Limnology
Chapter 4: Hydrodynamics

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Learning objectives

Chapter 4: Hydrodynamics

Today you will learn about :

1. Total derivative (Eulerian – Lagrangian frame)
2. Continuity equation (mass conservation)
3. Navier Stokes -equation for geophysical flows
4. Geophysical flows – classical examples
   a. Inertial currents
   b. Ekman flows / transport
   c. Global surface currents of the ocean

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Changes of currents in a moving fluid

Two different ways:

(i) With an anemometer on a weather station or by a moored current meter

At a fixed point
Spatial or Eulerian derivative

(ii) To measure on a floating platform such as a weather balloon or drifting / floating buoy

Moves with the fluid
Substantive, Lagrangian, material derivative

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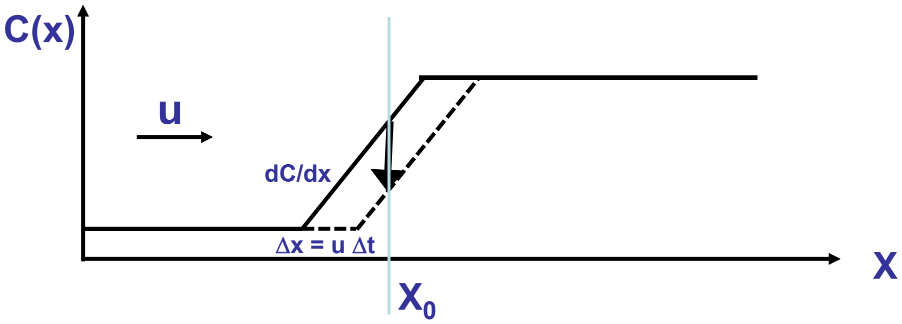
Total derivative – Euler versus Lagrange frame



Two views:

a) **Sitting on a water parcel:** → change: $DC/Dt = R$ (R = source/sink)

b) **Fixed at position x_0:** → $dC/dt = -u\, dC/dx + R = -\mathrm{div}(F_C) + R$

→ **Generalization**

$$DC/Dt = dC/dt + (dx/dt)(dC/dx) = dC/dt + u\,(dC/dx) = R$$

(Lagrange) (Euler) (Advection) (s/s)

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Total derivative = substantive derivative

1-dimensional:

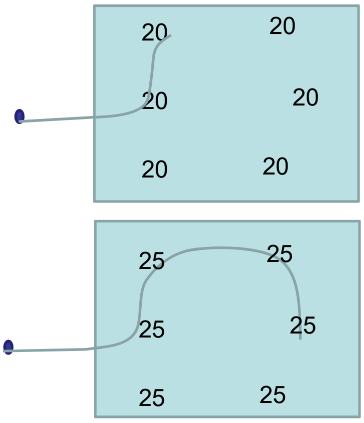
DC/Dt = dC/dt + u (dC/dx) = dC/dt + u grad(C) = R
(Lagrange) (Euler) (Advection) (s/s)

No variations in **space:**

1-dimensional:

DC/Dt = dC/dt + u (~~dC/dx~~)

Unsteady term



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Total derivative = substantive derivative

1-dimensional:

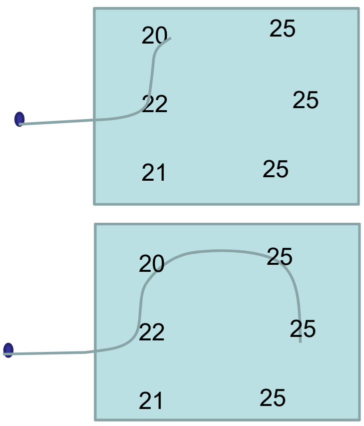
DC/Dt = dC/dt + u (dC/dx) = dC/dt + u grad(C) = R
(Lagrange) (Euler) (Advection) (s/s)

No variations in **time:**

1-dimensional:

DC/Dt = ~~dC/dt~~ + u (dC/dx)

Advection term



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Total derivative = substantive derivative

1-dimensional:

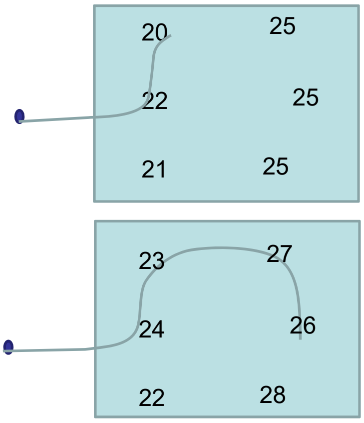
DC/Dt = dC/dt + u (dC/dx) = dC/dt + u grad(C) = R

(Lagrange) (Euler) (Advection) (s/s)

Variations in **time and space:**

1-dimensional:

DC/Dt = dC/dt + u (dC/dx)



Total, substantive derivative

3-dimensional: $$\frac{D}{Dt}(\star) \equiv \frac{\partial(\star)}{\partial t} + (\mathbf{v}\cdot\nabla)(\star) = \mathbf{R}$$

