3D Attitude representation – possibilities

- Many possibilities
  - Direction cosines / rotation matrices (DCM)
  - Euler/Cardan angles
  - Axis and angle
    - Quaternion
    - Rodrigues
    - Caley
    - …
  - Euler/Cardan angles
  - Others …

- Usage:
  - Interpretation: Euler angles
  - Derivation/algebraic operation: rotation matrices
  - Numerical “stability” with min. representation: quaternions

3D Attitude representation – SO(3)

Rotation matrices

- Unique description
- Decomposition to “fundamental” rotations about x_i-axis

$$\mathbf{R}_1(\theta)=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & -\sin\theta & \cos\theta\end{pmatrix}\quad \mathbf{R}_2(\theta)=\begin{pmatrix}\cos\theta & 0 & -\sin\theta\\ 0 & 1 & 0\\ \sin\theta & 0 & \cos\theta\end{pmatrix}\quad \mathbf{R}_3(\theta)=\begin{pmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}$$

- From p-frame to q-frame

$$\mathbf{R}_p^q=\mathbf{R}_3(\kappa)\mathbf{R}_2(\phi)\mathbf{R}_1(\omega)$$

- Parameterization by Euler-angles
  - Application dependent
  - Navigation: h-p-r (heading-pitch-roll) or p-r-h or r-p-h
  - Photogrammetry: o-p-k (omega-phi-kappa) or p-o-k
  - Guard against singularity!

3D Attitude representation - rotation axis + angle

- One rotation angle, θ, about an axis given by a vector

$$\mathbf{n} = \begin{pmatrix} n_1 & n_2 & n_3 \end{pmatrix}^T$$

- **n** = eigenvector of a Rotation matrix corresponding to real unique eigenvalue.

$$\mathbf{R} = \begin{pmatrix} \cos\theta + n_1^2(1-\cos\theta) & n_1 n_2(1-\cos\theta) + n_3\sin\theta & n_1 n_3(1-\cos\theta) - n_2\sin\theta \\ n_1 n_2(1-\cos\theta) - n_3\sin\theta & \cos\theta + n_2^2(1-\cos\theta) & n_2 n_3(1-\cos\theta) + n_1\sin\theta \\ n_1 n_3(1-\cos\theta) + n_2\sin\theta & n_2 n_3(1-\cos\theta) - n_1\sin\theta & \cos\theta + n_3^2(1-\cos\theta) \end{pmatrix}$$

3D Attitude representation - quaternion

- 4 algebraic parameters (non-unique!):

$$\overset{o}{q} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$$
$$= q_0 + \mathbf{q}$$

  - Bound by a condition: $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$

- A unite quaternion can be expressed by in terms of the angle and axis parameters

$$\overset{o}{q} = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)n_1\mathbf{i} + \sin\left(\frac{\theta}{2}\right)n_2\mathbf{j} + \sin\left(\frac{\theta}{2}\right)n_3\mathbf{k}$$

Additional attitude operations – rotation matrices

- Consecutive rotations

$$R_p^q = R_r^q \, R_s^r \, R_t^s \, R_p^t$$

- Inverse of a fundamental rotation matrix

$$\left(R_i\left(\theta\right)\right)^{-1} = \left(R_i\left(\theta\right)\right)^T = R_i\left(-\theta\right)$$

- Inverse of any rotation matrix

$$\left(R_p^q\right)^{-1} = \left(R_p^q\right)^T = R_q^p$$

- Infinitesimal rotations

$$\delta\boldsymbol{\alpha} = [\delta\alpha_1 \;\; \delta\alpha_2 \;\; \delta\alpha_3]^T$$

$$\delta\alpha_i \ll 1 \Rightarrow \cos\delta\alpha_i \simeq 1 \;\&\; \sin\delta\alpha_i \simeq \delta\alpha_i$$

$$R_1(\delta\alpha_1) \simeq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \delta\alpha_1 \\ 0 & -\delta\alpha_1 & 1 \end{bmatrix}$$

