1.10 Hydrostatic Equilibrium

Thus far we have examined the thermodynamic state of individual masses of air and seawater. Here we consider the state of the atmosphere and ocean in the presence of a gravitational field, particularly the height dependence of pressure, temperature, and density. The strength of the gravitational field, which depends primarily on the mass of the planet, is a central determinant of the mass of the atmosphere and ocean. Although both the atmosphere and ocean are bound to the earth by gravity, the ocean has a finite depth, while the atmosphere does not have a top and blends slowly into interplanetary space. The reason for this difference is that the atmosphere is compressible while the ocean is nearly incompressible, since the density of the atmosphere varies with pressure whereas the density of the ocean hardly varies at all.

The vertical variations of pressure in the atmosphere and ocean are observed to be much larger than either the horizontal or temporal variations. The decrease in pressure with height in the atmosphere, and the increase of pressure with depth in the ocean gives rise to a *vertical pressure gradient force*, $\mathscr{F}_p$

$$\mathscr{F}_p = -\frac{1}{\rho}\frac{\partial p}{\partial z} \qquad (1.32)$$

where z is depth/height from the surface and ρ is the density.

The vertical pressure gradient force results in a vertical acceleration in the direction of decreasing pressure (upwards). The vertical pressure gradient force is generally in very close balance with the downward force due to gravitational attraction. This is called *hydrostatic balance*, and is written as

$$g = -\frac{1}{\rho}\frac{\partial p}{\partial z} \qquad (1.33)$$

where g is the acceleration due to the Earth's gravity. The hydrostatic balance is applicable to most situations in the atmosphere and ocean, exceptions arising in the presence of large vertical accelerations such as are associated with thunderstorms.

Equation (1.33) can be integrated to determine a relationship between pressure and depth or height:

$$-\int dp = \int \rho g\, dz \qquad (1.34)$$

To integrate (1.34), it is commonly assumed that g is constant; however, g varies with distance from the Earth's center and also with latitude because of the nonsphericity of the Earth.

To account for these variations in g, the *geopotential* ϕ is often introduced

$$\phi(z) = \int_0^z g\, dz \qquad (1.35)$$

where the geopotential at sea level $\phi(0)$ is taken to be zero by convention. ϕ is the gravitational potential energy per unit mass, with units J kg^{-1}. Using the geopotential, we may write an alternative and equivalent statement of the hydrostatic balance:

$$dp = -\rho\, d\phi$$

The *geopotential height*, Z, can be defined for application to the atmosphere as

$$Z = \frac{\phi(z)}{g_0} = \frac{1}{g_0}\int_0^z g dz \qquad (1.36a)$$

where $g_0 = 9.8$ m s^{-2} is the globally averaged acceleration due to gravity at the Earth's surface. The force of gravity is thus perpendicular to surfaces of constant ϕ, while not exactly perpendicular to surfaces of constant z. Geopotential height is used as a vertical coordinate in many atmospheric applications. In the lower atmosphere, Z is very nearly equal to z; at a distance of z = 10 km above the Earth's surface at 40°N, $g = 9.771$ m s^{-2} and $Z = 9.986$ km. In oceanographic applications, the *dynamic depth*, D, is used analogously to the geopotential height in the atmosphere

$$D = \frac{\phi(z)}{g_0} = \frac{1}{g_0}\int_{p_0}^{p} v\, dp \qquad (1.36b)$$

where $D(p_0) = 0$ is assumed by convention and the *dynamic meter*, dm, is the common unit of dynamic depth. The pressure change is usually expressed in decibars (db, where 1 db = 100 mb), since a pressure of 1 db is equivalent to a change of dynamic depth of about 1 dm.

Since the focus of this text is on the ocean and lower atmosphere, we will assume that $g = g_0$ is a constant, which simplifies the integration and evaluation of (1.34). However, integration of (1.34) also requires some assumption about the vertical variation of the density, ρ.