Eulerian derivative (fixed frame) — $\frac{\partial(\star)}{\partial t}$

Changes due to the moving fluid — $(\mathbf{v}\cdot\nabla)(\star)$

Application to: $\star = \mathbf{u} = (u_1, u_2, u_3)$

(2.1.6) $$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\mathbf{grad})\,\mathbf{u}.$$

$$\frac{Du_i}{Dt} = \frac{\partial u_i}{\partial t} + \sum_j u_j \frac{\partial u_i}{\partial x_j}$$

Total derivative of vector (u_x, u_y, u_z)

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right)$$

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) =$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) =$$

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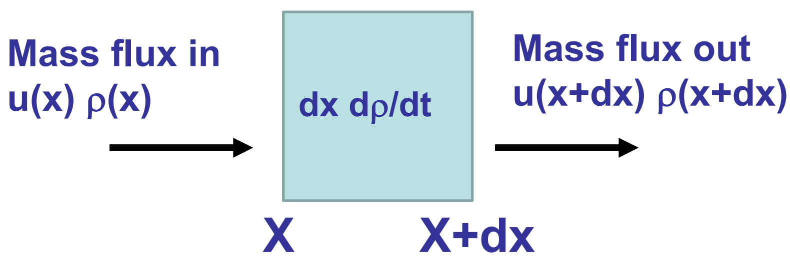
Continuity equation (conservation of mass)



Balance in box:

dρ/dt = [u(x) ρ(x) - u(x+dx) ρ(x+dx)] / dx = - d[u(x) ρ(x)] / dx

⇒ dρ/dt + d[u(x) ρ(x)] / dx = dρ/dt + div [u ρ] = 0

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Continuity equation (Mass conservation)

$$\frac{D\rho}{Dt} + \rho\nabla \cdot \mathbf{v} = 0$$

Incompressible fluid:
→ ρ does not vary along a path line
→ first term = 0

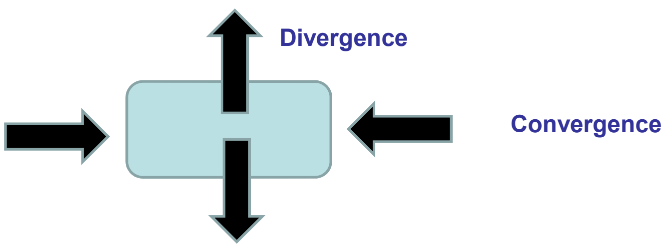
→ consequence $\nabla \cdot \mathbf{v} = 0$

Continuity equation (Mass conservation)

$$\nabla \cdot \mathbf{v} = 0$$

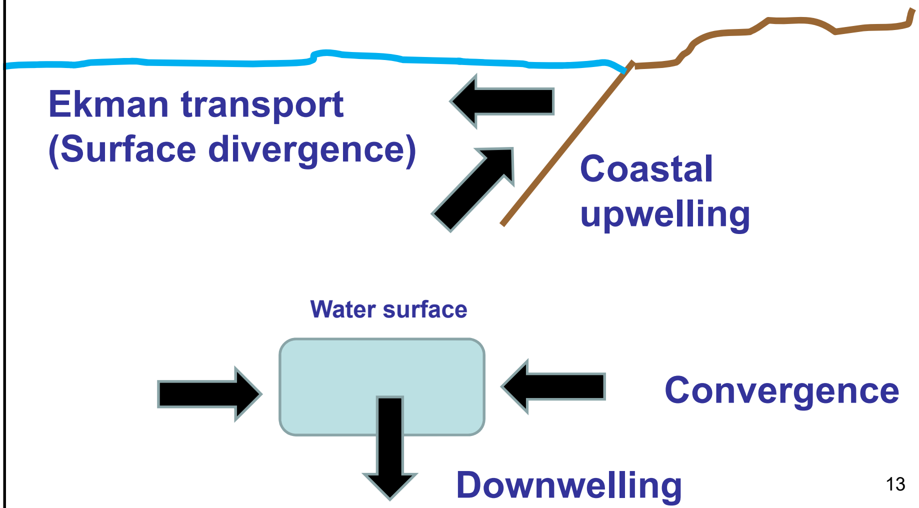
Div(u) = 0 = du/dx + dv/dy + dw/dz = 0

Div(u) = $du_1/dx + du_2/dy + du_3/dz = 0$



Continuity equation (Mass conservation)

$$\nabla \cdot \mathbf{v} = 0$$



Conservation of momentum

$$\frac{\partial}{\partial t}(\rho v_i) + \nabla \cdot (\rho v_i \mathbf{v}) = \rho f_i.$$

Forces f_i (right side)

- friction
- pressure force
- gravity g (z direction only)

Navier-Stokes equation

$$\rho\left(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right)=\mu\left[\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right]-\frac{\partial p}{\partial x}$$

$$\rho\left(\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}\right)=\mu\left[\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial^2 v_y}{\partial z^2}\right]-\frac{\partial p}{\partial y}$$

$$\rho\left(\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right)=\underbrace{\mu\left[\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}+\frac{\partial^2 v_z}{\partial z^2}\right]}_{\text{friction}}\underbrace{-\frac{\partial p}{\partial z}}_{\text{pressure}}\underbrace{+\rho g_z}_{\text{gravity}}$$

