$$\mu_X(t) = \int_{-\infty}^{\infty} x_t\, f(x_t)\, dx_t$$

Connections to random variables

Random variable

Probability density

$$f(x) \quad f(y)$$

Expectation

$$E(x) = \mu_X = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$E(y) = \mu_Y = \int_{-\infty}^{\infty} y\, f(y)\, dy$$

Covariance

$$cov(x, y) = E\left[(x - \mu_X)(y - \mu_Y)\right]$$

Random process

Probability density

$$f(x_i, t_i), f(x_j, t_j) \longrightarrow f_2(x_i, x_j)$$

Expectation

$$\mu_X(t) = \int_{-\infty}^{\infty} x_t\, f(x_t)\, dx_t$$

Joint expectation – autocovariance

$$\mathrm{acov}_X(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x_1 - \mu_{X_1})(x_2 - \mu_{X_2}) f(x_1, x_2)\, dx_1\, dx_2$$

Connections to random variables

Random variable

Probability density

$$f(x) \quad f(y)$$

Expectation

$$E(x) = \mu_X = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$E(y) = \mu_Y = \int_{-\infty}^{\infty} y\, f(y)\, dy$$

Covariance

$$cov(x, y) = E\left[(x - \mu_X)(y - \mu_Y)\right]$$

Variance

$$var(x) = cov(x, x) = E\left[(x - \mu_X)^2\right]$$

Random process

Probability density

$$f(x_i, t_i), f(x_j, t_j) \longrightarrow f_2(x_i, x_j)$$

Expectation

$$\mu_X(t) = \int_{-\infty}^{\infty} x_t\, f(x_t)\, dx_t$$

Joint expectation – autocovariance

$$\mathrm{acov}_X(t_1, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x_1 - \mu_{X_1})(x_2 - \mu_{X_2}) f(x_1, x_2)\, dx_1\, dx_2$$

Variance

$$\mathrm{acov}_X(\tau = 0) = \varphi_{xx}(0) = \sigma_X^2$$

Other characteristics

Random variable

Probability density

$f(x)$

Fourier transform of probability density (characteristic function)

$$\begin{aligned} g(\omega) &= E\left[e^{-i\omega x}\right] \\ &= \int_{-\infty}^{\infty} e^{-i\omega x} f(x)\, dx \end{aligned}$$

Random process

Probability density

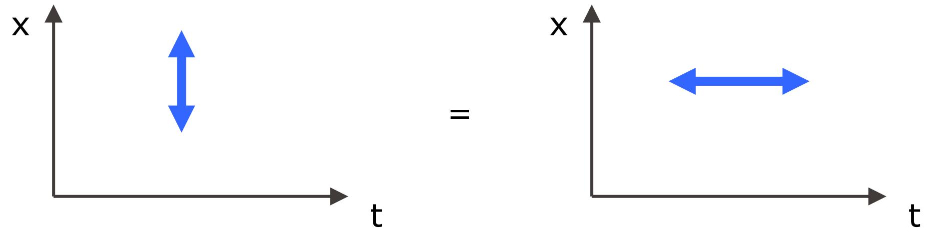
$$f(x_i, t_i), f(x_j, t_j) \longrightarrow f_2(x_i, x_j)$$

Power spectral density – PSD (Fourier transform of autocovariance function)

$$\Phi_X(\omega) = \int_{-\infty}^{\infty} \varphi_{xx}(\tau)\, e^{-i\, 2\pi\omega\, \tau} d\tau$$

Classification of random processes

- By **ergodicity***



$$E(x) = \lim_{t\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, dt$$

$$E(x^2) = \lim_{t\to\infty} \frac{1}{2T} \int_{-T}^{T} x^2(t)\, dt$$

$$\varphi_{xx}(t_1, t_2) = \lim_{t\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t_1)\, x(t_2)\, dt$$

* Process characteristics computed over one long process are expected to be same as those computed from many processes sampled at a specific time

Classification of random processes

By **stationarity**

- Process characteristics are constant in time (t)
- i.e. can be computed from a ”sample interval” (rather than absolute time)

$$\varphi_{xx}(\tau) = E\big(x(t)x(t+\tau)\big)$$

$$\varphi_{xx}(0) = E(x^2)$$

$$\varphi_{xx}(-\tau) = \varphi_{xx}(\tau)$$

Autocorrelation function (AC)

Autocorrelation function:

- Assumption **1**: Underlying process is **stationary** Gaussian random process
- Assumption **2**: Data (z) are sampled at **constant Δt**

$$m = \frac{1}{N}\sum_{i=1}^{N} z_i$$

$$\varphi_{zz}(l\Delta t) = \frac{1}{N-l-1}\sum_{i=1}^{N-l}(z_i - m)(z_{i+l} - m)^T ; l = 0,1\ldots N-2$$

Other time series (or frequency) analysis (later):

- AR, MA, ARMA, ARIMA, … + estimators

Stochastic Processes – agenda (Ch 5)

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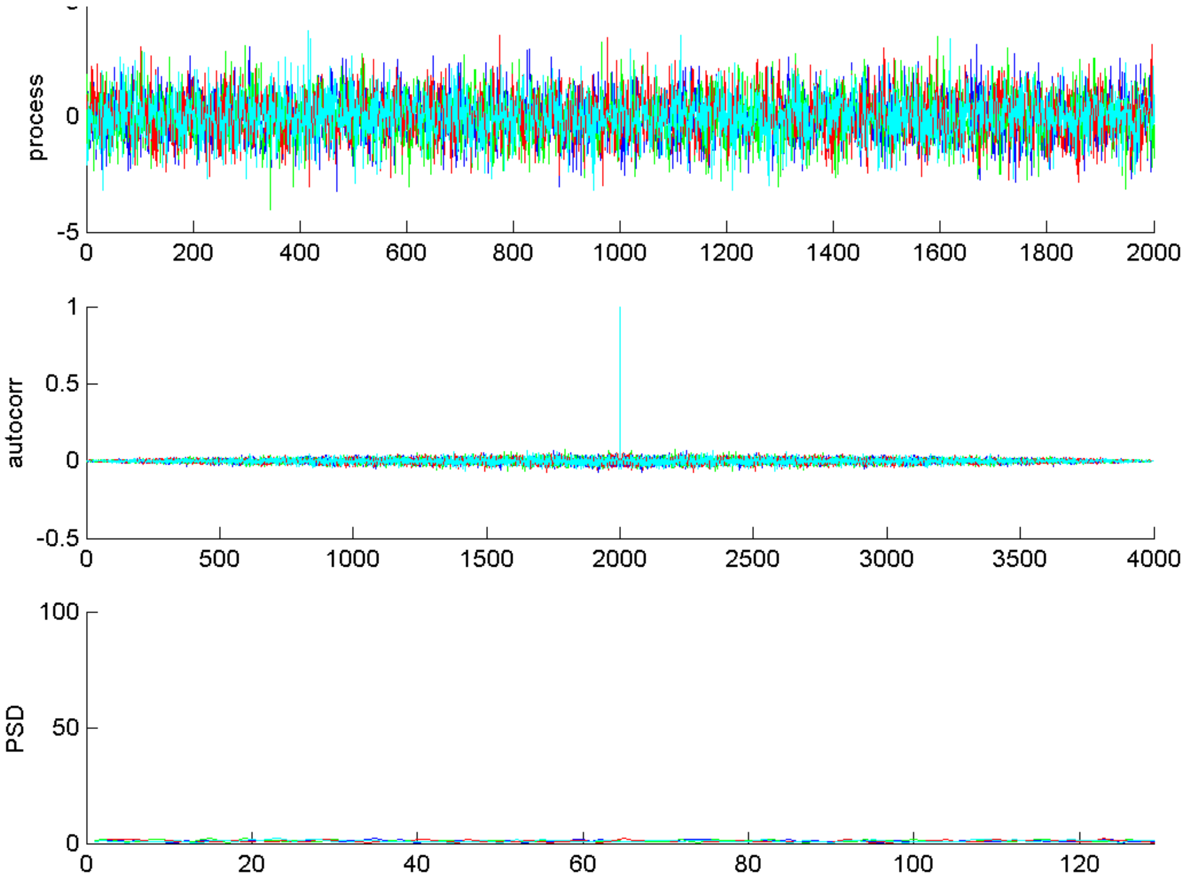
  - Random walk
  - Auto-regressive white noise (Gauss-Markov)

