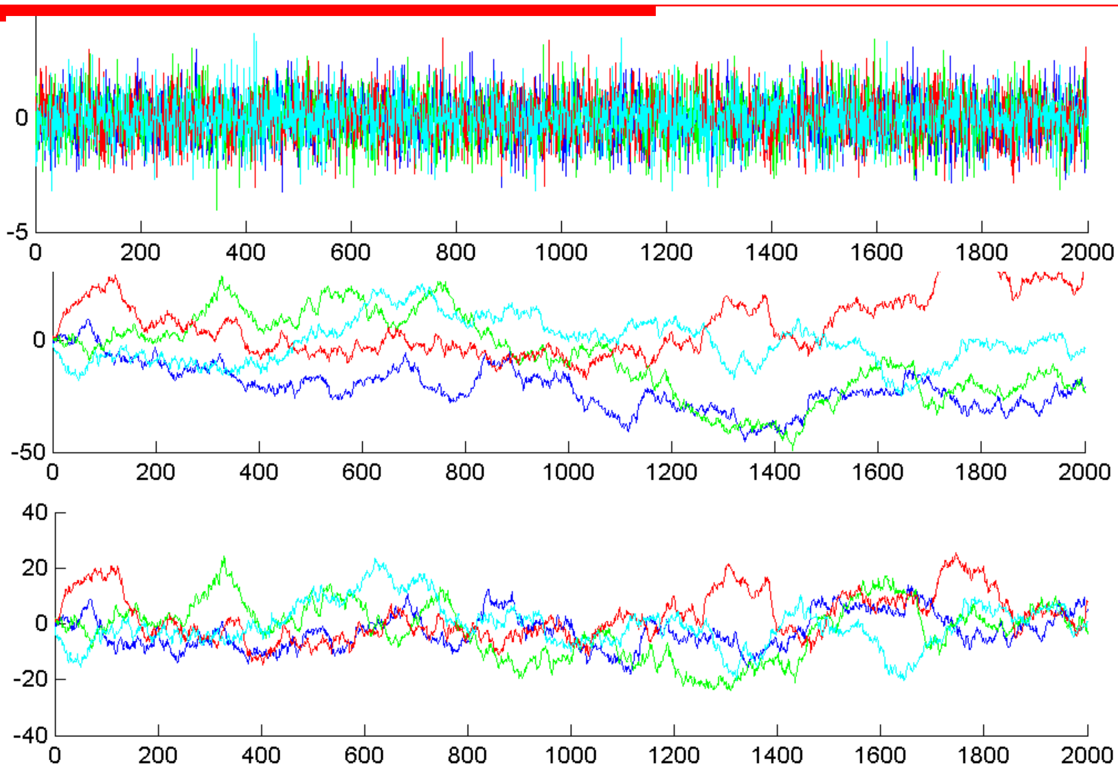
Stochastic Processes – agenda (Chap5)

- **Introduction**
  - Concept & classification
  - Model identification

- **Examples**
  - White Noise
  - Random bias
  - Autoregressive/shaped white noise (Gauss-Markov)

- **Characteristics**
  - Auto-correlation (**AC**)
  - Power Spectral Density (**PSD**)
  - Allan Variance (**AV**) / Wavelet Variance (**WV**)

Random process – an example



Random processes – general concept

- **Association** of random variable(s) with a deterministic parameter (e.g. time)
- Stochastic **collection** – values depend on the outcome of an experiment
- Time is a **parameter** but the process is ONLY LOOSELY/stochastic "function" of time
- Discrete or continuous

Random variable vs. random process

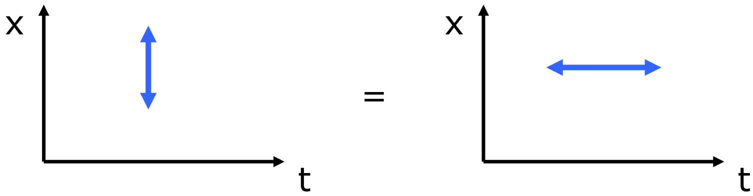
- Probability
  - Distribution
  - Density
- Characteristics

Variables	Processes
Moments	Autocorrelation func. (AC)
Characteristic function	Power spectral density (PSD)

TABLE – black b. / Ch.5

Classification of random processes

- By **ergodicity**



$$E(x) = \lim_{t\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\,dt$$

$$E(x^2) = \lim_{t\to\infty} \frac{1}{2T} \int_{-T}^{T} x^2(t)\,dt$$

$$\varphi_{xx}(t_1,t_2) = \lim_{t\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t_1)\,x(t_2)\,dt$$

Classification of random processes

- By **stationarity**
  - Characteristics = const.(t)
  - Can be computed from ”sample interval” (rather than absolute time)