Forces f_i (right side)

- **friction**
- **pressure force**
- **gravity g (x,y direction = 0)**

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Friction term

Second Fick's Law for Momentum

ρ · acceleration = - div (Momentum flux) = - div (τ)

ρ · acceleration = - div (τ) = - div (- ρ ν du/dz) = ρ ν d^2u/dz^2

Friction in 3-dimenions for u:

ρ · acceleration = ρ ν [d^2u/dx^2 + d^2u/dy^2 + d^2u/dz^2]

Same for v and w (in y and z direction)

Same for turbulent viscosity: ν → K

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Pressure term

Second Fick's Law for Momentum

ρ · acceleration = - div (Momentum flux) = - div (pressure)

ρ · acceleration = - div (pressure) = - dp/dx

Ditto for y and z →

ρ · acceleration = - div (pressure) = - dp/dy

ρ · acceleration = - div (pressure) = - dp/dz

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Take home message

1. Total derivative (Eulerian – Lagrangian frame)

   DC/Dt = dC/dt + u (dC/dx) = R

   (Lagrange) (Euler) (Advection)

2. Conservation of mass: Div(u) = 0

   → flow structure in upwelling and downwelling

3. Navier-Stokes equations:

   – Forces: friction, pressure and gravity

$$\rho\left(\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right)=\underbrace{\mu\left[\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}+\frac{\partial^2 v_z}{\partial z^2}\right]}_{\text{friction}}-\underbrace{\frac{\partial p}{\partial z}}_{\text{pressure}}+\overbrace{\rho g_z}^{\text{gravity}}$$

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In class exercice:

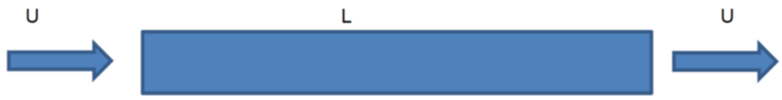
1) Analogy of diffusivity and viscosity

(a) Order the molecular viscosity ($\nu = \mu/\rho$), the molecular diffusivity of salt (D_S), and the molecular diffusivity of temperature (heat, D_T) in the order of their values. Check units.

(b) Explain this ranking order in (a) with physical arguments. Why are these three values so different although all three of them act as diffusion on a molecular level?

In class exercice:

2) Interpretation of the Péclet number



(a) Consider water flowing with U (m/s) through a pipe of length L (m). D_C is the molecular diffusivity of substance C. What is the interpretation of the **Péclet number $Pe_C = L\ U / D_C$** = ?

(b) And what is the interpretation of the **turbulent Pe_{Turb} = L U / K** = ?

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Euler vs Navier Stokes equations

1. Euler equations of motion: **inviscid fluid** (frictionless and non-viscous)
   1. Forces: pressure and gravity

$$\rho\left(\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right)= -\frac{\partial p}{\partial z}+\rho g_z$$

2. Navier Stokes equations :
   - Forces: **friction**, pressure and gravity

$$\rho\left(\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right)=\underbrace{\mu\left[\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}+\frac{\partial^2 v_z}{\partial z^2}\right]}_{\textbf{friction}}\underbrace{-\frac{\partial p}{\partial z}}_{\text{pressure}}+\overbrace{\rho g_z}^{\text{gravity}}$$

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Navier-Stokes equation with fluctuations

$$\frac{\partial u_i}{\partial t}+\sum_j u_j\frac{\partial u_i}{\partial x_j}=-\delta_{i3}g-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\nu\sum_j\frac{\partial^2 u_i}{\partial x_j^2}$$

acceleration advection gravity pressure friction

Now: u → U + u'
what happens after time averaging?

$$\frac{\partial u_i}{\partial t}+\sum_j u_j\frac{\partial u_i}{\partial x_j}=-\delta_{i3}g-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\nu\sum_j\frac{\partial^2 u_i}{\partial x_j^2}-\sum_j\frac{\partial}{\partial x_j}\left(\overline{u_i'u_j'}\right)$$

acceleration advection gravity pressure friction Reynolds stress

(4.2.3)

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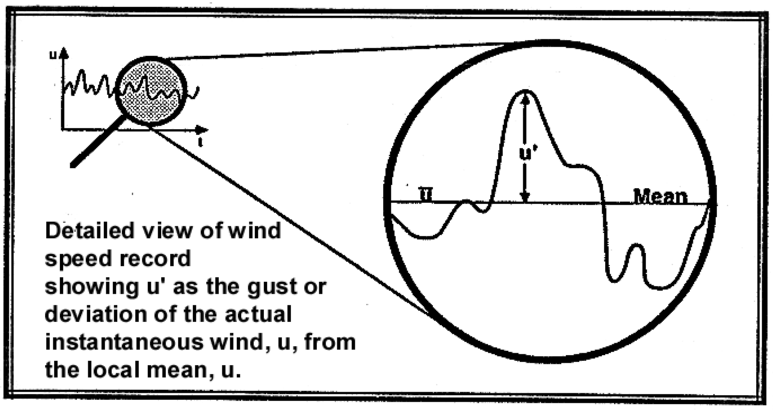
Navier-Stokes equation with fluctuations



Reynolds decomposition: $u_i(t) = U_i(t) + u_i'(t)$

mean fluctuations

What is the right timescale for averages?