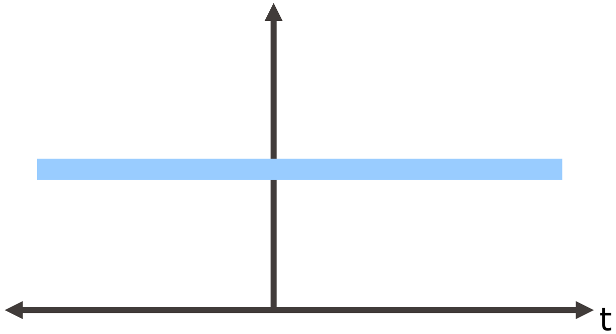
White noise - example



Random bias (constant)

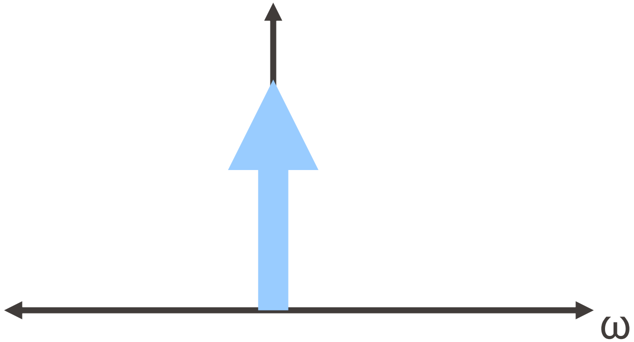
- State propagation: $x_{k+1} = x_k = m$
- Differential equation: $\dot{x}(t) = 0$

$\varphi_{xx}(\tau) = m^2$



Autocorrelation

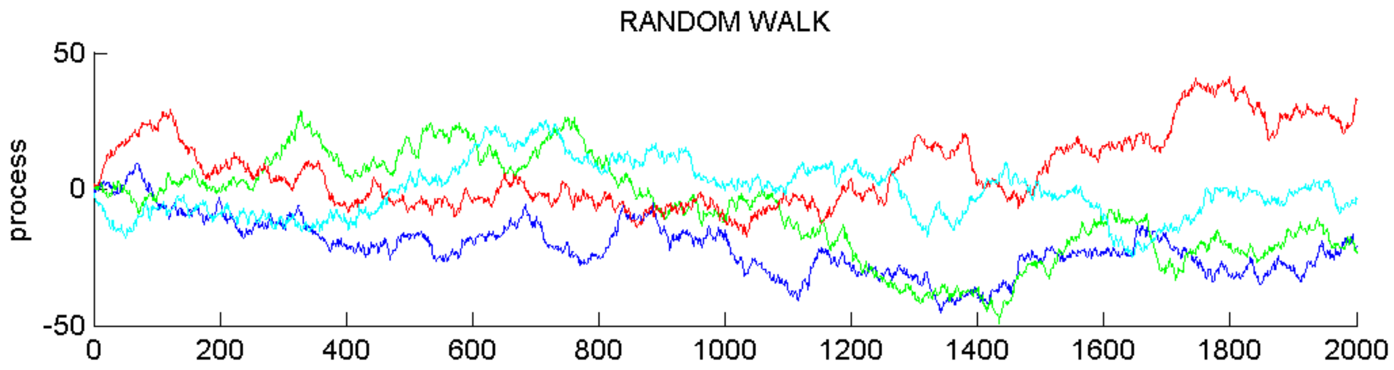
$\Phi_{xx}(\omega) = 2\pi m^2 \delta(\omega)$



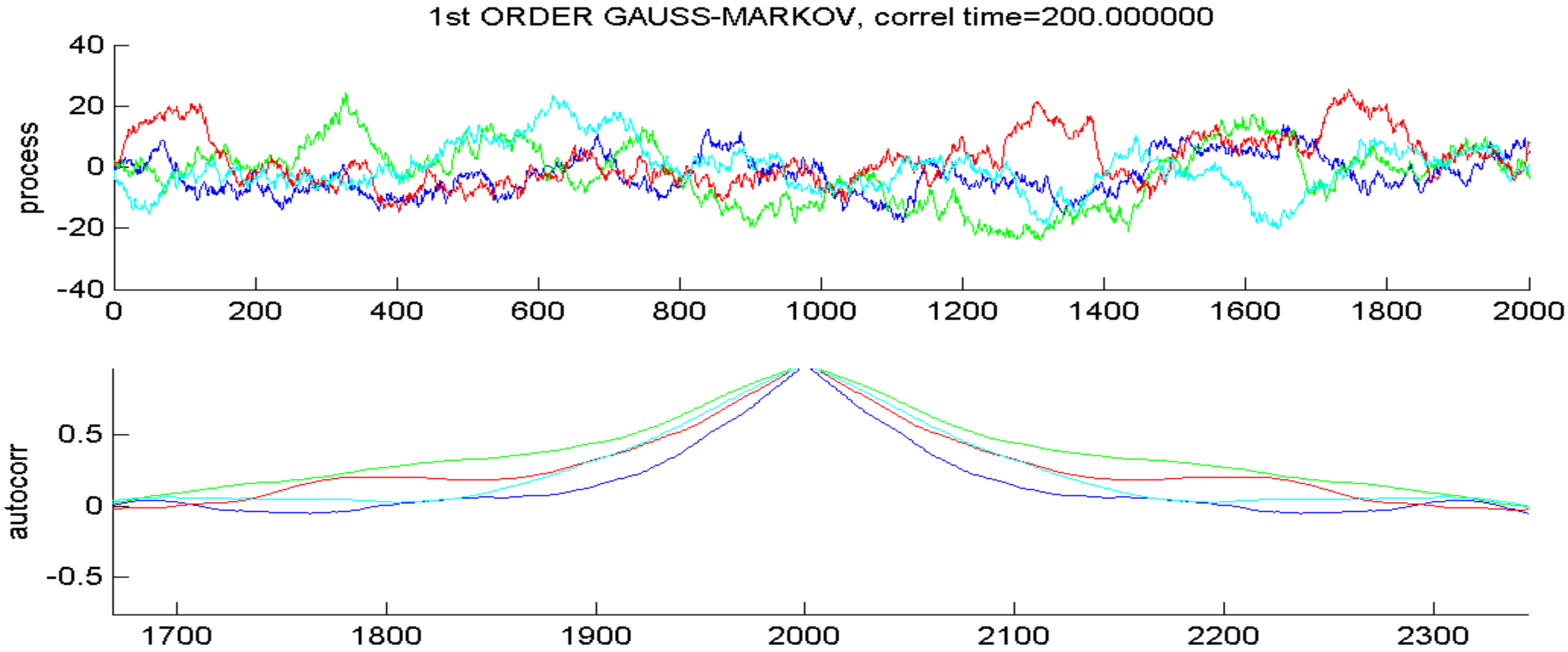
PSD

Random walk

- Differential equation: $$\dot{x}(t) = w(t)$$
- State propagation: $$x_{k+1} = x_k + w_k$$
- NOT a stationary process!



