where the relation $\dot{\mathbf{e}}^r = \dot{\psi}\,\mathbf{e}^\psi$ has been used. Similiarly, the second derivative is found to be

$$\begin{aligned} \ddot{\mathbf{x}}^m &= \ddot{r}\,\mathbf{e}^r + \dot{r}\,\dot{\mathbf{e}}^r + \dot{r}\,\dot{\psi}\,\mathbf{e}^\psi + r\,\ddot{\psi}\,\mathbf{e}^\psi + r\,\dot{\psi}\,\dot{\mathbf{e}}^\psi \\ &= (\ddot{r} - r\,\dot{\psi}^2)\,\mathbf{e}^r + (2\dot{r}\,\dot{\psi} + r\,\ddot{\psi})\,\mathbf{e}^\psi\,, \end{aligned} \tag{10}$$

where now the relation $\dot{\mathbf{e}}^\psi = -\dot{\psi}\,\mathbf{e}^r$ has been used. Thereby, one can easily identify the apparent accelerations that are due to the rotation of the $r\psi$-frame with respect to the inertial space: centrifugal acceleration $(-r\,\dot{\psi}^2)$, Coriolis acceleration $(2\dot{r}\,\dot{\psi})$, and tangential acceleration $(r\,\ddot{\psi})$.

In case of a circular motion with constant angular rate, some simplifications can be made that are in accordance with Eqs. (2) and (3). Thus, Eqs. (9) and (10) convert to

$$\dot{\mathbf{x}}^m = r\,\dot{\psi}\,\mathbf{e}^\psi\,, \tag{11}$$

$$\ddot{\mathbf{x}}^m = -r\,\dot{\psi}^2\,\mathbf{e}^r\,. \tag{12}$$

Thus, there is only an along-track velocity but no across-track component (the radius remains unchanged); in contrast, there is only an across-track acceleration (i.e., the apparent centrifugal acceleration) but there is no along-track component (the angular rate is constant).

Consequently, the nominal measurements are found to be

$$\mathbf{f}^b = \begin{bmatrix} f_1^b \\ f_2^b \end{bmatrix} = \begin{bmatrix} 0 \\ r\,\omega_0^2 \end{bmatrix} \tag{13}$$

for the accelerometers, where ω_0 has been used instead of $\dot{\psi}$; and

$$\boldsymbol{\omega}_{mb}^b = \begin{bmatrix} \omega_{mb}^b \end{bmatrix} = \begin{bmatrix} \omega_0 \end{bmatrix} \tag{14}$$

in case of the gyro. As the m-frame is assumed to be inertial and since there is only one gyro, ω_{mb}^b is used rather than the conventional quantity $\boldsymbol{\omega}_{ib}^b$.

Note that Eq. (13) clearly shows the effect of an apparent acceleration, because there is always a nonzero across-track acceleration although the radius remains constant. Hence, the observed accleration in the b-frame is only due to its rotation with respect to the m-frame.

1.3 Strapdown inertial navigation

As usual, it is necessary to define the initial conditions and to integrate the sensor measurements to obtain the current state vector of the vehicle.

Initial conditions

These are defined by five quantities, i.e., the initial position $\mathbf{x}_0^m$, the initial velocity $\mathbf{v}_0^m$, and the initial heading (or yaw) angle α. For the sake of simplicity, it will be assumed that there is no accleration phase – in other words: the measurements are only started when the vehicle is already in a “steady state” of motion. Furthermore, it is supposed that the along axis of the vehicle is always aligned with its velocity vector, i.e., there is no drift.

In this case, the initial conditions are given by $\mathbf{x}_0^m = [n_0, 0]^{\mathrm{T}}$, $\mathbf{v}_0^m = [0, v_0]^{\mathrm{T}}$, and $\alpha_0 = \psi_0 + \pi/2 = \pi/2$. From the latter, the initial attitude matrix is found by evaluating its general form at $\alpha = \alpha_0$:

$$\mathbf{R}_b^m = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \quad \Rightarrow \quad \mathbf{R}_b^m(\alpha_0) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} . \tag{15}$$

Note that this definition holds as $\mathbf{R}_m^b = \mathbf{R}_b^{m\mathrm{T}} = \mathbf{R}(\alpha)$. Further, note that it is not necessary in this example to introduce the local-level frame since it would always be parallel to the m-frame due to the 2D (planar) modeling.