$$R_2(\delta\alpha_2) \simeq \begin{bmatrix} 1 & 0 & -\delta\alpha_2 \\ 0 & 1 & 0 \\ \delta\alpha_2 & 0 & 1 \end{bmatrix}$$

$$R_3(\delta\alpha_3) \simeq \begin{bmatrix} 1 & \delta\alpha_3 & 0 \\ -\delta\alpha_3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R = R_3(\delta\alpha_3)\cdot R_2(\delta\alpha_2)\cdot R_1(\delta\alpha_1) = I - \left[\delta\boldsymbol{\alpha}\times\right]$$

$$\left[\delta\boldsymbol{\alpha}\times\right] = \begin{bmatrix} 0 & - & \delta\alpha_2 \\ \delta\alpha_3 & 0 & -\delta\alpha_1 \\ -\delta\alpha_2 & \delta\alpha_1 & 0 \end{bmatrix}$$

Additional attitude operations - quaternions

- Quaternions

$$\overset{o}{q} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$$

$$\overset{o}{p} = p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$$

$$\mathbf{v} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^T$$

$$R: \text{ rotation matrix equivalent to } \overset{o}{q}$$

- Quaternion product

$$\begin{aligned} \overset{o}{q} \otimes \overset{o}{p} = {} & q_0 p_0 - q_1 p_1 - q_2 p_2 - q_3 p_3 \\ & + (q_0 p_1 + q_1 p_0 + q_2 p_3 - q_3 p_2) \cdot \mathbf{i} \\ & + (q_0 p_2 + q_2 p_0 - q_1 p_3 + q_3 p_1) \cdot \mathbf{j} \\ & + (q_0 p_3 + q_3 p_0 + q_1 p_2 - q_2 p_1) \cdot \mathbf{k} \end{aligned}$$

- Conjugate and norm

$$\text{conjugate}\left(\overset{o}{q}\right) = \overset{o}{q}{}^{*} = q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}$$

$$\overset{o}{q}{}^{*} = -\frac{1}{2}\left(\overset{o}{q} + \mathbf{i}\,\overset{o}{q}\,\mathbf{i} + \mathbf{j}\,\overset{o}{q}\,\mathbf{j} + \mathbf{k}\,\overset{o}{q}\,\mathbf{k}\right)$$

$$\left\|\overset{o}{q}\right\| = \sqrt{\overset{o}{q}\,\overset{o}{q}{}^{*}} = \sqrt{\overset{o}{q}{}^{*}\,\overset{o}{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$$

- Rotation by quaternion

$$\mathbf{v}_{rotated} = R\mathbf{v}$$

$$\begin{bmatrix} 0 \\ \mathbf{v}_{rotated} \end{bmatrix} = \overset{o}{q}{}^{*} \begin{bmatrix} 0 \\ \mathbf{v} \end{bmatrix} \overset{o}{q}$$

Additional attitude operations – conversions

- Euler angles to rotation matrix

$$R_n^b = R_1(\varphi)R_2(\theta)R_3(\psi) = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \begin{matrix}\sin\varphi\sin\theta\cos\psi \\ -\cos\varphi\sin\psi\end{matrix} & \begin{matrix}\sin\varphi\sin\theta\sin\psi \\ +\cos\varphi\cos\psi\end{matrix} & \sin\varphi\cos\theta \\ \begin{matrix}\cos\varphi\sin\theta\cos\psi \\ +\sin\varphi\sin\psi\end{matrix} & \begin{matrix}\cos\varphi\sin\theta\sin\psi \\ -\sin\varphi\cos\psi\end{matrix} & \cos\varphi\cos\theta \end{bmatrix}$$