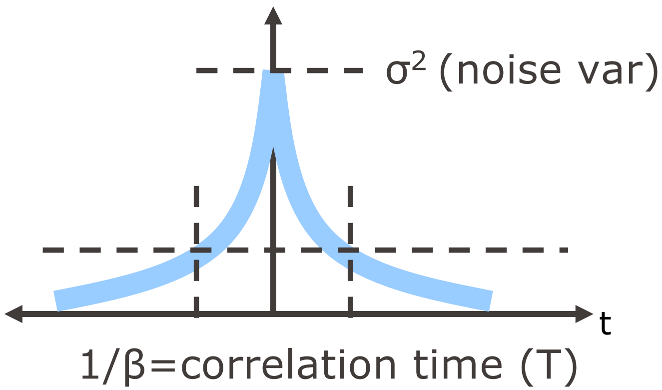
1st order Gauss-Markov process - example



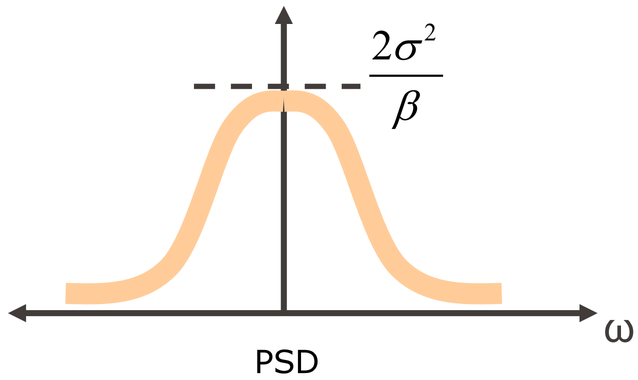
1st order Gauss-Markov process

- Differential equation: $\dot{x}(t) = -\beta x(t) + w(t)$
- State propagation: $x_{k+1} = e^{-\beta \Delta t} x_k + w_k$

$$\varphi_{xx}(\tau) = \sigma^2 e^{-\beta|\tau|}$$



$$\Phi_{xx}(\omega) = \frac{2\beta\sigma^2}{\omega^2 + \beta^2}$$



Other random processes

Random ramp

Quantization noise

Periodic (sinusoid)

Auto-Regressive (AR) process

Moving-Average (MA) process

Auto-Regressive Moving Average (ARMA)

…

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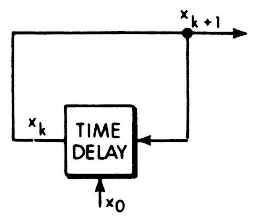
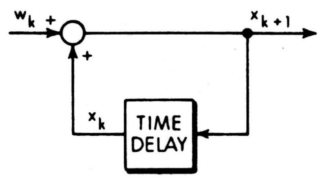
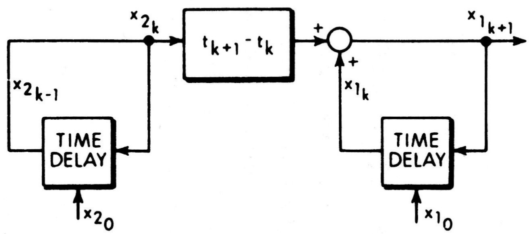
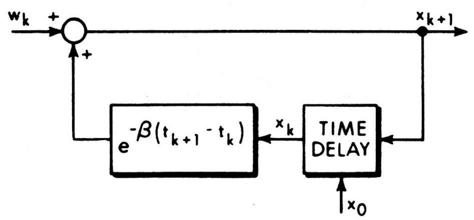
Error Models – Discrete Case (distributed)

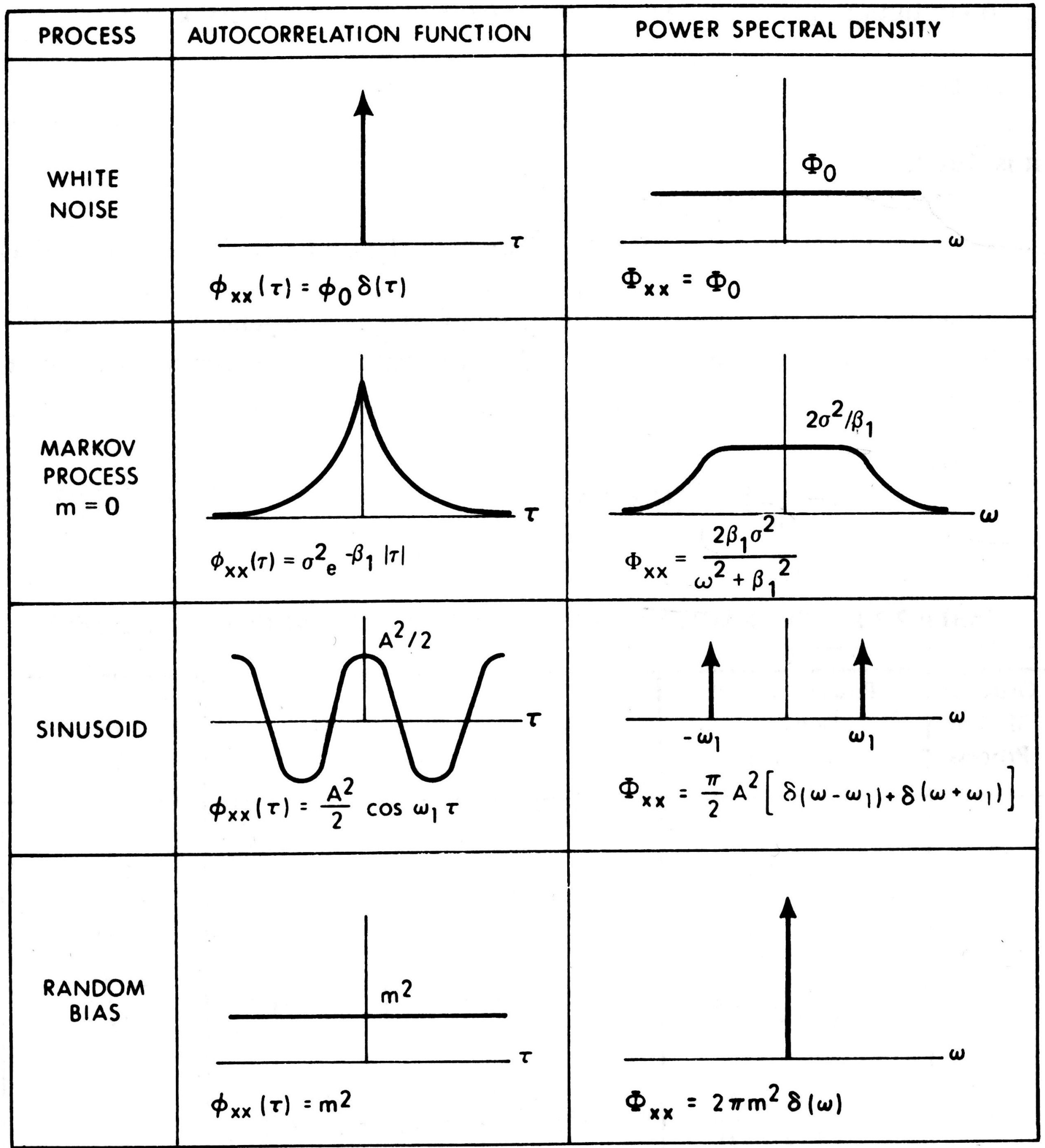
NAME	STATE DIFFERENCE EQUATION	BLOCK DIAGRAM	RELATION of DISCRETE q_k to CONTINUOUS q
RANDOM CONSTANT	$x_{k+1} = x_k$		$q_k = 0$
RANDOM WALK	$x_{k+1} = x_k + w_k$		$q_k = q(t_{k+1} - t_k)$
RANDOM RAMP	$x1_{k+1} = x1_k + \overset{\Delta t}{(t_{k+1}-t_k)}\, x2_k$ $x2_{k+1} = x2_k$		$q_k = 0$
EXPONENTIALLY CORRELATED RANDOM VARIABLE	$x_{k+1} = e^{-\beta \overset{\Delta t}{(t_{k+1}-t_k)}} x_k + w_k$		$q_k = \frac{q}{2\beta}\left[1 - e^{-2\beta(t_{k+1}-t_k)}\right]$