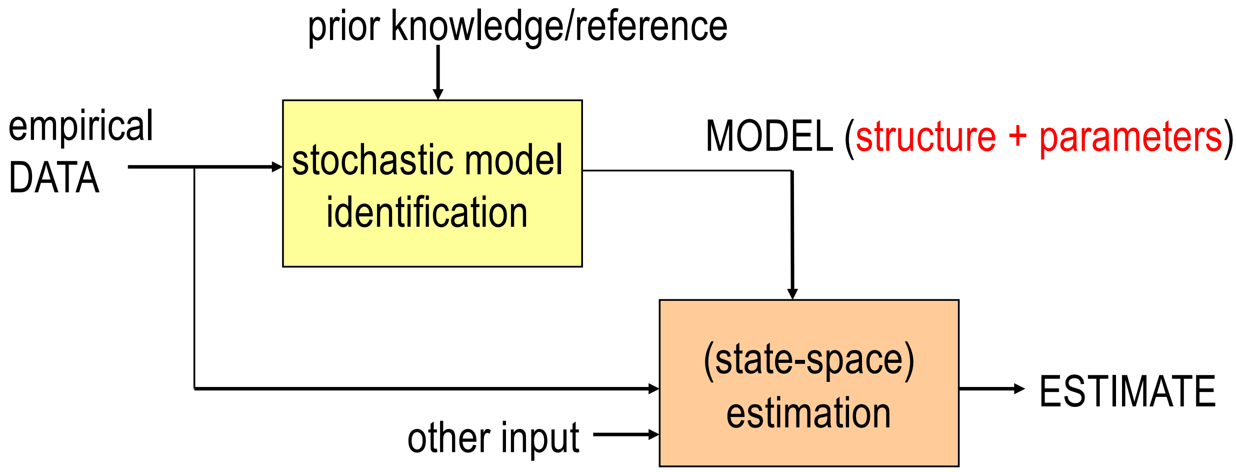
$$\varphi_{xx}(\tau) = E\big(x(t)\,x(t+\tau)\big)$$

$$\varphi_{xx}(0) = E(x^2)$$

$$\varphi_{xx}(-\tau) = \varphi_{xx}(\tau)$$

Empirical model identification

- Stochastic model
  - structure (type) + parameter (values)
  - prerequisite for sensor fusion (e.g. Kalman filtering)



Empirical model identification

- Autocorrelation techniques:
  - Assumption **1**: Underlying process is **stationary** Gaussian random process
  - Assumption **2**: Data (z) sampled at **constant Δt**

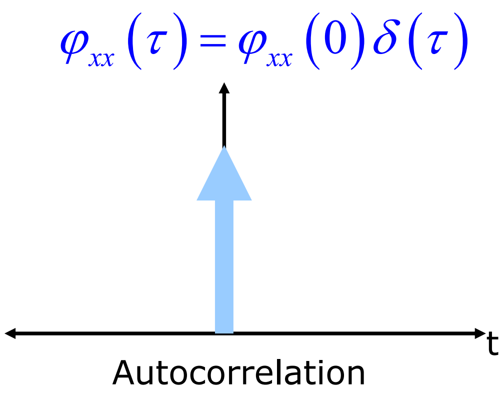
$$m = \frac{1}{N}\sum_{i=1}^{N} z_i$$

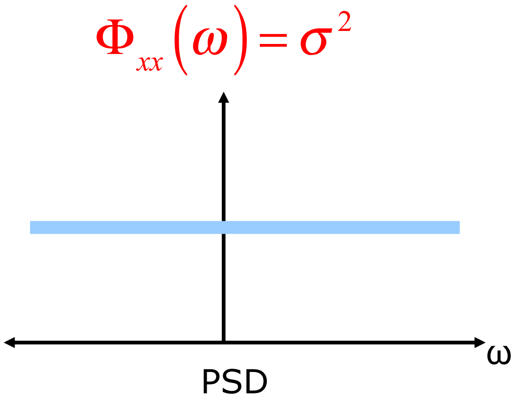
$$\varphi_{zz}\left(l\Delta t\right) = \frac{1}{N-l-1}\sum_{i=1}^{N-l}\left(z_i - m\right)\left(z_{i+l} - m\right)^T ; l = 0,1\ldots N-2$$

- Time series (or frequency) analysis:
  - AR, MA, ARMA, ARIMA, … + estimators
  - AV, WV, … + estimators

