Soil-Plant-Atmosphere Continuum (SPAC) Modeling

Material:

- **Template codes (Matlab):**

  Provided scripts:

  - template_SPAC_01_partA
  - template_SPAC_02_partB, balanceDRYDOWN
  - template_SPAC_03_partC

- **Reference Materials:**

  Lectures 2, 3, 4, 5, and 6

- **Supporting Data:**

  Papers from Daly et al. (2004): link to **Part I**, link to **Part II**

Additional resources and references:

Book *Ecohydrology: Dynamics of life and water in the critical zone* by Porporato and Yin (2022).

Objectives:

This session is designed to help students: (i) familiarize with modeling the coupled dynamics of photosynthesis, transpiration and soil water balance with the Soil-Plant-Atmosphere Continuum (SPAC), (ii) couple the system with a simple atmospheric boundary layer (ABL) model, and (iii) implement and test a simple model resolving a daily soil moisture balance forced by stochastic rainfall.

The document and exercise session are divided into three parts, in which guidelines are provided to revise the main equations and derivations introduced during the lectures, and use them to build the numerical model in Matlab.

Part A: partial closure of the SPAC

At the end of this section, you will obtain the relationship between the leaf water potential ψ_l and the transpiration rate E shown Figure 1 (see lecture 4). The intersections of dashed and solid lines satisfy the soil-root-plant-atmosphere continuum constraints and give transpiration as a function of leaf temperature T_l.

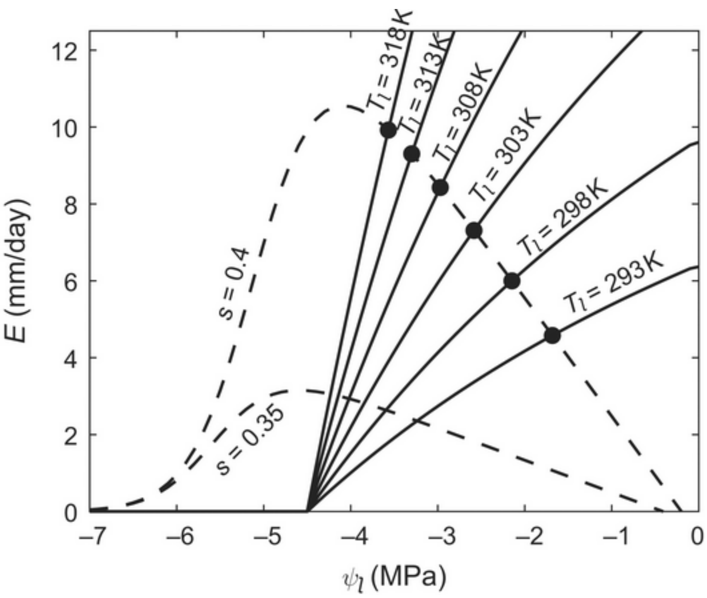

Figure 1: Relationship between ψ_l and E from the soil-to-leaf liquid water flux (dashed lines) and the leaf-to-atmosphere water vapor flux (solid lines).

In Figure 1, the air temperature was set as 293 K, the relative humidity was 30%, and the other SPAC parameters are provided in the following.

1. Identify the governing equations

Figure 1 depicts the solution of the following equations, for several leaf temperatures T_l and two values of the soil saturation s:

1) The soil-to-leaf water flux
$$E_1 = g_{srp}(\psi_s - \psi_l) \quad (1)$$
2) The leaf-to-atmosphere vapor flux
$$E_2 = g_{sa}(T_a, e_a, \psi_l)\frac{0.622}{p_0}[e_i(\psi_l, T_l) - e_a] \quad (2)$$

Both equations depend on the leaf water potential ψ_l, that ranges from -7 to 0 MPa in Figure 1.

2. Prepare the input data – on paper

Review the expressions needed to solve the problem (Eqs. (1-2)). After defining each term, fill in the missing equations according to the lectures.