After Gelb, A. (1974)

J. Skaloud, ESO

Common Random Processes

(distributed)



After Gelb, A. (1974)

Autocorrelation function (AC) - estimation

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Other time series (or frequency) analysis (later):

- AR, MA, ARMA, ARIMA, … + estimators

Autocorrelation function (AC) - estimation

MATLAB

xcorr(...,SCALEOPT)

- normalizes the correlation according to SCALEOPT:

1. 'biased' - scales the raw cross-correlation by 1/M (M=vec length).
2. 'unbiased' - scales the raw correlation by 1/(M-abs(lags)).
3. 'normalized' or 'coeff' - normalizes the sequence so that the auto-correlations at zero lag are identically 1.0.
4. 'none' - no scaling (this is the default).

Power spectrum (or PSD)

- PSD is the Discrete Fourier Transform (DFT) of AC:

$$\Phi_{xx}(\omega) = \sum_{\tau=-\infty}^{\infty} \varphi_{xx}(\tau) e^{-j\omega t}$$

- If we have a signal of length N:
  - Sample AC (AC estimate):

$$\hat{\varphi}_{xx}(\tau) = \frac{1}{N-\tau-1} \sum_{n=0}^{N-1-\tau} x_{n+\tau} x_n ; \qquad \tau = 0,1,\ldots,N-1$$

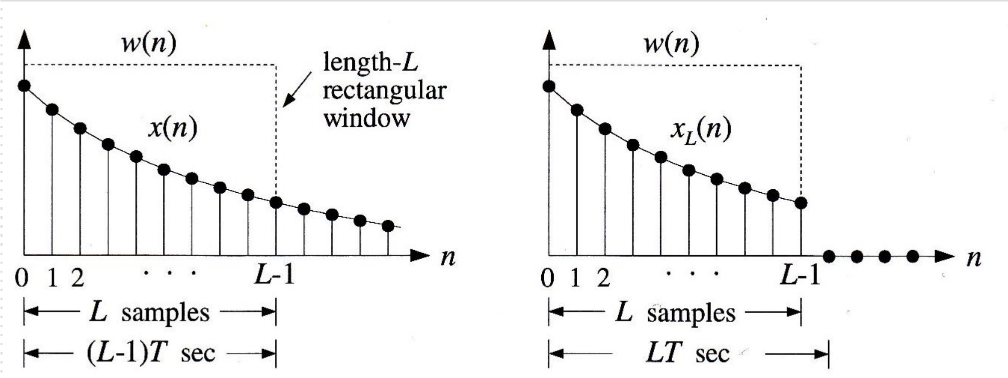
  - Periodogram spectrum (PSD estimate)

$$\hat{\Phi}_{xx}(\omega) = \sum_{\tau=-\infty}^{\infty} \hat{\varphi}_{xx}(\tau) e^{-j\omega\tau}$$

J. Skaloud, ESO

General note about windowing in DFT

- Time windowing -rectangular window for finite data



$$x(nT) = \varphi_{xx}(\tau)$$

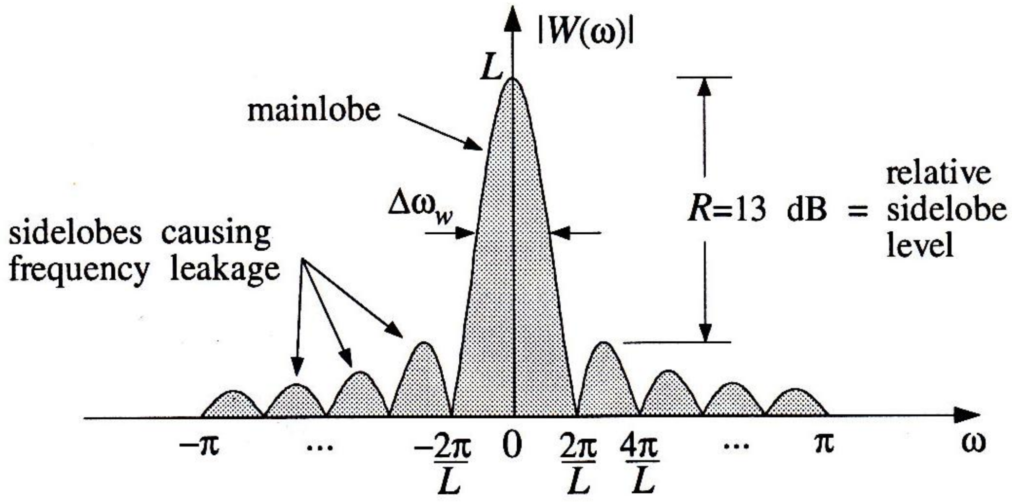
- Original and time-windowed spectrum:

$$\hat{\Phi}(\omega) = \sum_{n=-\infty}^{\infty} x(nT) e^{-j\omega nT}$$

$$\hat{\Phi}_L(\omega) = \sum_{n=0}^{L-1} x_L(nT) e^{-j\omega nT} = \sum_{n=-\infty}^{\infty} x_L(n) e^{-j\omega n}$$

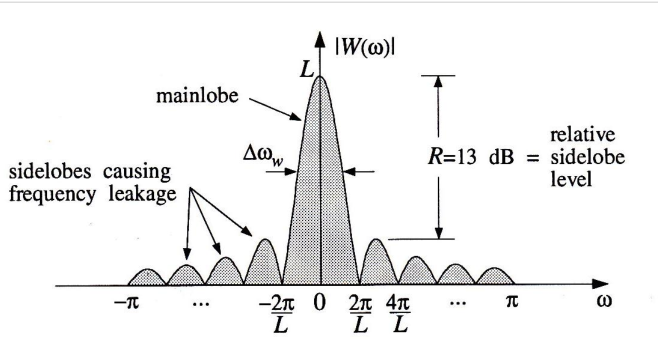
General note about windowing in DFT

- Magnitude spectrum of rectangular window:



$$|W(\omega)| = \left| \frac{\sin(\omega L/2)}{\sin(\omega/2)} \right|$$

General note about windowing in DFT



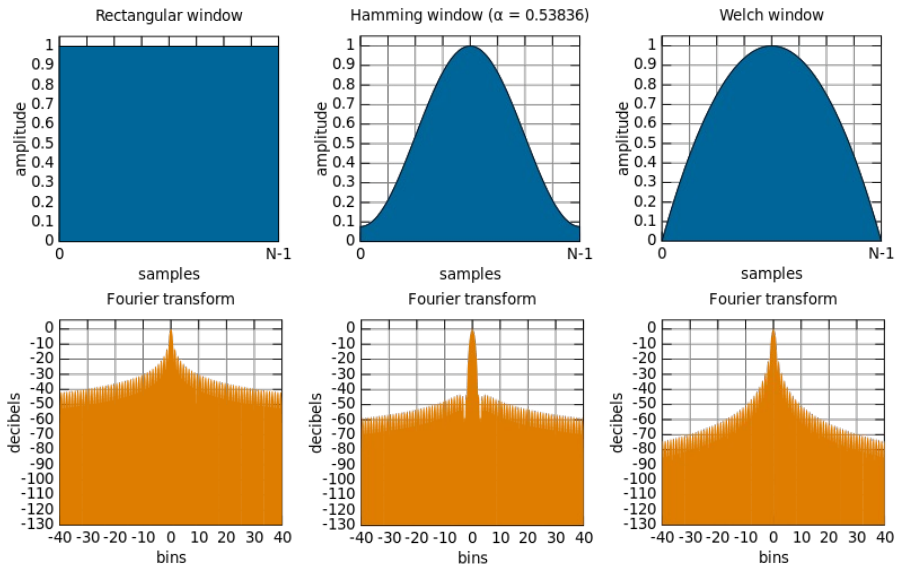
Reduces sidelobes

Reduces leakage

Narrows the main lobe

Some windowing function:

- Welch
- Hamming
- Hanning
- Kaiser
- …



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Allan variance (AV) - D. Allan 1966

Alternative measure of variability

= "variations of sample averages of different length τ"

$$\bar{X}_k(\tau) = \frac{1}{\tau}\sum_{j=0}^{\tau-1} X_{k-j} \qquad \sigma^2_{\bar{X}}(\tau) = \frac{1}{2}\mathbf{E}\left[\left(\bar{X}_k(\tau) - \bar{X}_{k-\tau}(\tau)\right)^2\right]$$

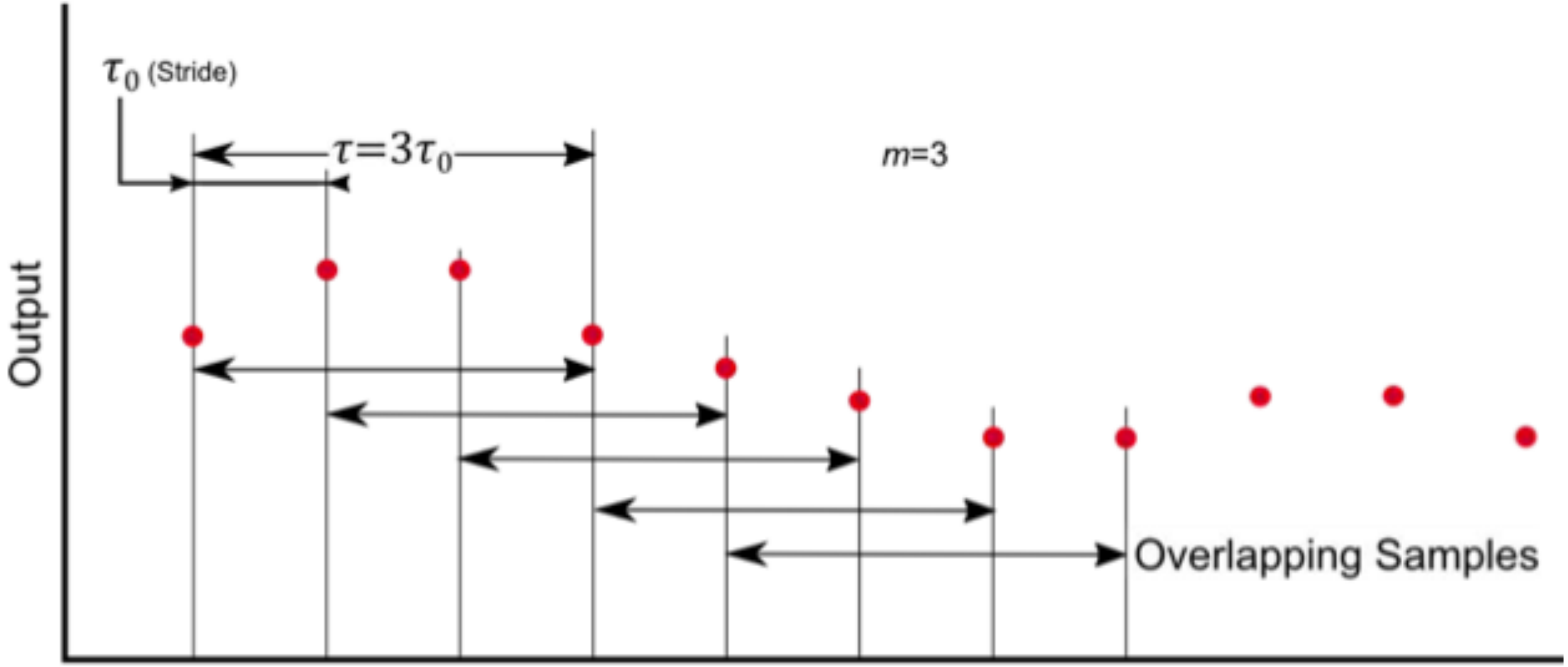
Greenhall (1991) estimator of AV

$$\hat{\sigma}^2_{\bar{x}}(\tau) = \frac{1}{2(N-2\tau+1)}\sum_{k=2\tau}^{N}\left(\bar{x}_k(\tau) - \bar{x}_{k-\tau}(\tau)\right)^2$$