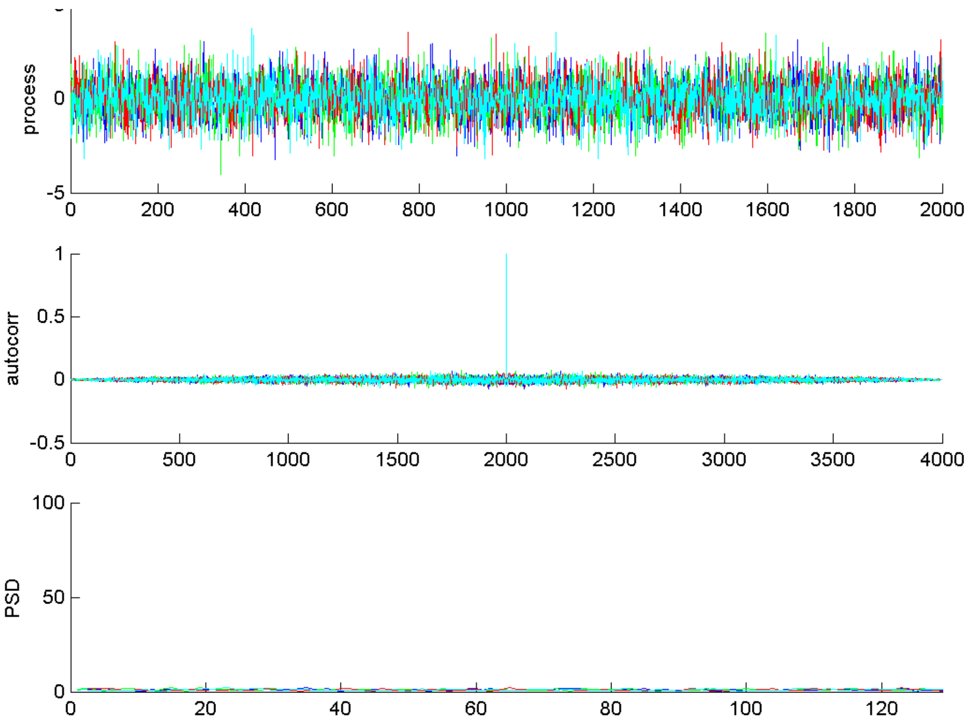
White noise process

- State propagation: $x_{k+1} = w_k$
- Differential equation: theoretically not differentiable





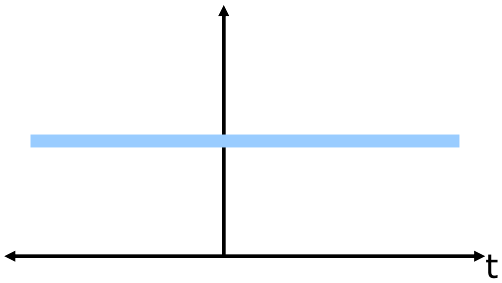
White noise - example



Random bias/constant

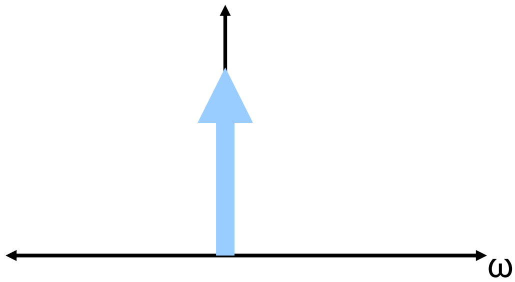
- State propagation: $x_{k+1} = x_k = m$
- Differential equation: $\dot{x}(t) = 0$

$$\varphi_{xx}(\tau) = m^2$$



Autocorrelation

$$\Phi_{xx}(\omega) = 2\pi m^2 \delta(\omega)$$



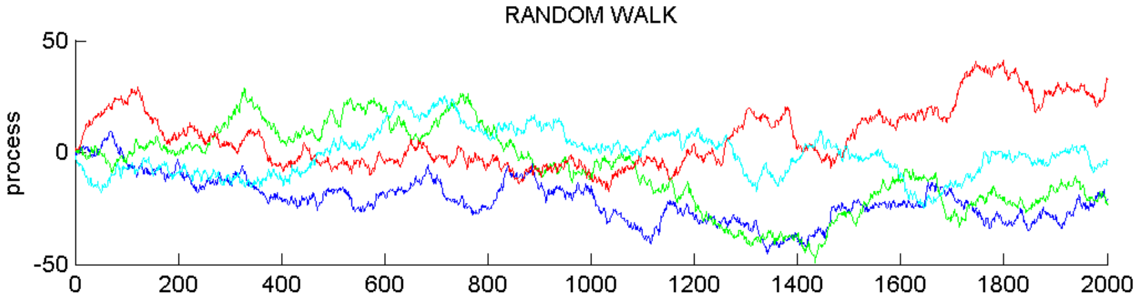
PSD

Shaped white noise

- Amplitude:
  - Gaussian probability density -> results of number of independent random variables.
- Correlation in time (frequency)
  - **White noise**: not correlated in time
  - **Colored noise**: white noise put through a small linear system can duplicate virtually many forms of time-correlated noise.
- Linear transformation of white noise = **Shaping Filter**