<u>Equation (1) for E_1</u>

- $g_{srp}[\text{MPa}] = \frac{g_p \cdot LAI\ g_{sr}}{g_p \cdot LAI + g_{sr}}$, where:
  - $g_p = g_{p,max} \cdot \exp\left[-\left(-\frac{\psi_l}{d_1}\right)^c\right]$ (Vulnerability curve)

    → Plant parameters can be found in Table 2.
  - $g_{sr} = \ldots.$ (Soil-root conductance, see lecture 3)

    → Soil parameters in Table 3.

    Please use this expression for $K = K_s \cdot s^{2b+3}$

    Important: make sure that units are consistent.
- $\psi_s = \bar{\psi}_s \cdot s^{-b}$ → Soil parameters in Table 3.

<u>Equation (2) for E_2</u>

- $g_{sa} = \frac{g_a\ g_s}{g_a + g_s}$, where:
  - $g_a = \ldots.$ (Aerodynamic or atmospheric conductance from the turbulence similarity theory, see lecture 4)

    → Atmospheric parameters in Table 4.
  - $g_s = \ldots.$ (Stomatal conductance according to Jarvis model, see lecture 4)

    → Plant parameters in Table 2.
- $e_i = e_{sat}(T_i) \exp\left[\frac{v_w(\psi_l \cdot 10^6 - g\rho_w\ z_i)}{R \cdot T_i}\right]$, with:
  - $T_i = T_l$
  - v_w, g, ρ_w given in Table 1.
  - z_i is equal to the canopy height (h_c, Table 2)
  - The multiplication by 10^6 is needed to make units consistent.
  - $e_{sat}(T_l) = 611.71 \exp\left[\frac{17.27 \cdot T_l}{237.3 + T_l}\right] \cdot 10^{-6}$ [MPa] (Jones, 1992)

    Important: in this expression T_l should be in degrees Celsius.

Table 1: General constants

Parameter	Value	Units	Description
g	9.8	m/s^2	Gravitational acceleration
ρ_w	998	kg/m^3	Water density
k	0.41	-	Von Karman Constant
R	8.314	$J/(mol\ K)$	Universal Gas Constant
v_w	$18/1000/\rho_w$	m^3/mol	Molecular volume of liquid water

Table 2: Plant parameters

Parameter	Value	Units	Description
Z_r	60	cm	Rooting depth
LAI	1.4	-	Leaf area per unit ground area
RAI_w	5	-	Root Area Index under well watered conditions
A	8	-	Parameter accounting for root growth
RAI	$= RAI_w \cdot s^{-a}$	-	Root Area Index
h_c	20	m	Canopy height
C	2	-	Parameter of Vulnerability Curve
d_1	2	MPa	Parameter of Vulnerability Curve
$g_{p,max}$	$11.7 \cdot 10^{-6}$	$m/(MPa\ s)$	Maximum Plant Conductance
k_1	0.005	m^2/W	Parameter for f_{PAR} in Jarvis' model
k_2	0.0016	$1/K^2$	Parameter for f_{T_a} in Jarvis' model
T_{opt}	298	K	Parameter for f_{T_a} in Jarvis' model
D_x	$1250 \cdot 10^{-6}$	MPa	Parameter for f_D in Jarvis' model
ψ_{l0}	-4.5	MPa	Parameter for f_{ψ_l} in Jarvis' model
ψ_{l1}	-0.05	MPa	Parameter for f_{ψ_l} in Jarvis' model
$g_{s\,max}$	$25 \cdot 10^{-6}$	m/s	Maximum stomatal conductance

Table 3: Soil parameters (loam)

Parameter	Value	Units	Description
$\bar{\psi}_s$	$-1.43 \cdot 10^{-3}$	MPa	Air entry suction point (Brooks Corey)
b	5.39	-	Parameter related to pore size distribution (Brooks Corey)
K_s	$20/100/(60 \cdot 60 \cdot 24)$	m/s	Saturated hydraulic conductivity