$$x_k : k = 1,\ldots,N$$

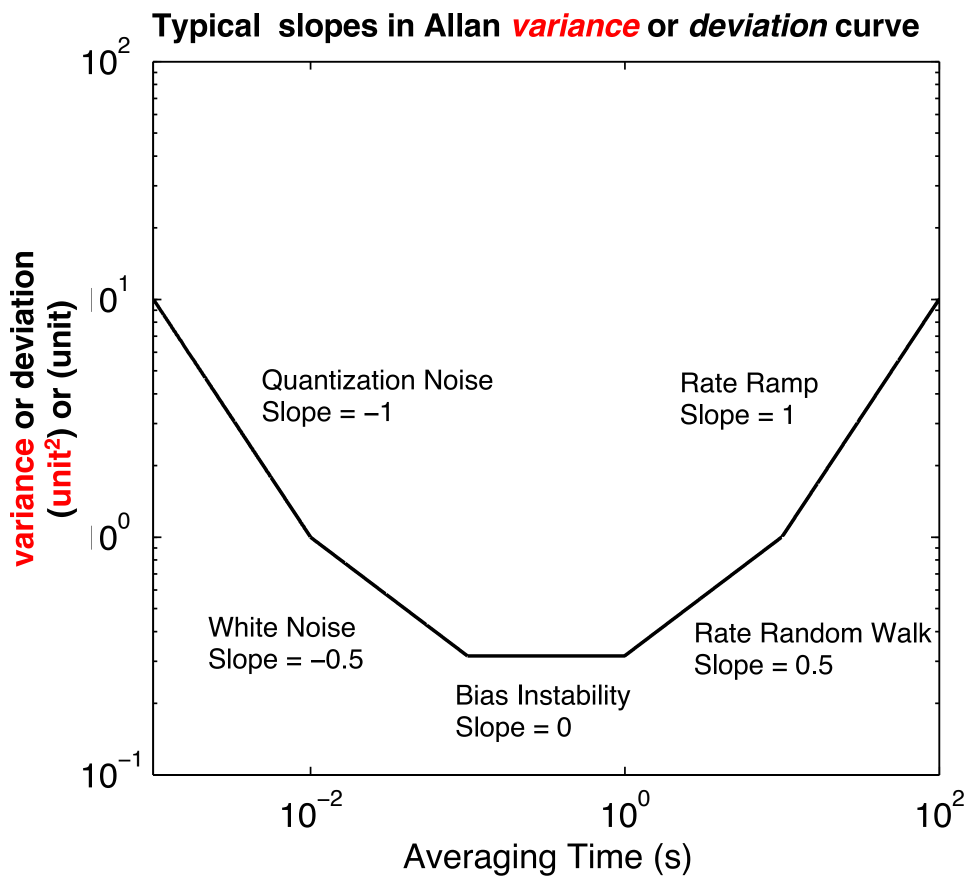
Allan variance – with sample overlap

Concept of AV with overlapping samples



source: Freescale Semiconductor, Application note #AN5087, Rev. 0,2/2015

Process identification in AV plots

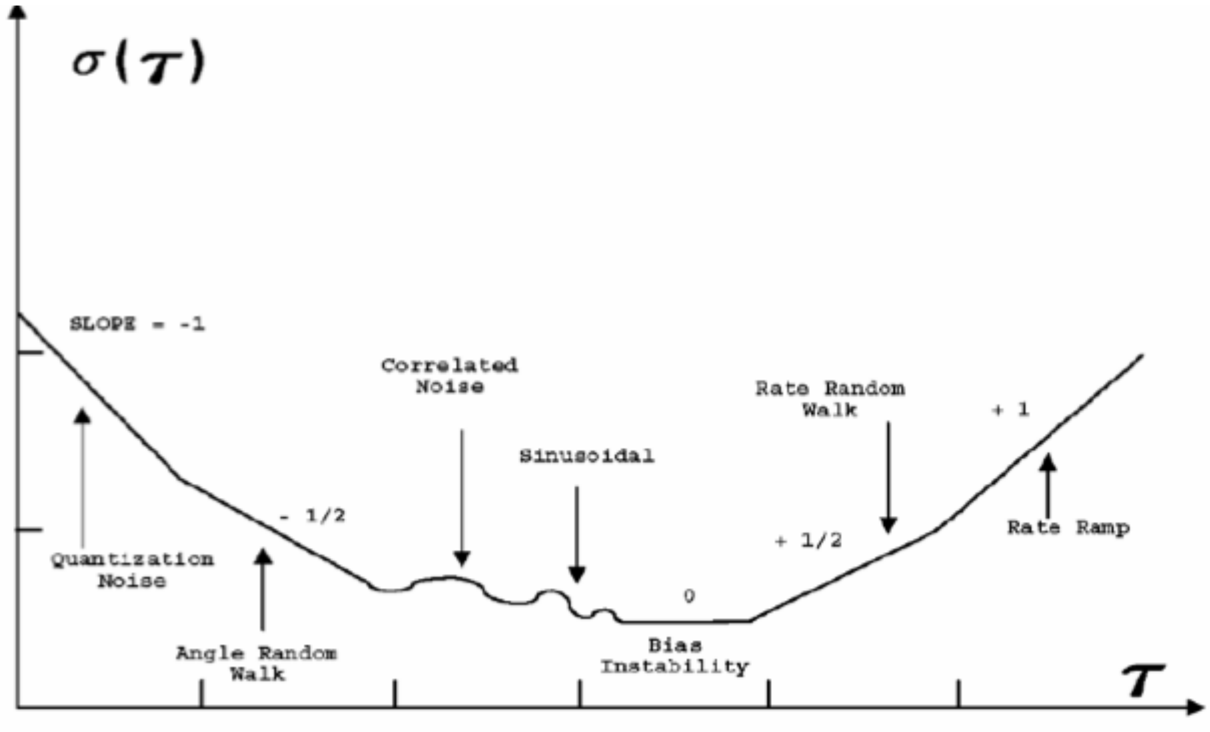


Remarks:

- Forward mapping:
  - Easy
  - IEEE standard
- Inverse:
  - not always
  - not a standard
  - Use GMWM …

Allan deviation – process signature

Complex error structure -> parameter estimation difficult



AV of a “static” gyro, source: scientific publication

EPFL

Latent time series models estimation methods

Transformation into a “non-latent” model (e.g. ARMA)

- Does not work in general
- Difficult to “inverse”

“Graphical” / **lin. regression** method

- Limited to a few possible models
- Not consistent in general and “inefficient”

Maximum Likelihood Estimation (**MLE**) / EM algorithm

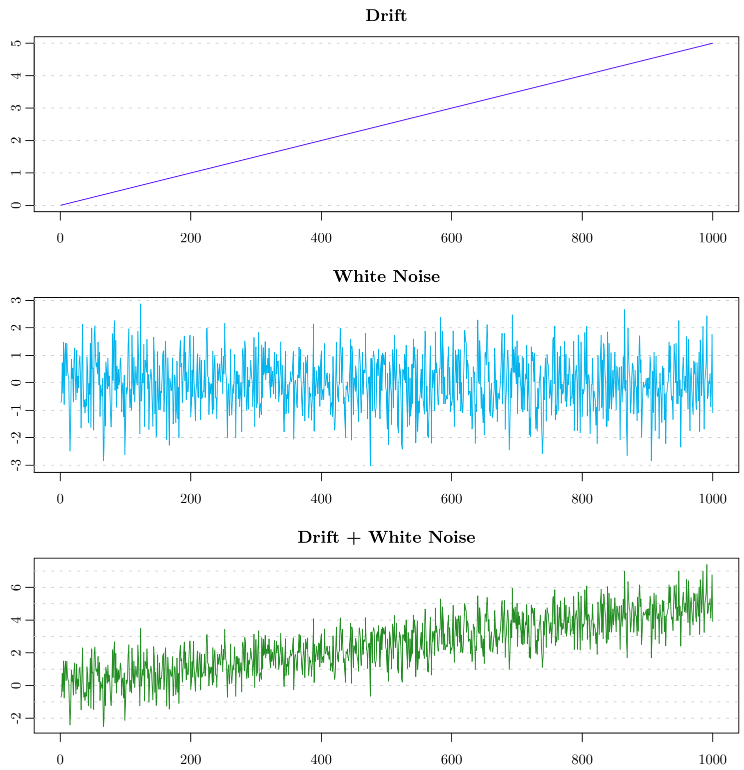
- Computationally intensive, laborious for new models
- Diverges with “complex” models

General Method of Wavelet Moment (**GMWM**)

- Rigorous, precise and efficient (*from students in this course*)
- Released as a freely available package in “R” (v1. 2015)
- Released as a GUI online tool (v1. 2018)
- http://ggmwm.smac-group.com/