Random walk

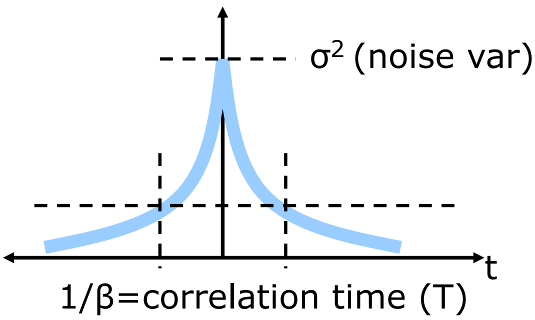
- Differential equation: $\dot{x}(t) = w(t)$
- State propagation: $x_{k+1} = x_k + w_k$
- Not a stationary process!



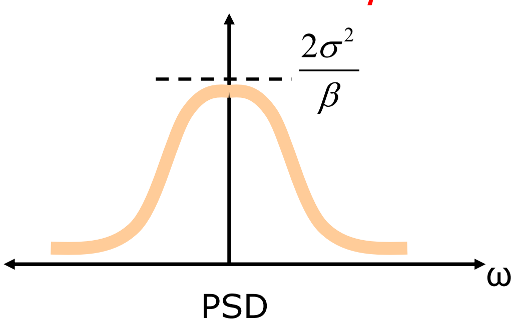
1st order Gauss-Markov process

- Differential equation: $\dot{x}(t) = -\beta x(t) + w(t)$
- State propagation: $x_{k+1} = e^{-\beta \Delta t} x_k + w_k$

$$\varphi_{xx}(\tau) = \sigma^2 e^{-\beta|\tau|}$$

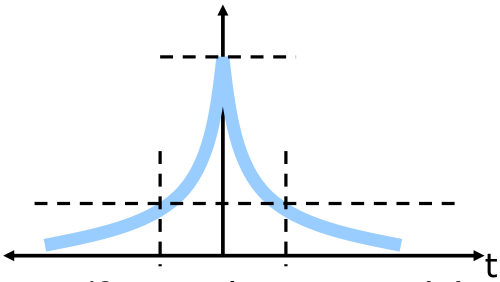


$$\Phi_{xx}(\omega) = \frac{2\beta\sigma^2}{\omega^2 + \beta^2}$$



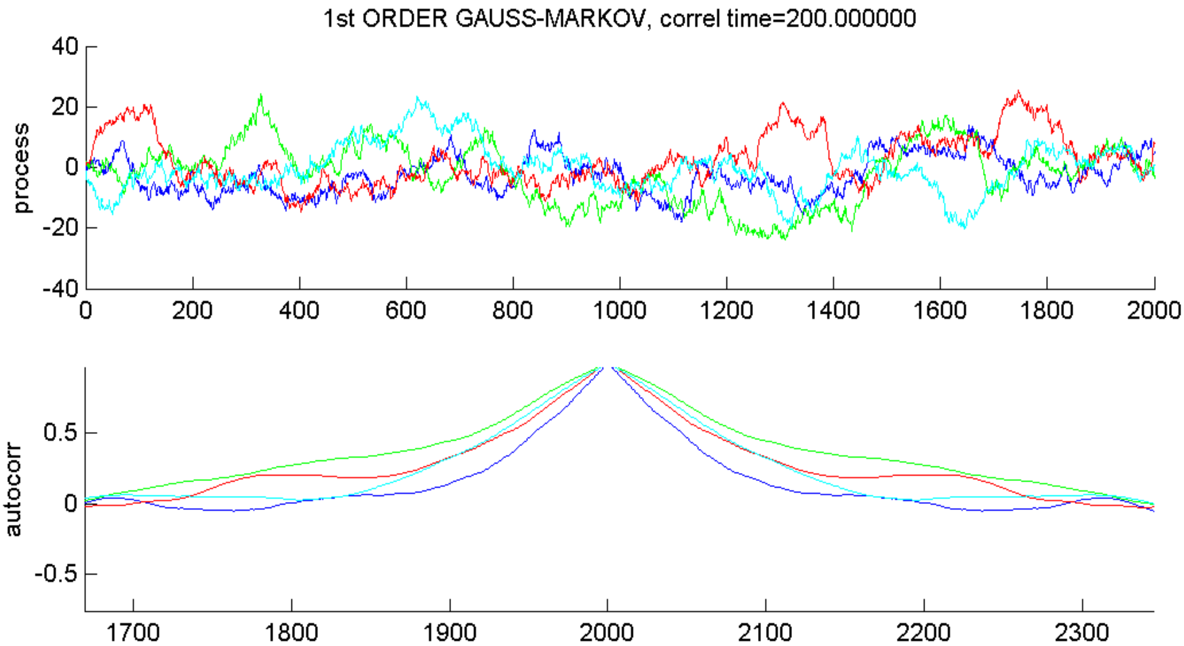
Question:

$$\varphi_{xx}\left(T\right)=\varphi_{xx}\left(1/\beta\right)=?$$



1/β=correlation time (T)

1st o. Gauss-Markov process - example



Other random processes

- Random ramp
- Quantization noise
- Auto-Regressive (AR) process
- Moving-Average (MA) process
- Auto-Regressive Moving Average (ARMA)
- …

Power spectrum (or PSD)

- PSD is the discrete Fourier transform (DFT) of AC:

$$\Phi_{xx}(\omega) = \sum_{\tau=-\infty}^{\infty} \varphi_{xx}(\tau) e^{-j\omega t}$$

- If we have a signal of length N:
  - Sample AC (AC <u>estimate</u>):

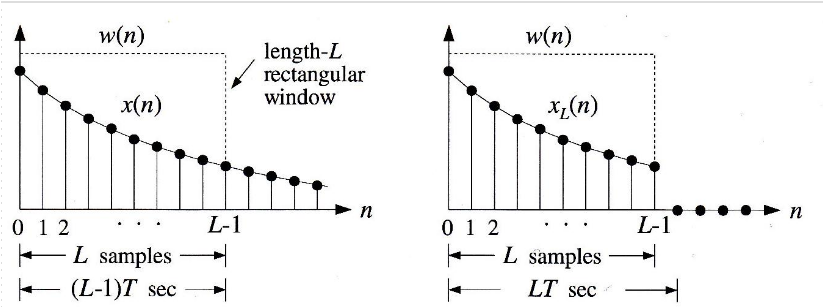
$$\hat{\varphi}_{xx}(\tau) = \frac{1}{N-\tau-1} \sum_{n=0}^{N-1-\tau} x_{n+\tau} x_n ; \qquad \tau = 0,1,\ldots,N-1$$

  - Periodogram spectrum (PSD <u>estimate</u>)

$$\hat{\Phi}_{xx}(\omega) = \sum_{\tau=-\infty}^{\infty} \hat{\varphi}_{xx}(\tau) e^{-j\omega\tau}$$

General note about windowing in DFT

- Time windowing -rectangular window, $x(nT) = \varphi_{xx}(\tau)$



  - Original and time-windowed spectrum:

$$\hat{\Phi}(\omega) = \sum_{n=-\infty}^{\infty} x(nT)e^{-j\omega nT}$$

$$\hat{\Phi}_L(\omega) = \sum_{n=0}^{L-1} x_L(nT)e^{-j\omega nT} = \sum_{n=-\infty}^{\infty} x_L(n)e^{-j\omega n}$$

General note about windowing in DFT

- Windowing has two (side) effects:
  - Reduction of the frequency resolution of the computed spectrum

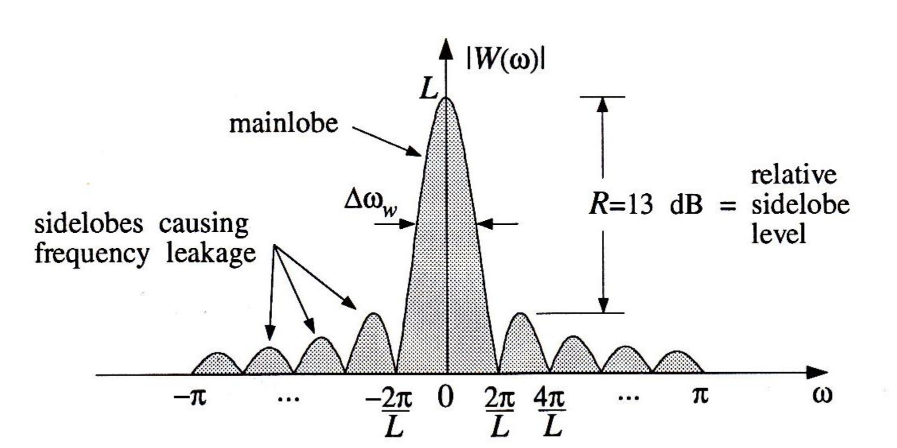
$$\Delta\omega = \frac{1}{T_L}$$

  - Frequency leakage
    - Introduction of high-frequency components into the spectrum
    - Caused by the sharp clipping of the signal at the left & right sides of the rectangular window