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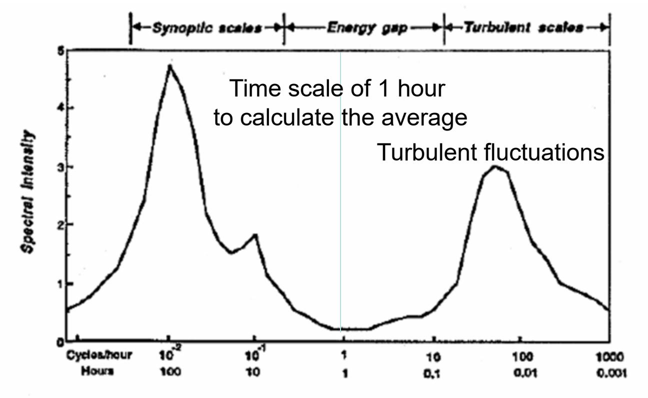
Spectral energy (TKE) density

Turbulent kinetic energy equation

$$\text{TKE} = 0.5 \cdot \overline{(u')^2} \, .$$



This natural gap often missing in natural waters

Large-scale flow
Weather changes
4 days

Turbulent mixing
wind gusts
2 min

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Reynolds decomposition

Mean plus fluctuation: **w → W + w‘**
c → C + c‘

Rules:
<w‘> = 0 **<c‘> = 0** **(assignment)**
<w> = W **<c> = C**

Average of <w · c> = <(W + w‘)·(C + c‘)> = <(W·C + W·c’ + w‘·C + w‘·c‘)> = W·C + 0 + 0 + <w‘·c‘>
= Mean advection + turbulent flux

Eddy formulation: <w‘·c‘> = -K · dC/dz

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N-S equation with fluctuations

(4.2.3)

$$\frac{\partial u_i}{\partial t} + \sum_j u_j \frac{\partial u_i}{\partial x_j} = -\delta_{i3} g - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \sum_j \frac{\partial^2 u_i}{\partial x_j^2} - \sum_j \frac{\partial}{\partial x_j}\left(\overline{u_i' u_j'}\right)$$

(I) (II) (III) (IV) (V) (VI)

I: local acceleration (storage of **average** momentum; inertia)
II advection of the average momentum by the **average** flow field
III: gravity
IV: acceleration due to the **average** pressure-gradient forces
V: friction due to the viscous stress of the **average** flow field
VI: **Reynolds stress (covariance of fluctuating velocity components).**

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Summary: Effect of turbulence

D → D + K (for matter, heat, etc)

ν → ν + K (for momentum, velocity)

Eddy formulation: <w‘·c‘> = -(D+K) · dC/dz

<u‘·w‘> = -(ν+K) · du/dz

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2. Navier-Stokes equation: Geophysical flow

(4.3.8)
$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv + \nu_h \nabla_h^2 u + \nu_z \frac{\partial^2 u}{\partial z^2}$$

(4.3.9)
$$\frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu + \nu_h \nabla_h^2 v + \nu_z \frac{\partial^2 v}{\partial z^2}$$

Total derivative | pressure | Coriolis | Horizontal viscosity | Vertical viscosity

(4.3.10)
$$\frac{\partial p}{\partial z} = -\rho g$$
vertical N-S equation ≈ hydrostatic pressure distribution

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Geophysical flow: effect of the Coriolis acceleration

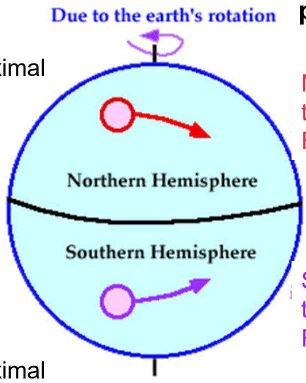
Coriolis acceleration: due to Earth's rotation around the N-S axis

Coriolis acceleration is:

maximal

0

maximal



Coriolis forces acts perpendicular to the flow:

Northern Hemisphere:
to the right of the flow
Fluid to orbit in clockwise circle

Southern Hemisphere:
to the left of the flow
Fluid to orbit in anti-clockwise circle

f = 2 $\omega \cdot \sin(\theta)$, with θ is the latitude
ω = angular frequency = 2 π / day

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2a. Inertial currents

Assumptions:

- negligible pressure gradients ($\delta p/\delta x = \delta p/\delta y = 0$)
- almost frictionless ($v_h = v_z = 0$)
- spatially homogeneous ($\delta u/\delta x = \delta u/\delta y = \delta v/\delta x = \delta v/\delta y = 0$)