Table 4: Atmospheric variables and parameters

Parameter	Value	Units	Description
ϕ	400	W/m^2	Leaf available energy (i.e., the PAR used in Jarvis' f_{PAR})
RH	30	%	Relative Humidity
T_a	293	K	Air temperature
U_w	5	m/s	Wind speed
p_0	$101325 \cdot 10^{-6}$	MPa	Atmospheric pressure
$e_{sat}(T_a)$	$= 611.71 \exp\left[\frac{17.27 \cdot T_a}{237.3 + T_a}\right] \cdot 10^{-6}$	MPa	Saturated vapor pressure at T_a (T_a in degrees Celsius)
e_a	$= \frac{RH}{100} \cdot e_{sat}(T_a)$	MPa	Atmospheric vapor pressure
VPD	$= e_{sat}(T_a) - e_a$	MPa	Vapor pressure deficit (i.e., the D used in Jarvis' f_D)
d	$\cong 0.75\, h_c$	m	Displacement height
ε	$\cong d/10$	m	Momentum roughness height
ε_q	$\cong 0.2 \cdot \varepsilon$	m	Water vapor roughness height
g_a	$= \dots f(U_w, k, h_c, d, \varepsilon, \varepsilon_q)$	MPa	Aerodynamic conductance

3. Prepare the input data – on computer

Open the template code "template_SPAC_01_partA" in Matlab and complete it with the parameters and equations from section 2.

Implement all the equations for <u>one</u> leaf temperature and <u>one</u> soil saturation, namely:

- $T_l = 293$ K
- $s = 0.35$

4. Run the model – on computer

To span different values of ψ_l, you need to implement a for loop across ψ_l values ranging from -7 to 0 MPa (x-axis in Figure 1). You can choose any discretization for ψ_l. In the example below and Matlab script the spacing between points is set to 0.01 MPa and the for loop is already implemented.

```
...                 % all the equations and parameters needed
psi_l = -7:0.01:0; % MPa, Leaf water potential
```

```
for i=1:length(psi_l)

    % SOIL-TO-LEAF water flux
    g_srp(i) = ...
    E_1(i)   = ...

    % LEAF-TO-ATMOSPHERE flux (Jarvis’ model, 1976)
    if psi_l(i)<psi_l0
       f_psil(i) = 0;
    elseif
       ...
    else
       ...
    end
    g_s(i)  = ...
    g_sa(i) = ...
    e_i(i)  = ...
    E_2(i)  = ...

end
% Plot psi_l against E_1 s=0.35 (dashed line in Fig. 1)

plot(psi_l , E_1(:)*...,'--k'); % “...” because E should be in mm/day
hold on

% Plot psi_l against E_2 for Tl=293 (solid black line in Fig. 1)

plot(psi_l , E_2(:)*...,'--k');
```

5. Run the model for a wider range of T_l and s and plot the results

Now repeat the computations for $s = 0.4$ and for all the leaf temperatures reported in Figure 1. In doing so, it might be convenient to add a for loop across s, and another one across T_l, as reported in the example script below.

Note that all the variables that depend on s and T_l need to be associated with the corresponding loop index. For example, the soil-root conductance g_{sr} is a function of the hydraulic conductivity K and the Root Area Index (RAI), both of which depend on the value of s. As such, since the values of s are provided in a vector, i.e. $s = [0.35\ 0.4]$, the resulting g_{sr} will be a vector too.

Plot the results all together, similar to Figure 1.

```
psi_l = -7:0.01:0;                                  % MPa, Leaf water potential
s     = [0.35 0.4];                                 % -   , Values of soil saturation
Tl    = [Ta Ta+5 Ta+10 Ta+15 Ta+20 Ta+30 Ta+60]; % K   , Leaf temperatures (Ta=293 K)

...        % all the equations and parameters needed

for j=1:length(s)
```

```
        for i=1:length(psi_l)

            % SOIL-TO-LEAF water flux
            g_srp(i,j) = ...
            E_1(i,j)   = ...

            % LEAF-TO-ATMOSPHERE flux (Jarvis' model, 1976)
            if psi_l(i)<psi_l0
               f_psil(i,j) = 0;
            elseif
               ...
            else
               ...
            end
            g_s(i,j)  = ...
            g_sa(i,j) = ...
            for kk=1:length(Tl)
                e_i(i,j,kk)  = ...
                E_2(i,j,kk)  = ...
            end
        end
        % Plot psi_l against E_1 for different s (dashed red lines in Fig. 1)
        plot(... , ...,'--r'); % E should be in mm/day
        hold on

end
% Plot psi_l against E_2 for different temperatures (solid black lines in Fig. 1)
for kk=1:length(Tl)
    plot(... , ...,'-k');     % E should be in mm/day
end
```

6. Question: *Why do we observe a dramatic decrease in transpiration with lower saturation values?*

7. Run the model for different soil textures

Let's now consider several soil types characterized by the parameters listed in Table 5. Run the model by adopting these parameters.