$$|W(\omega)| = \left|\frac{\sin(\omega L/2)}{\sin(\omega/2)}\right|$$

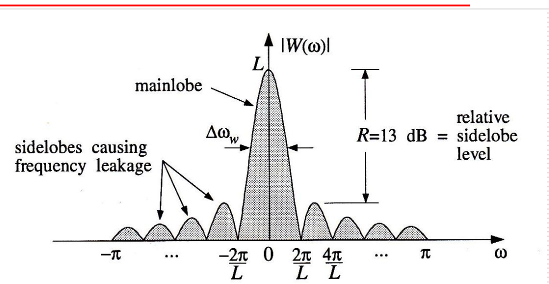
General note about windowing in DFT

- Magnitude spectrum of rectangular window:

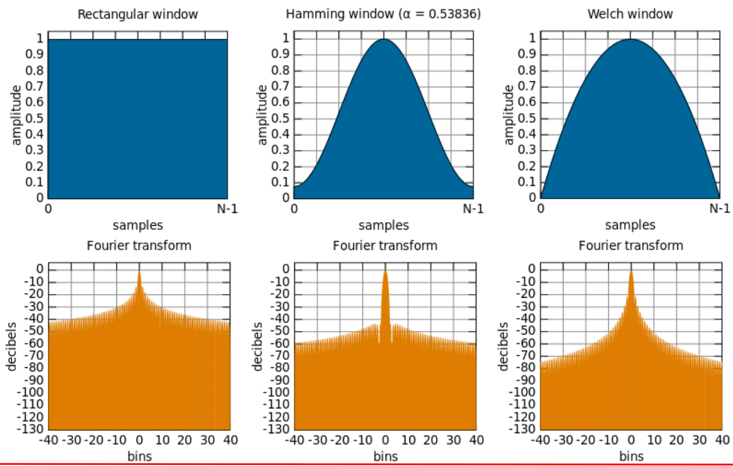


$$|W(\omega)| = \left|\frac{\sin(\omega L/2)}{\sin(\omega/2)}\right|$$

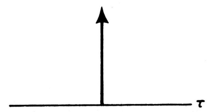
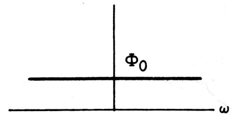
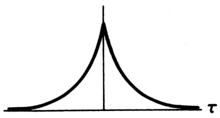
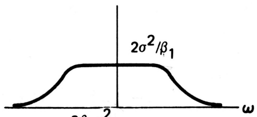
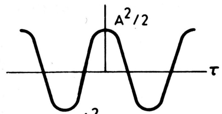
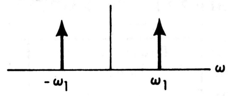
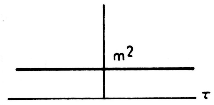
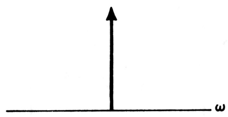
General note about windowing in DFT



- Reduces sidelobes
- Reduces leakage
- Narrow main lobe
- Some windowing function:
  - Welch
  - Hamming
  - Hanning
  - Kaiser
  - …

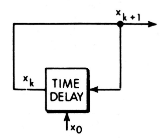
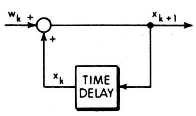
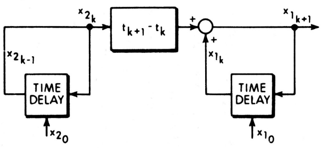
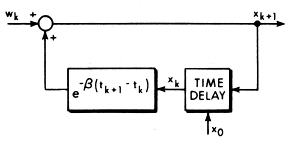


Common Random Processes (distributed)

PROCESS	AUTOCORRELATION FUNCTION	POWER SPECTRAL DENSITY
WHITE NOISE	 $\phi_{xx}(\tau) = \phi_0 \delta(\tau)$	 $\Phi_{xx} = \Phi_0$
MARKOV PROCESS $m = 0$	 $\phi_{xx}(\tau) = \sigma^2 e^{-\beta_1 \lvert\tau\rvert}$	 $\Phi_{xx} = \frac{2\beta_1\sigma^2}{\omega^2 + \beta_1^2}$
SINUSOID	 $\phi_{xx}(\tau) = \frac{A^2}{2} \cos \omega_1 \tau$	 $\Phi_{xx} = \frac{\pi}{2} A^2 \left[\delta(\omega - \omega_1) + \delta(\omega + \omega_1)\right]$
RANDOM BIAS	 $\phi_{xx}(\tau) = m^2$	 $\Phi_{xx} = 2\pi m^2 \delta(\omega)$