(4.4.3) $$\frac{\partial u}{\partial t} = fv$$

f = 2 $\omega \cdot \sin(\theta)$

(4.4.4) $$\frac{\partial v}{\partial t} = -fu$$

Ex: θ=30 °N → f = 0.73×10^{-4} s^{-1}

With the solution:

(4.4.5) $$u = U\cos(ft), \qquad v = -U\sin(ft)$$

Radius = U/f (m)   Period = $2\pi/f$ (s)

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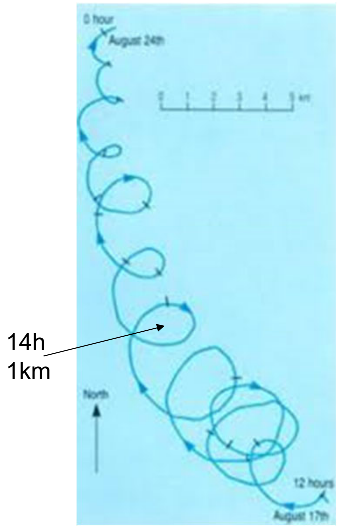
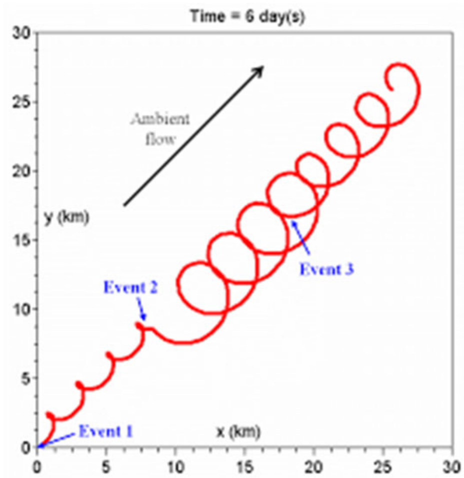
2a. Inertial currents: examples

Radius = U/f (m) Period = 2π/f (s)



Baltic Sea



Southern Hemisphere

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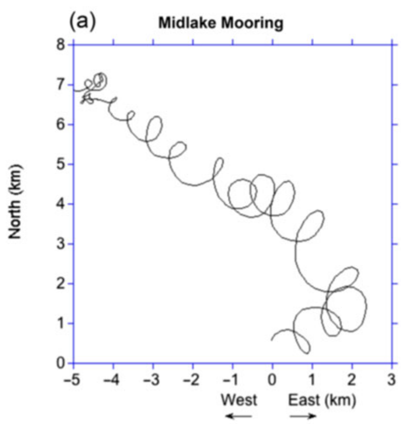
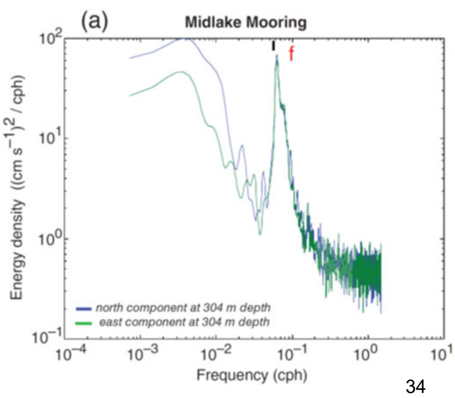
2a. Inertial currents in Lake Geneva

Current measurements in mid-lake at deepest location in May 2008

$f = 2\,\omega \cdot \sin(47\,^oN) = 1.06 \times 10^{-4}\ s^{-1}$

Period = $2\pi/f$ = 16.5 h
R = 0.025 / f = 240 m





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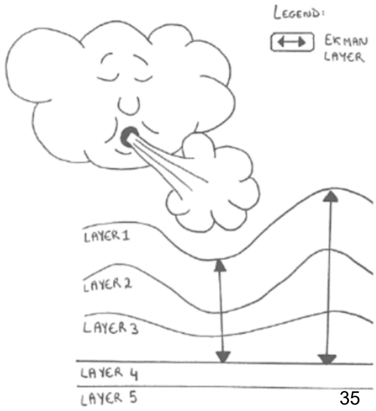
2b. Wind shear at the surface

Wind stress = shear stress exerted by the wind on the surface of water bodies

$$\tau = \rho_{air} \overline{U'W'} = \rho_w \overline{u'w'}$$

(4.5.2) $$\tau = \rho_{air} C_{10} U_{10}^2$$



Example:

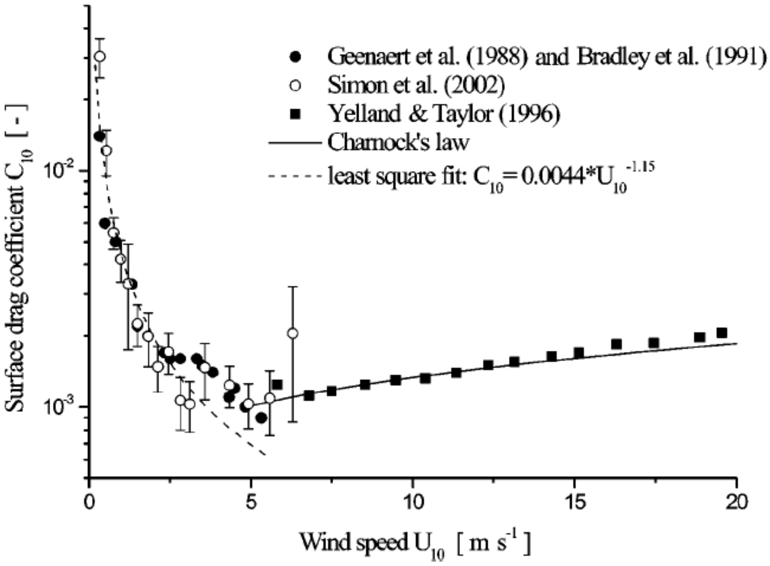
U_{10} = 5 m s^{-1}

ρ_a = 1.25 kg m^{-3}

C_{10} = 1x10^{-3}

→ τ = 0.03 N m^{-2}

2b. Drag coefficient C_{10} as F(U_{10})



In class exercise:

1) Set-up of water surface by wind

Wind, with a velocity of U_{10} = 10 m s^{-1}, blows in the main axis of a 10 km long water body, which is a rectangular prism with depth = 10 m. What is the surface inclination of the homogeneous lake water (no density differences)?
(Hint: Use $C_{10} = 1.5x10^{-3}$ and calculate first the force onto the lake surface and set it equal to the pressure force on the volume).

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2b. Ekman flow

Governing equations: only the Coriolis and friction terms are taken into account:

(4.5.10) $$\frac{\partial^2 U}{\partial z^2} = -\frac{f}{\nu_z} V$$

(4.5.11) $$\frac{\partial^2 V}{\partial z^2} = \frac{f}{\nu_z} U$$

Solutions:

(4.5.14) $$U = V_0 \exp\left(\frac{z}{D_E}\right) \cos\left(\frac{\pi}{4} + \frac{z}{D_E}\right)$$

(4.5.15) $$V = V_0 \exp\left(\frac{z}{D_E}\right) \sin\left(\frac{\pi}{4} + \frac{z}{D_E}\right)$$

Where $V_0 = \frac{\tau}{\rho\sqrt{\nu_z f}}$ = scale of velocity $D_E = \sqrt{\frac{2\nu_z}{f}}$ = depth of Ekman layer

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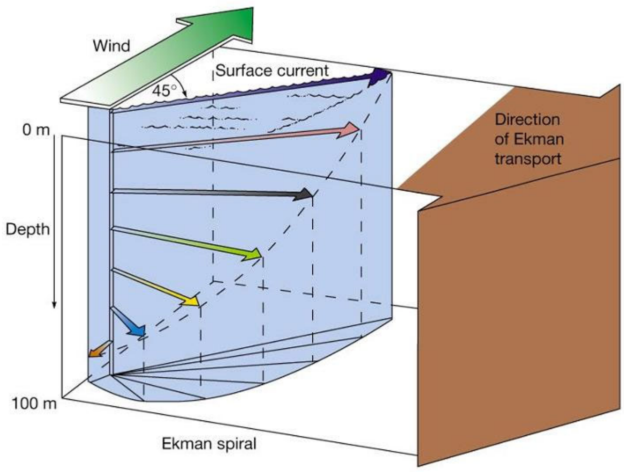
2b. Ekman transport relative to wind



→ **the net water transport over the Ekman depth is perpendicular to the wind**

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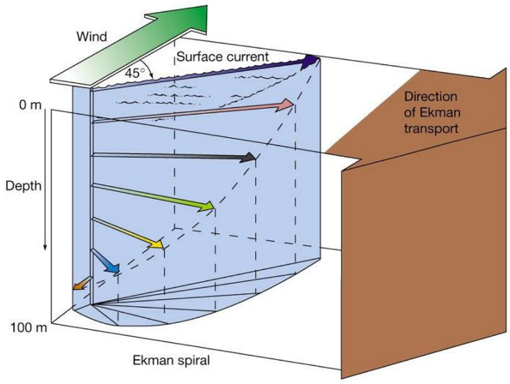
2b. Depth of the Ekman layer

Ekman depth: when the current is at 180° from the original position (half turn)

$$D_E = \sqrt{\frac{2\nu_z}{f}}$$



If vertical viscosity Vz ↑
→ deeper Ekman layer

f is how much the current turns to the right

If f ↑ , shallower Ekman layer

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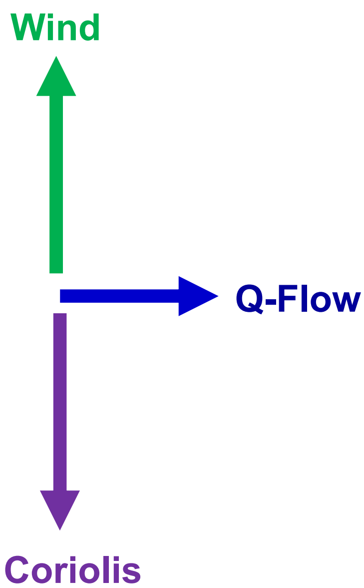
2b. Ekman flow