Dead reckoning

Attitude computation

As a first step, the current attitude matrix must be computed. This is achieved by numerical integration of the corresponding differential equation which is obtained by differencing the first part of Eq. (15) and rearranging the result:

$$\dot{\mathbf{R}}_b^m = \begin{bmatrix} -\sin\alpha & -\cos\alpha \\ \cos\alpha & -\sin\alpha \end{bmatrix} \dot{\alpha} = \mathbf{R}_b^m \begin{bmatrix} 0 & -\dot{\alpha} \\ \dot{\alpha} & 0 \end{bmatrix} = \mathbf{R}_b^m \, \mathbf{\Omega}_{mb}^b , \tag{16}$$

where $\dot{\alpha} = \omega_{mb}^b$ is the gyro measurement.

When using a sufficiently small integration interval, it may be assumed that the angular rate remains constant during that interval. Thus, the numerical integration of Eq. (16) is given by

$$\mathbf{R}_b^m(t_k) = \mathbf{R}_b^m(t_{k-1}) \exp\left(\mathbf{\Omega}_{mb}^b(t_k) \cdot (t_k - t_{k-1})\right) . \tag{17}$$

Alternatively, the attitude angle itself may be computed from the first- or second-order approximations

$$\alpha(t_k) = \alpha(t_{k-1}) + \dot{\alpha}(t_k) \cdot (t_k - t_{k-1}) , \tag{18}$$

$$\alpha(t_k) = \alpha(t_{k-1}) + \frac{1}{2}\big(\dot{\alpha}(t_k) + \dot{\alpha}(t_{k-1})\big) \cdot (t_k - t_{k-1}) . \tag{19}$$

Navigation computation

As the m-frame is supposed to be inertial and the gravitational field is neglected, the navigation equations are given by

$$\dot{\mathbf{v}}^m = \mathbf{f}^m , \qquad \dot{\mathbf{x}}^m = \mathbf{v}^m . \tag{20}$$

Using the integrated attitude matrix, the specific-force measurements of the accelerometers need to be resolved in the m-frame, i.e., $\mathbf{f}^m = \mathbf{R}_b^m \, \mathbf{f}^b$. Afterwards, the transformed measurement can be integrated to obtain the current velocity vector. When using a first-order approximation, the velocity vector follows from

$$\mathbf{v}^m(t_k) = \mathbf{v}^m(t_{k-1}) + \mathbf{R}_b^m(t_k) \, \mathbf{f}^b(t_k) \cdot (t_k - t_{k-1}) . \tag{21}$$

Again, a second-order approximation is obtained from

$$\begin{aligned} \mathbf{v}^m(t_k) &= \mathbf{v}^m(t_{k-1}) + ... \\ &+ \frac{1}{2}\Big(\mathbf{R}_b^m(t_k) \, \mathbf{f}^b(t_k) + \mathbf{R}_b^m(t_{k-1}) \, \mathbf{f}^b(t_{k-1})\Big) \cdot (t_k - t_{k-1}) . \end{aligned} \tag{22}$$

Finally, the current position is found in analogy to the velocity vector by using either a first- or second-order approximation:

$$\mathbf{x}^m(t_k) = \mathbf{x}^m(t_{k-1}) + \mathbf{v}^m(t_k) \cdot (t_k - t_{k-1}) \,, \tag{23}$$

$$\mathbf{x}^m(t_k) = \mathbf{x}^m(t_{k-1}) + \frac{1}{2}\left(\mathbf{v}^m(t_k) + \mathbf{v}^m(t_{k-1})\right) \cdot (t_k - t_{k-1}) \,. \tag{24}$$