After Gelb, A. (1974)

Error Models – Discrete Case (distributed)

NAME	STATE DIFFERENCE EQUATION	BLOCK DIAGRAM	RELATION of DISCRETE q_k to CONTINUOUS q
RANDOM CONSTANT	$x_{k+1} = x_k$		$q_k = 0$
RANDOM WALK	$x_{k+1} = x_k + w_k$		$q_k = q(t_{k+1} - t_k)$
RANDOM RAMP	Δt $x_{1_{k+1}} = x_{1_k} + (t_{k+1} - t_k) x_{2_k}$ $x_{2_{k+1}} = x_{2_k}$		$q_k = 0$
EXPONENTIALLY CORRELATED RANDOM VARIABLE	Δt $x_{k+1} = e^{-\beta(t_{k+1} - t_k)} x_k + w_k$		$q_k = \frac{q}{2\beta}\left[1 - e^{-2\beta(t_{k+1} - t_k)}\right]$

After Gelb, A. (1974)

Allan variance (AV) - D. Allan 1966

- Alternative measure of variability

- = "variations of sample averages of different length τ"

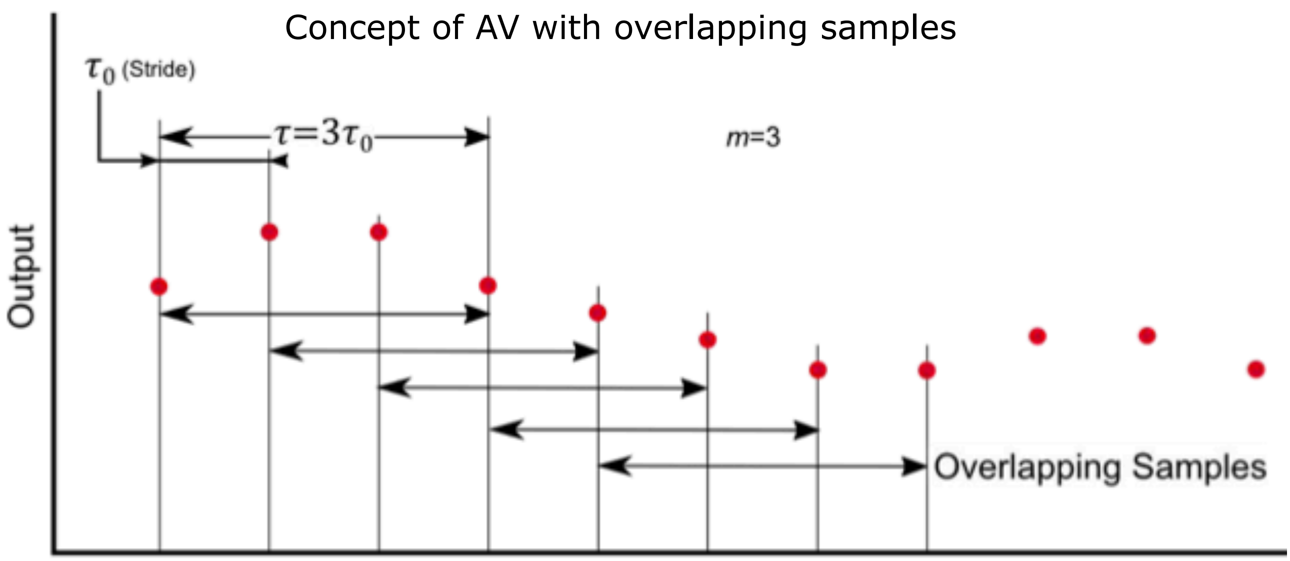
$$\bar{X}_k(\tau) = \frac{1}{\tau}\sum_{j=0}^{\tau-1} X_{k-j} \qquad \sigma^2_{\bar{X}}(\tau) = \frac{1}{2}\mathbf{E}\left[\left(\bar{X}_k(\tau) - \bar{X}_{k-\tau}(\tau)\right)^2\right]$$

- Greenhall (1991) estimator of AV

$$\hat{\sigma}^2_{\bar{x}}(\tau) = \frac{1}{2(N-2\tau+1)}\sum_{k=2\tau}^{N}\left(\bar{x}_k(\tau) - \bar{x}_{k-\tau}(\tau)\right)^2$$