Simple example 1

- Stochastic process: WN + ramp/drift



- Simple linear regression model:

$$y_t = \omega t + \varepsilon_t$$

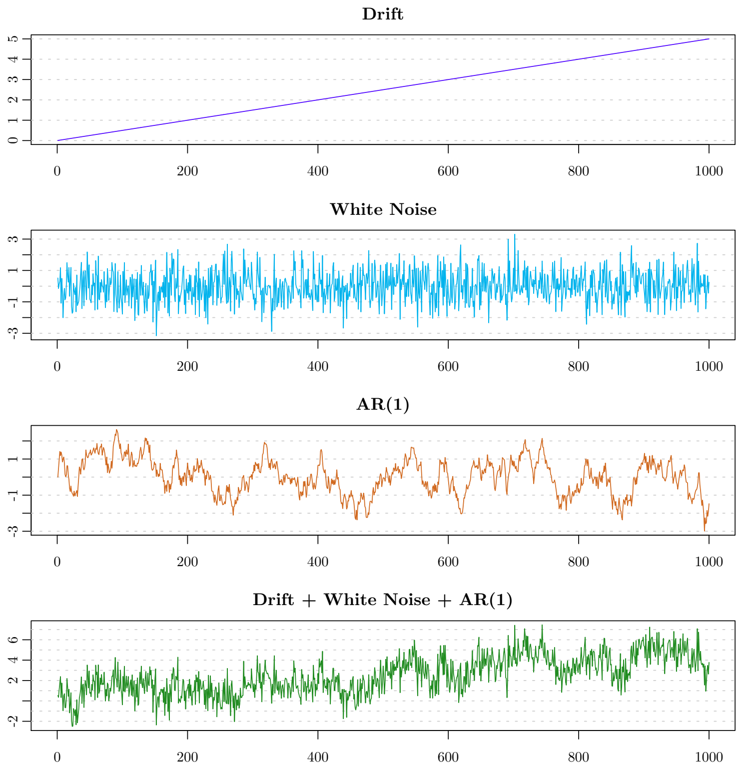
$$\varepsilon_t \sim N\left(0, \sigma^2\right)$$

- MLE is perfectly fine

- What about if we add AR(1) (e.g. GM) process?

Simple example 2

- Stochastic process: WN + ramp/drift + AR1(GM1)



- Not a linear regression model but **state-space model**
- Computing the likelihood is not an easy task (e.g. Kalman filter)
- Practice: MLE (in fact EM-KF) fails

GMWM idea

Fact: AV is a special case of the wavelet variance (WV)

- AV = 2 * WV (for Haar wavelet)

Match Wavelet Variances

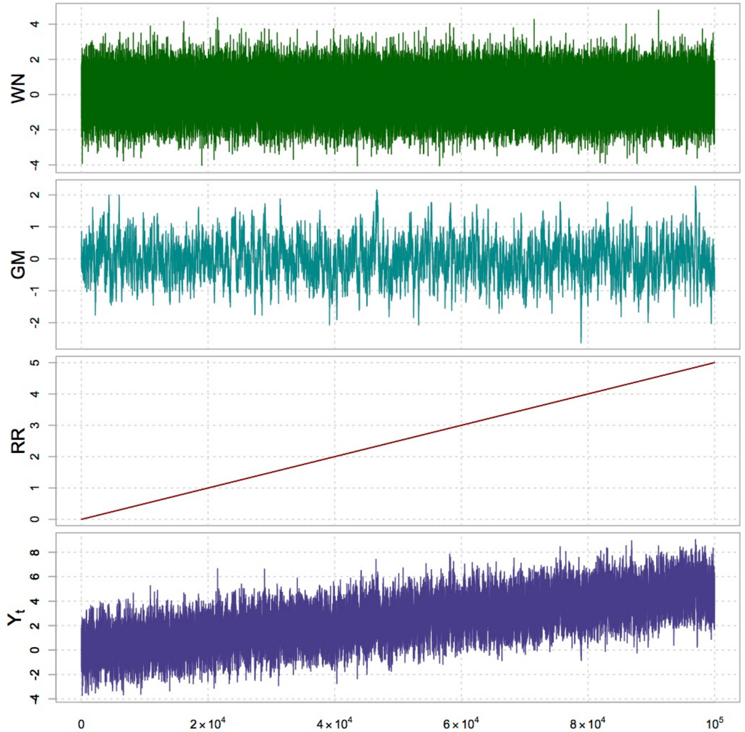
- Exploit the known relation between a model θ and its "WV signature" $\nu(\theta)$
- Inverse the mapping by minimizing the discrepancy between WV from data: $\hat{\nu}$ and $\nu(\theta)$ from model:

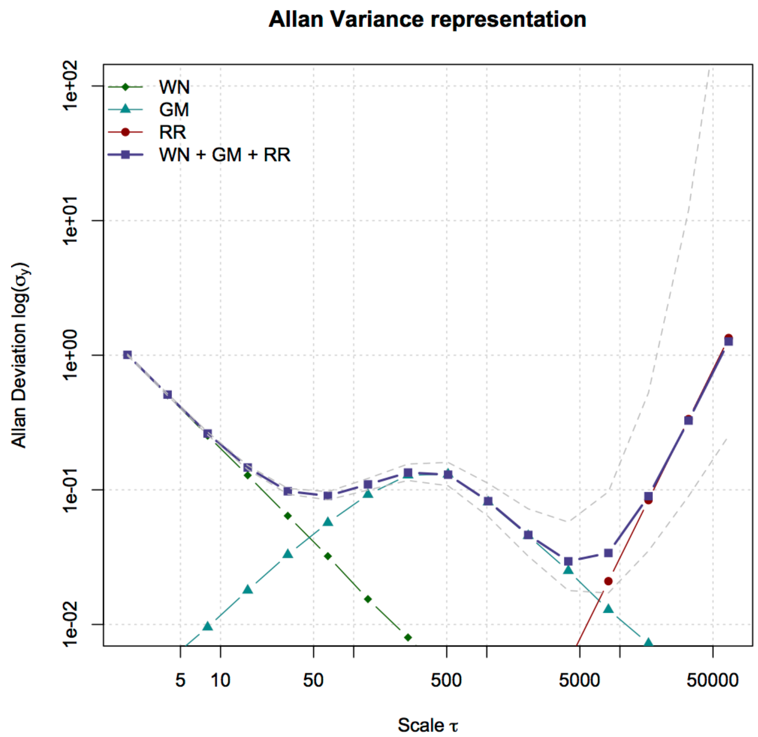
Solution:

$$\hat{\theta} = \underset{\theta \in \Theta}{\operatorname{argmin}} \left(\hat{\nu} - \nu(\theta)\right)^T \Omega \left(\hat{\nu} - \nu(\theta)\right)$$

