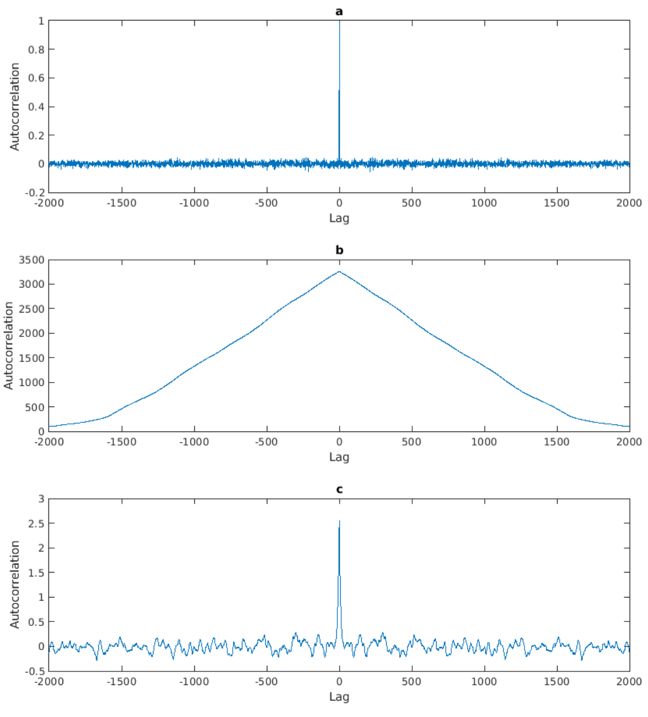
Sensor Orientation
Exercise 2

2025

Q1

You are provided with three graphs representing the auto-correlation functions of different signals. Your task is to identify the type of noise present in each signal and justify your answers based on the characteristics of auto-correlation functions.



1. Identify the type of noise present in each graph
2. Provide a brief explanations for your answers

Solution

1. White noise, random walk, Gauss-Markov
2. a) The autocorrelation is a delta function, showing no correlation, which is a characteristic of the white noise; b) The plot shows high correlation, making it resemble a random walk; c) The plot shows the autocorrelation to be decaying exponential, which is a characteristic of a Gauss-Markov process.

Q2

You are provided with a plot representing a signal at a frequency of 1Hz. The signal values are integers. In the discrete case, the auto-correlation function of a signal is defined as:

$$a[l\Delta t] = \frac{1}{N-l-1} \sum_{i=1}^{N-l} (x_i - m)(x_{i+l} - m) \tag{1}$$

Here N is the number of samples and m is the mean of the signal and $l \in [0, 1, ..., N-2]$.