- Rotation matrix to Euler angles (arctan2 is recommended)

$$R_{13} \neq \pm 1 : \theta = \arcsin(R_{13}), \quad \varphi = \arctan\left(\frac{R_{23}/\cos\theta}{R_{33}/\cos\theta}\right), \quad \psi = \arctan\left(\frac{R_{12}/\cos\theta}{R_{11}/\cos\theta}\right)$$

$$R_{13} = +1 : \theta = -\frac{\pi}{2}, \quad \psi + \varphi = \arctan\left(\frac{-R_{21}}{R_{22}}\right) \xrightarrow{\varphi=0 \text{ (arbitrary)}} \psi = \arctan\left(\frac{-R_{21}}{R_{22}}\right)$$

$$R_{13} = -1 : \theta = +\frac{\pi}{2}, \quad \psi - \varphi = \arctan\left(\frac{-R_{21}}{R_{22}}\right) \xrightarrow{\varphi=0 \text{ (arbitrary)}} \psi = \arctan\left(\frac{-R_{21}}{R_{22}}\right)$$

Additional attitude operations – conversions

- Euler angles to quaternion

$$\overset{o}{q} = \begin{bmatrix} cos(\phi/2)cos(\theta/2)cos(\psi/2) + sin(\phi/2)sin(\theta/2)sin(\psi/2) \\ sin(\phi/2)cos(\theta/2)cos(\psi/2) - cos(\phi/2)sin(\theta/2)sin(\psi/2) \\ cos(\phi/2)sin(\theta/2)cos(\psi/2) + sin(\phi/2)cos(\theta/2)sin(\psi/2) \\ cos(\phi/2)cos(\theta/2)sin(\psi/2) - sin(\phi/2)sin(\theta/2)cos(\psi/2) \end{bmatrix}$$

- Quaternion to Euler angles (arctan2 is recommended)

$$\varphi = \arctan\left(\frac{2\left(q_0 q_1 + q_2 q_3\right)}{1 - 2\left(q_1^2 + q_2^2\right)}\right), \quad \theta = \arcsin\left(2\left(q_0 q_2 - q_1 q_3\right)\right), \quad \psi = \arctan\left(\frac{2\left(q_0 q_3 + q_1 q_2\right)}{1 - 2\left(q_2^2 + q_3^2\right)}\right)$$

Additional attitude operations – conversions

- Quaternion to rotation matrix

$$R_n^b = \begin{bmatrix} (q_0^2+q_1^2-q_2^2-q_3^2) & 2(q_1q_2-q_0q_3) & 2(q_1q_3+q_0q_2) \\ 2(q_1q_2+q_0q_3) & (q_0^2-q_1^2+q_2^2-q_3^2) & 2(q_2q_3-q_0q_1) \\ 2(q_1q_3-q_0q_2) & 2(q_2q_3+q_0q_1) & (q_0^2-q_1^2-q_2^2+q_3^2) \end{bmatrix}$$

- Rotation matrix to quaternion

$$q_0 = \pm\frac{1}{2}\sqrt{1+R_{11}+R_{22}+R_{33}}$$

$$\overset{o}{q}_{1,2} = \frac{1}{4q_0}\begin{bmatrix} 4q_0^2 \\ R_{32}-R_{23} \\ R_{13}-R_{31} \\ R_{21}-R_{12} \end{bmatrix}$$

$$q_1 = \pm\frac{1}{2}\sqrt{1+R_{11}-R_{22}-R_{33}}$$

$$\overset{o}{q}_{3,4} = \frac{1}{4q_1}\begin{bmatrix} R_{32}-R_{23} \\ 4q_1^2 \\ R_{12}+R_{21} \\ R_{13}+R_{31} \end{bmatrix}$$

$$q_2 = \pm\frac{1}{2}\sqrt{1-R_{11}+R_{22}-R_{33}}$$

$$\overset{o}{q}_{5,6} = \frac{1}{4q_2}\begin{bmatrix} R_{13}-R_{31} \\ R_{12}+R_{21} \\ 4q_2^2 \\ R_{23}+R_{32} \end{bmatrix}$$

$$q_3 = \pm\frac{1}{2}\sqrt{1-R_{11}-R_{22}+R_{33}}$$

$$\overset{o}{q}_{7,8} = \frac{1}{4q_3}\begin{bmatrix} R_{21}-R_{12} \\ R_{13}+R_{31} \\ R_{23}+R_{32} \\ 4q_3^2 \end{bmatrix}$$