You can select one or more values of s and one or more values of T_l.

Table 5: Soil parameters for several soil types

Soil type	Air entry suction point (Brooks Corey) $\bar{\psi}_s$ (MPa)	Parameter related to pore size distribution (Brooks Corey)	Saturated hydraulic conductivity K_s (cm/day)

		$b\ (-)$	
Sand	$-0.34 \cdot 10^{-3}$	4.05	200
Loamy sand	$-0.17 \cdot 10^{-3}$	4.38	100
Sandy loam	$-0.70 \cdot 10^{-3}$	4.90	80
Clay	$-1.82 \cdot 10^{-3}$	11.4	10

- How much do the values of g_{srp} and E_1 change among the different cases?

- Make a plot in which you overlap the differences that you find among different soil types. How does soil type affect the results?

- Try to justify the changes among different cases.

8. Run the model for different vegetation types

Let's now consider a soil type of your choice (Table 3 or Table 5) but with a different vegetation cover, whose rooting depth (Z_r) and leaf area index (LAI) are typical of a temperate forest (Table 6). For simplicity, consider all the other plant parameters as in Table 2.

Table 6: Plant parameters (temperate forest)

Parameter	**Value**	**Units**	**Description**
Z_r	0.8	m	Rooting depth
LAI	5	-	Leaf area per unit ground area

- Make a plot to graphically assess the changes. Again, you are free to select one or multiple values for s and T_l. Justify the observed changes.

Part B: Full coupling of the SPAC-ABL model

Here, we close the problem by coupling the water transport (from the soil through the leaf to the atmosphere) to the energy balance. At the end of this section, you will reproduce results shown in Figure 2 and will be able to assess the role of soil, plant, and boundary layer features on the temporal dynamics of the main variables characterizing the SPAC during dry-down experiment (see lecture 5).