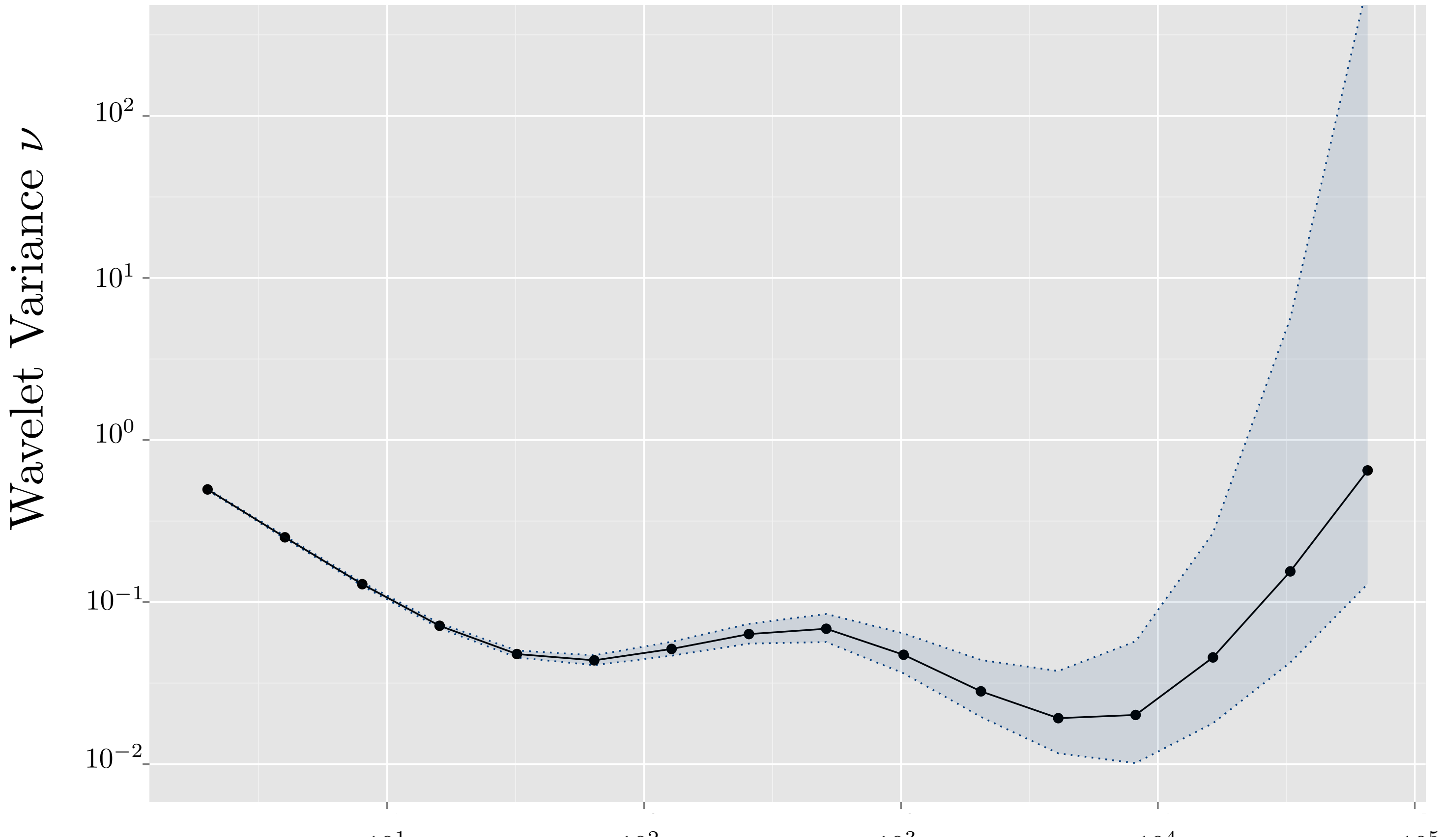
Example

- Composite stochastic process: WN + GM1 + RAMP

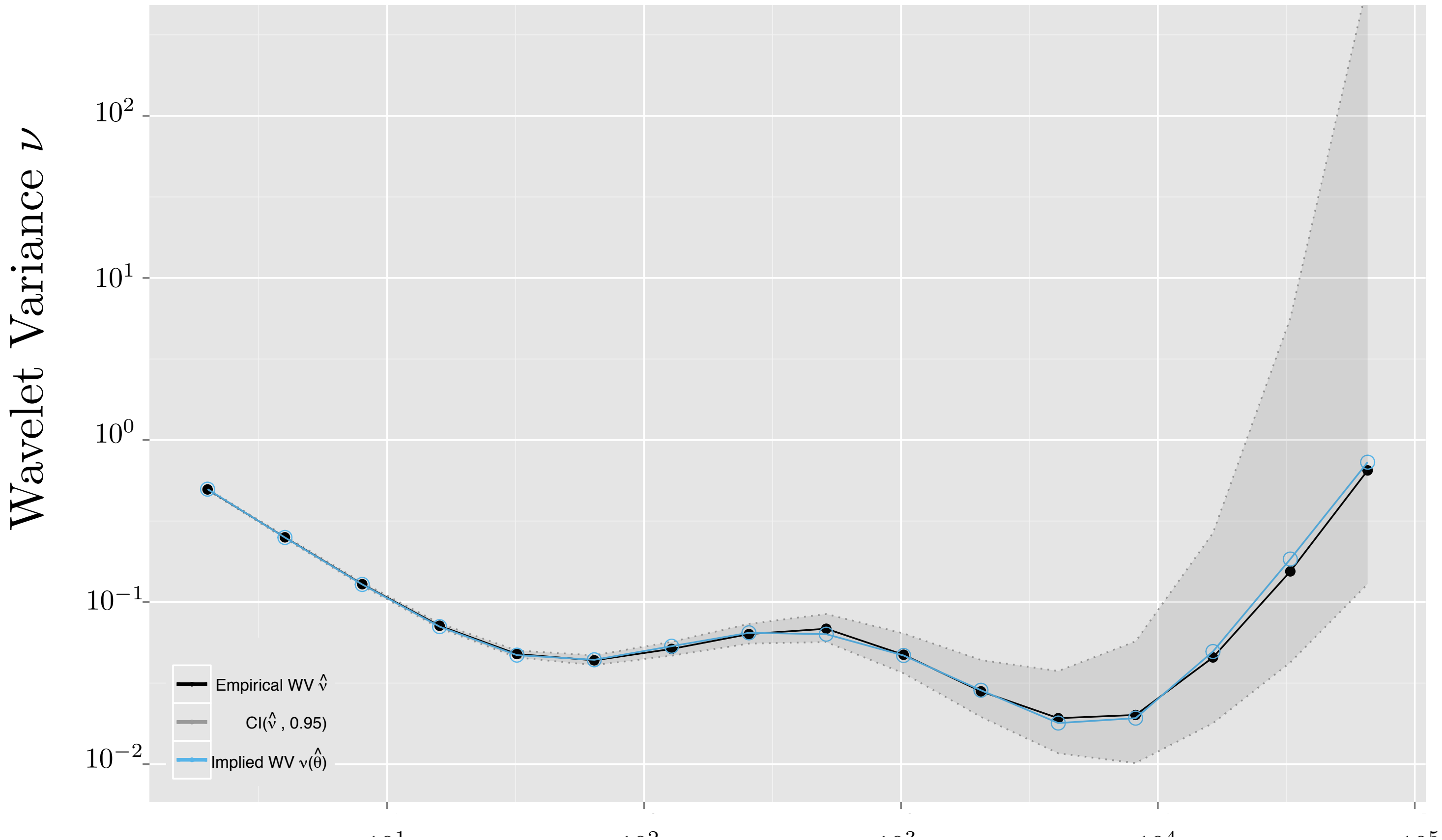




Empirical Wavelet Variance (= 2 x AV) of the example



GMWM estimation results



GMWM estimation results

- Estimated parameters

	θ_0	$\hat{\theta}$	$\mathrm{IC}\,(\theta_0, 0.95)$
σ^2_{WN}	1.00	1.00	$(0.99; 1.01)$
σ^2_{GM}	0.60	0.58	$(0.55; 0.61)$
β	10^{-2}	$1.07 \cdot 10^{-2}$	$(0.99 \cdot 10^{-2}; 1.12 \cdot 10^{-2})$
ω	$5 \cdot 10^{-5}$	$4.87 \cdot 10^{-5}$	$(4.67 \cdot 10^{-5}; 5.07 \cdot 10^{-5})$

- Goodness of fit test:
  - p-value ~ 1 (i.e. we cannot reject the proposed model)