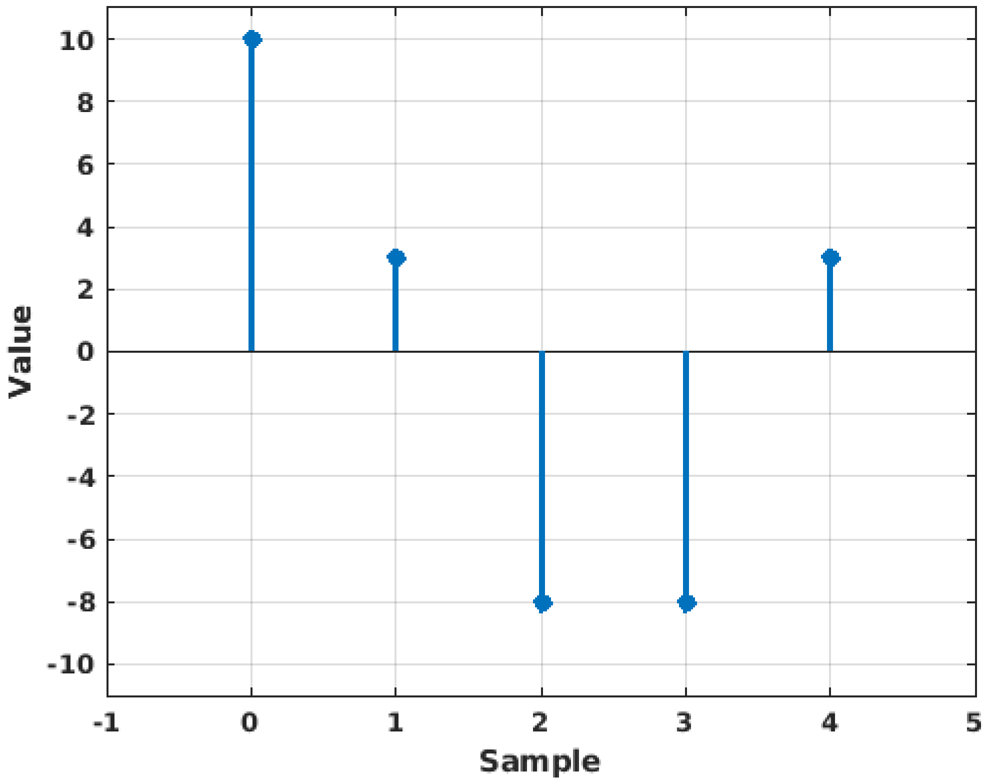


Figure 1: Caption

1. Without programming (by hand or with a simple calculator), calculate the unbiased auto-correlation function of this signal
2. Draw an approximate graph of the auto-correlation of the signal
3. Calculate the variance of the signal from the auto-correlation plot
4. Calculate the auto-correlation time of the signal (Assume linear interpolation between two data points), if this signal is modeled as a first-order Gauss Markov process?
5. What will be the standard deviation of driving white noise, if this signal is modeled as a first-order Gauss Markov process?

Solution

1.
$$\mu_x = \frac{10+3-8-8+3}{5} = 0$$

$$a[0] = \frac{9+64+64+9}{4} = 36.5 \quad (2)$$
$$a[1] = \frac{-24+64-24}{3} = 5.3 \quad (3)$$
$$a[2] = \frac{-24-24}{2} = -24 \quad (4)$$
$$a[3] = \frac{9}{1} = 9 \quad (5)$$

2. Draw a graph
3. Variance equals the value at $a[0] = 36.5$
4. When $\tau = \tau_c$,
$$a(\tau_c) = e^{-1}a[0] = 13.42 \quad (6)$$
Looking at the plot between the lag(τ) and $acov_X$, we can see that this value occurs between the lags 0 and 1. Using a linear interpolation (as asked in the question), we can find the value of τ_c:
$$a_{interp} = \alpha t + \beta \quad (7)$$
$$\alpha = a[1] - a[0] = -31.2 \quad (8)$$
$$\beta = a[0] = 36.5 \quad (9)$$
Solving for t, one finds $\tau_c = 0.74$
5. Using equation 5.34 from polycopie
$$\sigma^2_{\text{wn}} = \sigma^2_{\text{gm}}\left(1 - e^{-2\frac{\Delta t}{\tau_c}}\right) \quad (10)$$
$$\Delta T = 1;\ \tau_c = 0.74;\ \sigma^2_{\text{gm}} = 36.5 \quad (11)$$
$$\sigma_{\text{wn}} = 5.84 \quad (12)$$

Q3

1. Imagine you're in charge of a groundbreaking satellite mission, monitoring Earth from a geostationary orbit. At noon, your geostationary satellite[1] is perfectly positioned with coordinates (a, b, c)

[1] a satellite that stays stationary over the surface of the Earth

in the Earth-Centered, Earth-Fixed (ECEF) frame. Your task is to determine the new coordinates of the satellite in the Earth-Centered Inertial (ECI) frame after 6 hours.

Following assumptions are made

- Both the ECEF and ECI frames align precisely at midnight
- Earth completes a full rotation of 360 degrees every 24 hours at a constant angular velocity.

2. Now you would like to account for the Earth rotation around the sun as well. What is the full Earth rotation rate now in ECI? What is the corrected position of your satellite ?