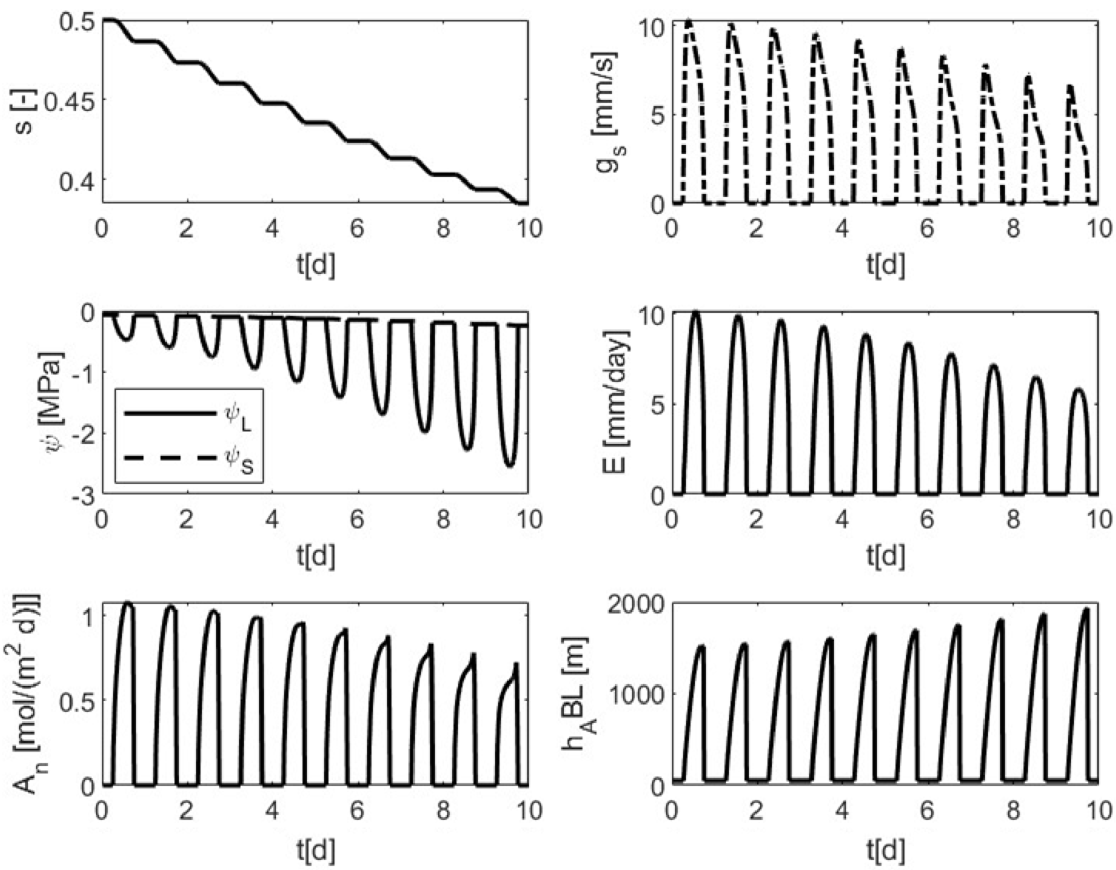

Figure 2: Temporal dynamics of some of the main variables characterizing the SPAC during a dry-down of 10 days, starting from an initial value of $s_{t=1}$ equal to 0.5.

1. Overview and identification of the governing equations

To couple the equations (1) – (2) analyzed in Part A for the SPAC to the energy balance, we need a third equation (Eq. (3)). Thus, the closed system reads:

$$ET = g_{srp}(\psi_s - \psi_l) \quad (1)$$

$$\rho_w ET = \rho g_{sa}(T_a, e_a, \psi_l)\frac{0.622}{p_0}[e_i(\psi_l, T_l) - e_a] \quad (2)$$

$$\lambda_w \rho_w ET = Q - H \quad (3)$$

and it can be solved for the unknown ET, ψ_l and T_l.

In Eq. (3), Q is the radiation and $H = \rho c_p g_a (T_l - T_a)$ is the sensible heat flux.

If the atmospheric variables T_a and e_a are unknown, the SPAC system can be coupled to an ABL model that considers constant profiles of q (specific humidity) and θ (potential temperature) with the ABL (section 3 below).

To solve the system of equations (1) – (3) it is convenient to eliminate one variable and write a system of two equations in the unknown ψ_l and T_l.

(1) $ET = g_{srp}(\psi_s - \psi_l)$

(2) $ET = \frac{\rho}{\rho_w} g_{sa} \frac{0.622}{p_0}(e_i - e_a)$

(3) $\rho c_p g_a (T_l - T_a) + \lambda_w \rho_w ET - Q = 0$

In particular, we can compare the first two equations to eliminate ET:

$$ET_{(1)} - ET_{(2)} = 0 \qquad \rightarrow \qquad g_{srp}(\psi_s - \psi_l) - \frac{\rho}{\rho_w} g_{sa} \frac{0.622}{p_0}(e_i - e_a) = 0 \tag{4}$$

And plug the expression for E given by (2) into (3):

$$\rho c_p g_a (T_l - T_a) + \lambda_w \rho_w \cdot \left[\frac{\rho}{\rho_w} g_{sa} \frac{0.622}{p_0}(e_i - e_a)\right] - Q = 0 \tag{5}$$

In doing so, this nonlinear system of equations can be solved for ψ_l and T_l.
Note that T_a and e_a are atmospheric variables, namely the temperature and vapor pressure of the ABL (t_{ABL} and e_{ABL}, respectively).

In the following, we will proceed step by step to couple them with the ABL dynamics and, therefore, to build our SPAC model - following a procedure similar to that described in Daly et al. (2004)'s papers.

2. Discretization of soil water balance at the hourly time scale

To assess the evolution of s and the other variables (shown in Figure 2) during the dry-down, we need to implement a temporal loop ranging across 10 days. Indeed, differently from Part A, all the variables now change over time.

The soil water balance equation at the hourly timescale during interstorm periods (rainfall equal to zero) reads:

$$n\, Z_r \frac{ds}{dt} = IN - OUT$$

$$n\, Z_r \frac{ds}{dt} = -E - ET - L \tag{6}$$

where E is the soil evaporation, ET is the transpiration from vegetation and L is the leakage.