Wind force (north)

balances out with

Coriolis force (south)

→ Transport to the right (east)

In Northern Hemisphere

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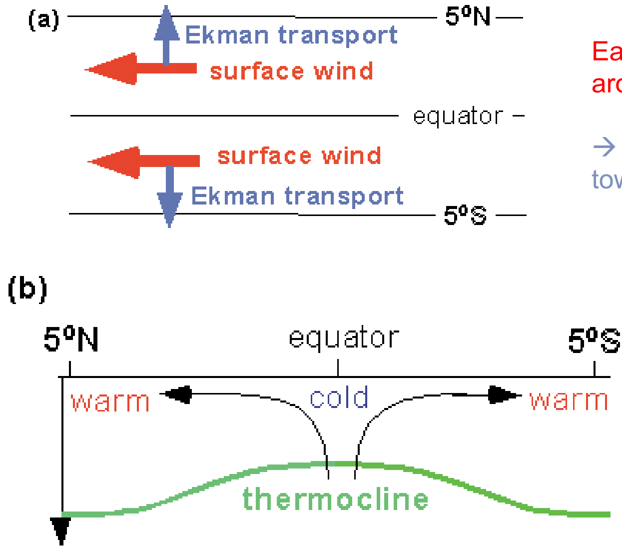
2b. Equatorial divergence



Easterly trade winds around equator

→ Moves surface water towards the poles

Surface water divergence

→ Upwelling from subsurface

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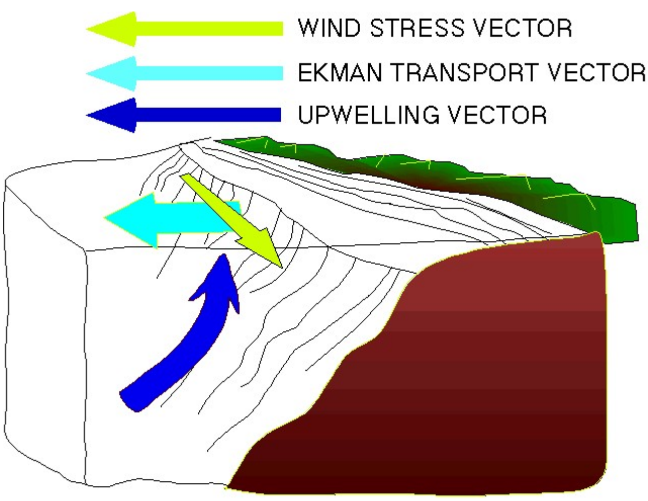
2b. Coastal upwelling

Winds usually blow parrallel to the shores → coastal upwelling



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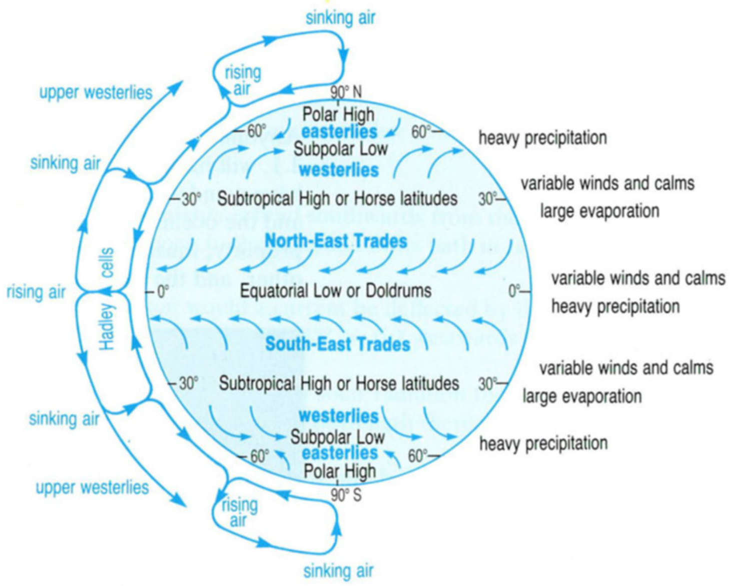
2c. Global wind distribution



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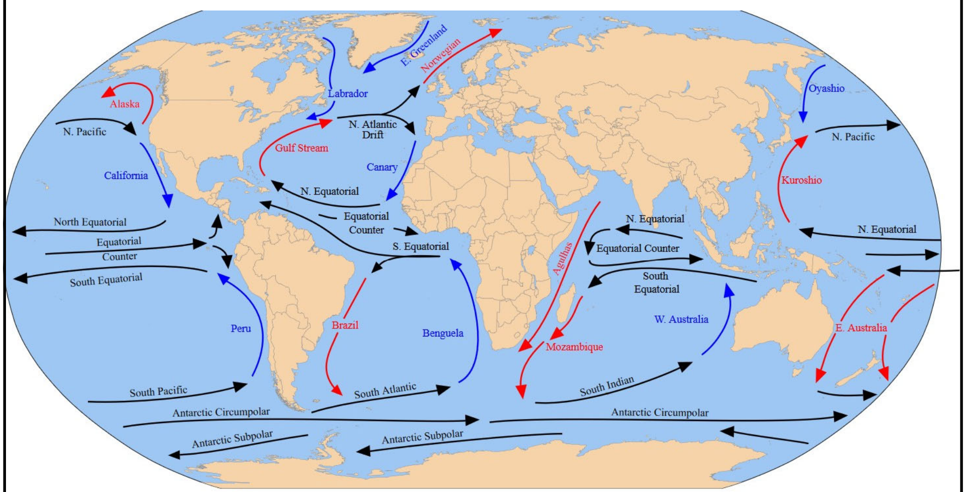
2c. Ocean surface currents