Because seawater is nearly incompressible, density is nearly constant within the ocean, and thus there is a nearly linear relationship of pressure with depth (Figure 1.4). In the ocean, the following integrated form of the hydrostatic equation is used:

$$p(-z) = p_0 + \rho g z \tag{1.37a}$$

where p_0 is the atmospheric pressure. If we assume that $\rho = 1036$ kg m^{-3} and $g = 9.8$ m s^{-2}, we may then write

$$p(-z) = p_a + 10153\, z \tag{1.37b}$$

where z is in meters and p is in Pa. Because 10^5 Pa = 1 bar, it is easily seen that the oceanic pressure increases at approximately 1 db per meter of depth. Ocean pressures given in db are numerically equivalent to the depth in meters to within 1–2%. However, if seawater were actually incompressible, the sea level would rise by more than 30 m, because the hydrostatic pressure in the deep ocean is so great.

Because air is compressible and density decreases with height in the atmosphere (Figure 1.5), integration of (1.34) for the atmosphere is more complicated than for the ocean. However, useful insights can be derived from examining an idealized *homogeneous atmosphere*, where density is assumed constant. Consideration of a homogeneous atmosphere with finite surface pressure implies a finite total height for the atmosphere, which is called the *scale height H*. Assuming that density is constant, we can integrate (1.34) from sea level, where the pressure is p_0, to a height H, where the pressure is zero, to obtain

$$p_0 = \rho g H \tag{1.38}$$

The height of the homogeneous atmosphere (often referred to as the *scale height*) is therefore

$$H = \frac{p_0}{\rho g} = \frac{R_d T_0}{g} \tag{1.39}$$

where T_0 is the surface temperature and H can be evaluated from the surface temperature and known constants to be approximately 8 km. From the ideal gas law, it is easily inferred that temperature must decrease with height in the homogeneous atmosphere. The lapse rate of the homogeneous atmosphere is obtained by differentiating the ideal gas law with respect to z, holding density constant

$$\frac{\partial p}{\partial z} = \rho R_d \frac{\partial T}{\partial z} \tag{1.40}$$

Combining (1.40) with the hydrostatic equation (1.33) leads to the result

$$\Gamma = -\frac{\partial T}{\partial z} = \frac{g}{R_d} = 34.1°\text{C km}^{-1} \tag{1.41}$$

Thus the lapse rate of a homogeneous atmosphere is constant and about six times larger than the lapse rate normally observed in the atmosphere (which is $\Gamma \approx 6.5$°C km^{-1}). The lapse rate for the homogeneous atmosphere is referred to as the *autoconvective lapse rate* for the following reason: if the lapse rate exceeds the autoconvective value, it is implied that the lower air is less dense than the air above, causing the atmosphere to overturn and the spontaneous initiation of convection. Values of the atmospheric lapse rate as large as the autoconvective value are observed over desert surfaces in summer when the solar heating is high; however, lapse rates in the atmosphere typically do not exceed $\Gamma \approx 10$°C km^{-1}.

Further insight is gained by examining the characteristics of yet another idealized atmosphere, called the *isothermal atmosphere*. After substitution of the ideal gas law for density, we can write the hydrostatic equation in the following form:

$$\partial p = -\frac{pg}{R_d T}\partial z \tag{1.42}$$

This equation is easily integrated for a constant temperature from sea level ($z = 0$, $p = p_0$) to some arbitrary height z

$$\int_{p_0}^{p} \frac{dp}{p} = -\frac{g}{R_d T}\int_0^z dz \tag{1.43}$$

or

$$\ln\frac{p}{p_0} = -\frac{gz}{R_d T} \tag{1.44a}$$

Taking antilogs and using $H = RT/g$, we have

$$p = p_0 \exp\left(-z/H\right) \tag{1.44b}$$

Thus pressure decreases exponentially with height in an isothermal atmosphere, and there is no definite upper boundary to this atmosphere. Note that when $z = H$, the pressure is $1/e$ of its surface value. The isothermal atmosphere resembles the real atmosphere more closely than does the homogeneous atmosphere; however, (1.44b) is not applicable to the real atmosphere except when applied over a shallow layer above the ground.