Equation (6) needs to be discretized for every time step t to get the temporal evolution of soil moisture:

$$s(t) = s(t-1) - \frac{dt}{n\,Z_r} \cdot [E(t-1) + ET(t-1) + L(t-1)] \tag{7}$$

- Here, for the sake of simplicity, we will use a simplified description of soil evaporation E (Figure 3, as in Daly et al.'s paper) given by:

$$E(s) = \begin{cases} 0 & s < s_h \\ \frac{s-s_h}{s_w-s_h} E_W & s_h \leq s \leq s_w \\ E_W & s > s_w \end{cases} \tag{8}$$

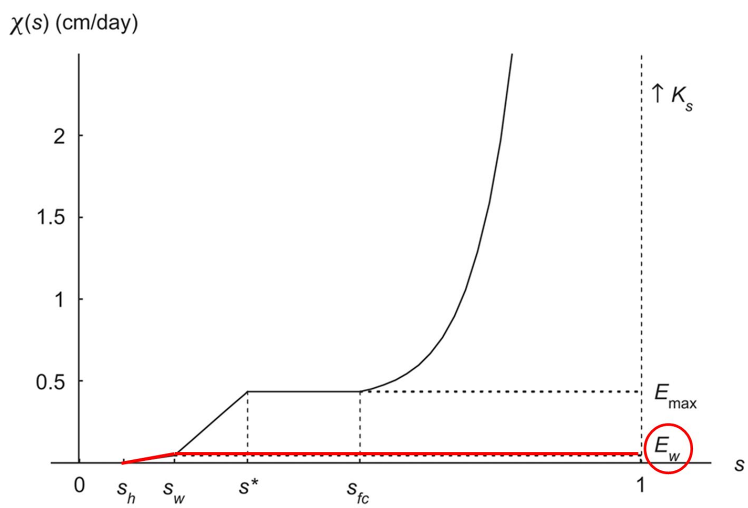

Figure 3: Piecewise approximation function for the soil evaporation (in red).

- The leakage term L is assumed to follow the behavior of hydraulic conductivity as:

$$L = K = K_s s^{2b+3} \tag{9}$$

with K_s the soil saturated hydraulic conductivity and b the exponent of the Brooks and Corey retention curve $\psi_s = \overline{\psi}_s s^{-b}$ (see lecture 2).

- ET is the transpiration from vegetation to the atmosphere given by Eq. (2):

$$ET = \frac{\rho}{\rho_w} g_{sa}(q_i(T_l, \psi_l) - q_{ABL})$$

where q_i and q_{ABL} are the specific humidity values inside and outside the leaf. Remember the useful relationship between specific humidity and vapor pressure, namely $q \cong 0.622\frac{e}{p}$.

The value of q_{ABL} will be obtained from the coupling with the ABL dynamics, described in next section.

3. ABL dynamics: equations and discretization

Finally, to include the ABL dynamics in the SPAC system, we need to consider the equations for the potential temperature θ and humidity q (Figure 4).

Storage term / Surface flux / Entrainment flux (from free atm) $\rho c_p h \frac{d\vartheta}{dt} = H + \rho c_p(\vartheta_f - \vartheta)\frac{dh}{dt}$	**Energy balance** eq. of ABL
$\rho h \frac{dq}{dt} = \rho_w E + \rho(q_f - q)\frac{dh}{dt}$	**Water balance** Eq. of ABL
$\frac{dh}{dt} = \frac{(1+2\beta)H}{\rho c_p \gamma_\vartheta h}$	**ABL growth** Eq. based on turbulent kinetic energy closure at the capping inversion

Figure 4: Equations for the ABL dynamics (see Lecture 5 – The atmosphere).