Equatorial and coastal upwelling combined → Mid-gyre downwelling

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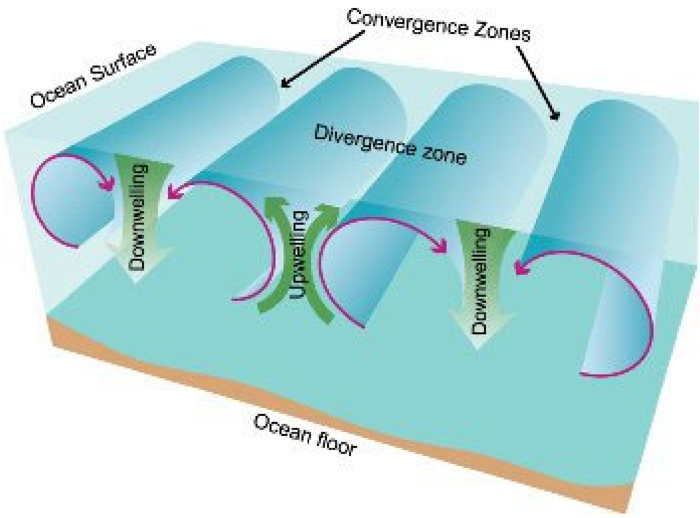
2c. Downwelling in convergence zones



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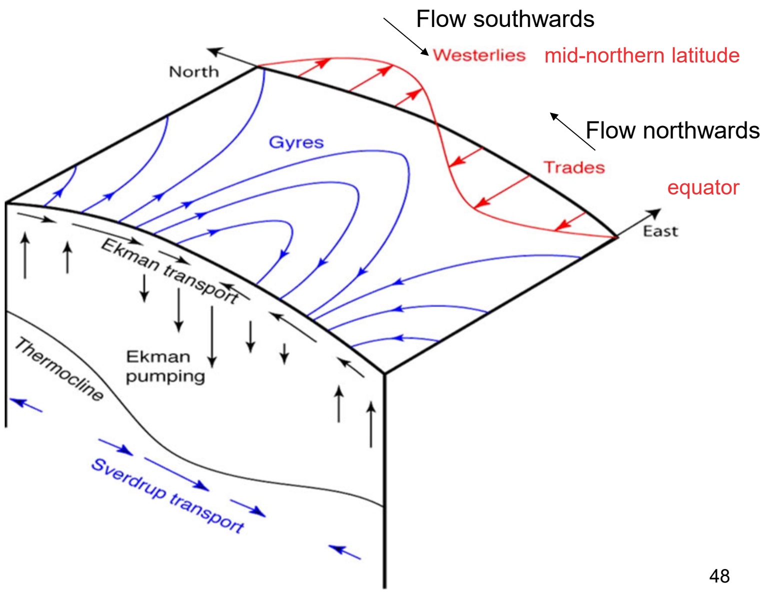
2c. Mid-Atlantic convergence



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In class exercise:

2. Wind-induced transport of water

In the North-Atlantic at 30°N, wind blows towards east with a velocity U_{10} of 8 m s^{-1} (measured 10 m above water surface). Use $C_{10} = 1.5x10^{-3}$, and $\omega = 7.29x10^{-5}\ s^{-1}$.

a) Calculate the velocity and the direction of the current at the surface under the assumption of $v_z = 0.1\ m^2 s^{-1}$.

b) How deep is the Ekman layer? How do you interpret the “depth of the Ekman layer”?

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In class exercice:

3. Primary production and the global circulation in the ocean

a) Where in the ocean occurs upwelling and where downwelling?
b) Primary production needs nutrients for the formation of organic matter. There are two significantly different sources of nutrients in the ocean – explain them.
c) Where in the ocean will primary production be maximal and where will it be minimal?
d) Are these two sources also relevant in lakes? Explain the difference in the nutrient replenishment functioning in lakes and in oceans.

In class exercise:

4. Vertical structure of elements in the ocean (Nozaki Periodic Table)

Look at the “*Periodic Table of the Elements in the Ocean*” (the so called Nozaki Periodic Table). Do the vertical profiles of N, P, Si, Fe, C and O, Na, Cl as well as the noble gases make sense to you? See More details on: http://www.mbari.org/science/upper-ocean-systems/chemical-sensor-group/periodic-table-of-elements-in-the-ocean/