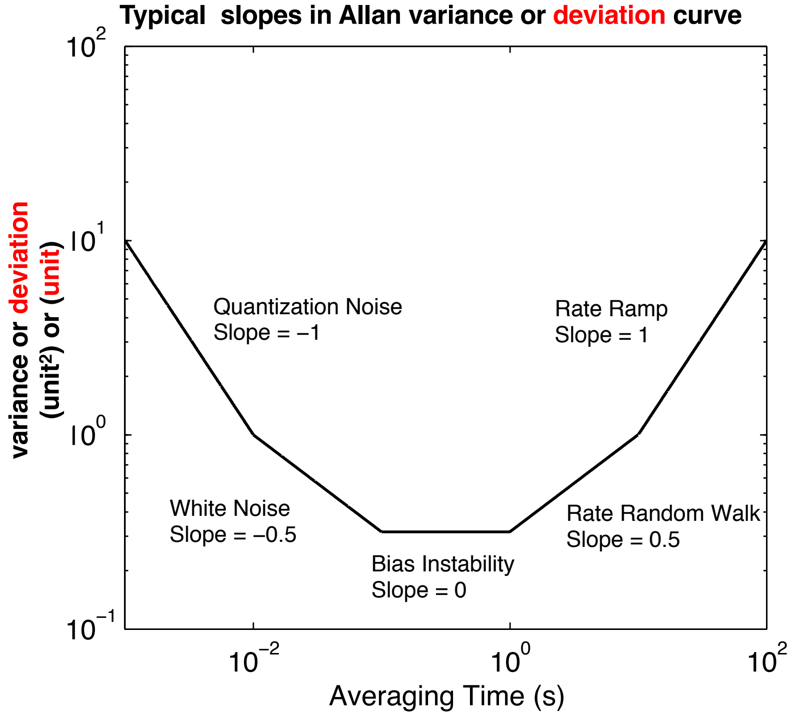
$$x_k : k = 1,\ldots,N$$

Allan variance – with sample overlap



source: Freescale Semiconductor, Application note #AN5087, Rev. 0,2/2015

Process identification in AV plots

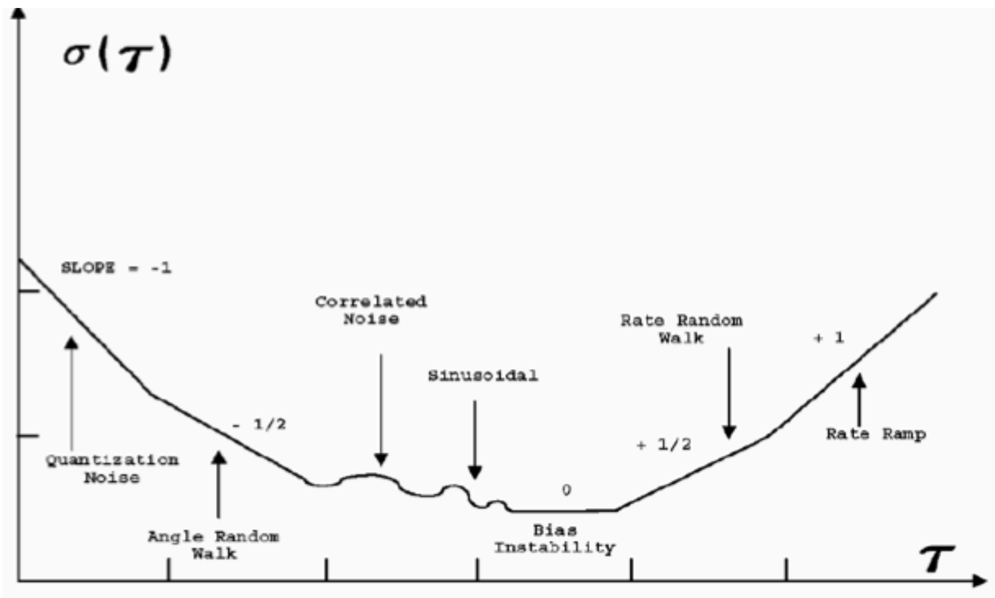


Remarks:

- Forward mapping:
  - Easy
  - IEEE standard
- Inverse:
  - not always
  - not a standard
  - Use GMWM …

Allan deviation – process signature

Complex error structure -> parameter estimation difficult



AV of a “static” gyro, source: scientific publication

Latent time series models estimation methods

- **Transformation** into a “non-latent” model (e.g. ARMA)
  - Does not work in general
  - Difficult to “inverse”
- “Graphical” / **lin. regression** method
  - Limited to a few possible models
  - Not consistent in general and “inefficient”
- Maximum Likelihood Estimation (**MLE**) / EM algorithm
  - Computationally intensive, laborious for new models
  - Diverges with “complex” models
- General Method of Wavelet Moment (**GMWM**)
  - Rigorous, precise and efficient (from students in this course)
  - Released as a freely available package in “R” (2015-09)
  - Released as a GUI online tool (2017-08)
  - http://ggmwm.smac-group.com/