Many meteorological applications require an accurate relationship between atmospheric pressure and height, which necessitates considering the variation of temperature with height. These applications include: determination of the elevation at which the observations of pressure, temperature, and humidity are obtained from balloons carrying radiosondes; conversion between pressure and height as a vertical coordinate in numerical models of the atmosphere; reduction of surface pressure to sea-level pressure over land; and determination of the thickness between pressure levels. The vertical variations of the temperature profile can be accounted for by integrating (1.42) in a piecewise manner, between height levels that are close enough so that a mean atmospheric temperature in the layer can be defined. Thus we have

$$\int_{z_1}^{z_2} g\,dz = -\int_{p_1}^{p_2} \frac{R_d T_v}{p}\,dp$$

Assuming that $\overline{T}_v$ is constant within the layer, we can integrate to obtain

$$z_2 - z_1 = -\frac{R_d \overline{T}_v}{g} \ln \frac{p_2}{p_1} \tag{1.45}$$

or

$$p_2 = p_1 \exp\left[\frac{g}{R_d \overline{T}_v}(z_1 - z_2)\right] \tag{1.46}$$

Equation (1.45) is referred to as the *hypsometric equation*. From (1.45), it is seen that the *thickness* $\Delta z = z_2 - z_1$ of a layer bounded by two isobaric surfaces is proportional to the average virtual temperature of the layer ($\overline{T}_v$). Figure 1.12 shows the variation with latitude of the relative thickness of isothermal atmospheric layers. Since temperature decreases with latitude away from the equator, the distance between two isobaric surfaces decreases from equator to pole.

An additional application of the hydrostatic equation to the atmosphere is integration under the assumption of a constant lapse rate. Assuming that temperature varies linearly with height with a lapse rate Γ, we have

$$T = T_0 - \Gamma z \tag{1.47}$$

Substituting (1.47) into (1.42) yields

$$\frac{dp}{p} = -\frac{g}{R_d} \frac{dz}{T_0 - \Gamma z}$$

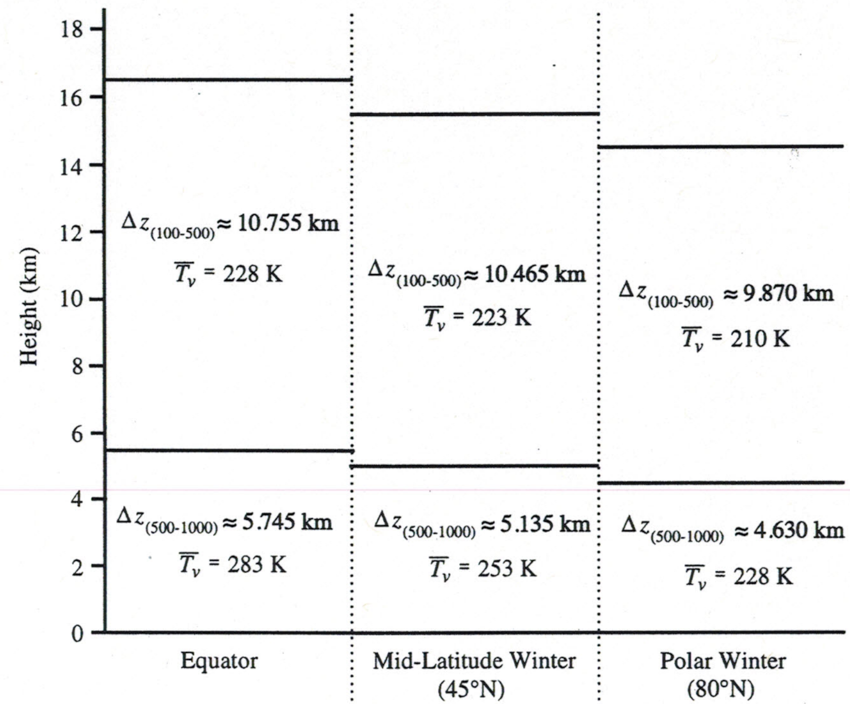

Figure 1.12 Representation of the thicknesses of the 1000–500 hPa and 500–100 hPa layers and their variation with latitude. The thickness of the layer between two isobaric surfaces is determined by the mean virtual temperature in the layer, according to (1.45), resulting in layers of decreasing thickness from equator to pole.

This equation is easily integrated between the limits ($z = 0, p = p_0$) and (z, p) to obtain

$$\ln \frac{p}{p_0} = \frac{g}{R_d \Gamma} \ln\left(\frac{T_0 - \Gamma z}{T_0}\right)$$

or

$$p = p_0 \left(\frac{T}{T_0}\right)^{g/R_d \Gamma} \tag{1.48}$$

Note that the exponent in (1.48) is equal to the ratio of the autoconvective lapse rate (1.41) to the actual lapse rate.