Solution

1.

$$\mathbf{R}_e^i(\theta) = \mathbf{R}_3(-\theta) = \begin{bmatrix} \cos(-\theta) & \sin(-\theta) & 0 \\ -\sin(-\theta) & \cos(-\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{13}$$

$$\theta_h = \frac{3\pi}{2} \quad \text{[270 deg from 00h00(midnight) to 18h00]} \tag{14}$$

$$\mathbf{x}_e^i = \mathbf{R}_e^i(-\theta_h) \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b \\ -a \\ c \end{bmatrix} \tag{15}$$

2. Considering that the Earth do a full revolution around itself in 24h and $\frac{1}{365}$ of its revolution around the sun, its rotation rate becomes:

$$\omega_E = \frac{2\pi}{24 * 3600} * (1 + \frac{1}{365}) = 7.292e^{-5} rad/s \tag{16}$$

Computing θ and x_e^i can then be computed as earlier

Q4

Imagine yourself standing on the equator, precisely on the Prime meridian(0^0E), marveling at the wonders of navigation and geodesy. Since you like to challenge yourself, you decided to define your own coordinate system where the X axis is pointing North, Y axis is pointing East and Z axis is pointing towards the Earth (In geodesy, we call this coordinate system as a local-level frame - NED). You decide to put the origin of the coordinate system on the surface of the earth. Now, imagine you have an object of interest located at coordinates [P, Q, R] km in your NED coordinate system.

- What would be the coordinate of this object in ECEF (in km)? Assume the radius of the earth to be uniform with a value of 6500km

Solution

It's important to note that the origins of the two frames, ECEF (Earth-Centered, Earth-Fixed) and NED (North-East-Down), differ. ECEF has its origin at the center of the Earth, while NED has its origin on the surface of the Earth. Consequently, there exists a translation when converting between the two frames. The initial step involves obtaining a rotation matrix between the two frames. If we denote the rotation matrix as $\mathbf{R}_B^A$, each column of $\mathbf{R}_B^A$ represents the projections of the axes of frame B onto axes of frame A.

Orientation of axes in ECEF was discussed in the course. Based on this information and the definition of the NED frame, along with your specific location on Earth, we can express the rotation matrix as follows:

$$\mathbf{R}_l^e(0,0) = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{17}$$

The provided matrix arises because the X-axis of the NED frame aligns with the Z-axis of the ECEF frame, while the Z-axis of the NED frame points opposite to the X-axis of the ECEF frame. The Y-axes of both frames are parallel to each other. This alignment holds true specifically because you are positioned at the intersection of the equator and the prime meridian. For locations elsewhere on Earth, you'll need to either calculate the corresponding projections or use Equation 3.19.

Applying the transformation equation, we get

$$x^e = \begin{bmatrix} 6500 \\ 0 \\ 0 \end{bmatrix} + \mathbf{R}_l^e(0,0) \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} 6500 - R \\ Q \\ P \end{bmatrix} km \tag{18}$$

Q5

Prove that PSD of a first-order Gauss-Markov stochastic process is defined by

$$\Phi_X(f) = \frac{2\sigma^2\beta}{4\pi^2 f^2 + \beta^2}$$

given that, the auto-correlation function is

$$a(\tau) = \sigma^2 e^{-\beta|\tau|}$$

Solution

$$\Phi_X(f) = \int_{-\infty}^{\infty} \sigma^2 e^{-\beta|\tau|} e^{-j2\pi f\tau} d\tau \tag{19}$$

$$\Phi_X(f) = \int_{-\infty}^{0} \sigma^2 e^{\beta\tau} e^{-j2\pi f\tau} d\tau + \int_{0}^{\infty} \sigma^2 e^{-\beta\tau} e^{-j2\pi f\tau} d\tau \tag{20}$$

$$\Phi_X(f) = \int_{-\infty}^{0} \sigma^2 e^{(\beta-j2\pi f)\tau} d\tau + \int_{0}^{\infty} \sigma^2 e^{-(\beta+j2\pi f)\tau} d\tau \tag{21}$$

$$\Phi_X(f) = \sigma^2 \left.\frac{e^{(\beta-j2\pi f)\tau}}{\beta - j2\pi f}\right|_{-\infty}^{0} - \sigma^2 \left.\frac{e^{(\beta-j2\pi f)\tau}}{\beta + j2\pi f}\right|_{0}^{\infty} \tag{22}$$

Solving this integral gives the result straight off