They can be discretized over time:

$$\theta(t) = \theta(t-1) + \frac{dt}{\rho c_p\, h_{ABL}(t)} \cdot \left[H + \rho c_p\big(\theta_{FA}(t) - \theta(t-1)\big)\frac{dh_{ABL}}{dt}\right] \tag{10}$$

$$q(t) = q(t-1) + \frac{dt}{\rho\, h_{ABL}(t)} \cdot \left[\rho_w ET(t) + \rho\big(q_{FA}(t) - q(t-1)\big)\frac{dh_{ABL}}{dt}\right] \tag{11}$$

Considering that:

- θ is the temperature of the ABL, so we call it t_{ABL}
- θ_{FA} is the temperature of the free atmosphere, i.e. $t_{FA}(t) = \frac{4.78}{1000} h_{ABL}(t) + 293.6$ (Appendix A in Daly et al., 2004)
- q is the humidity of the ABL, so we call it q_{ABL}
- q_{FA} is the humidity of the free atmosphere, i.e. $q_{FA}(t) = -\frac{0.00285}{1000} h_{ABL}(t) + 0.01166$ (Appendix A in Daly et al., 2004)
- The ABL height is given by $\frac{dh_{ABL}}{dt} = \frac{(1+0.2)H}{\rho\, c_p\, \gamma_\theta\, h_{ABL}}$, with $\gamma_\theta = \frac{4.78}{1000}$

We obtain:

$$t_{ABL}(t) = t_{ABL}(t-1) + \frac{dt}{\rho c_p\ h_{ABL}(t)} \cdot \left[H + \rho c_p \big(t_{FA}(t) - t_{ABL}(t-1)\big) \frac{1.2\ H(t)}{\rho\ c_p\ \ 0.00478\ \ h_{ABL}(t)}\right] \quad (12)$$

$$q_{ABL}(t) = q_{ABL}(t-1) + \frac{dt}{\rho\ h_{ABL}(t)} \cdot \left[\rho_w EV(t) + \rho \big(q_{FA}(t) - q(t-1)\big) \frac{1.2\ H(t)}{\rho\ c_p\ \ 0.00478\ \ h_{ABL}(t)}\right] \quad (13)$$

4. Build the SPAC-ABL model in Matlab

4.1 Prepare the input data

Open the template code "template_SPAC_02_partB" in Matlab and complete it with the parameters listed in the following Tables. Make sure to adjust the units to those requested in the Matlab code!

Table 7: General constants

Parameter	Value	Units	Description
g	9.8	m/s^2	Gravitational acceleration
ρ_w	998	kg/m^3	Water density
k	0.41	-	Von Karman Constant
R	8.314	$J/(mol\ K)$	Universal Gas Constant
v_w	18/1000/ρ_w	m^3/mol	Molecular volume of liquid water
ρ	1.2	kg/m^3	Air density
c_p	1012	$J/(kg\ K)$	Specific heat of air
cc	$3 \cdot 10^8$	m/s	Speed of light
NA	$6.022 \cdot 10^{23}$	$1/mol$	Avogadro's constant
h_{planck}	$6.63 \cdot 10^{-34}$	$J\ s$	Planck's constant
λ_w	$2.26 \cdot 10^6$	J/kg	Latent heat of vaporization of water

Table 8: Soil parameters (loam)

Parameter	Value	Units	Description
$\bar{\psi}_s$	$-1.43 \cdot 10^{-3}$	MPa	Air entry suction point (Brooks Corey)
b	5.39	-	Parameter related to pore size distribution (Brooks Corey)
K_s	20	cm/day	Saturated hydraulic conductivity
n	0.45	-	Soil porosity
E_W	0.2	mm/day	Maximum soil evaporation
s_w	0.2	-	Wilting point
s_h	0.05	-	Hygroscopic point
$s_{t=1}$	0.5	-	Initial soil moisture (at time $t = 1$)

Table 9: Plant parameters

Parameter	Value	Units	Description
Z_r	60	cm	Rooting depth
LAI	1.4	-	Leaf area per unit ground area
RAI_w	5.6	-	Root Area Index under well watered conditions
a	8	-	Parameter accounting for root growth
d_{root}	0.2	mm	Root diameter
h_c	20	m	Canopy height
c_1	2	-	Parameter of Vulnerability Curve
d_1	2	MPa	Parameter of Vulnerability Curve
$g_{p,max}$	$11.7 \cdot 10^{-6}$	$m/(MPa\ s)$	Maximum Plant Conductance
k_1	0.005	m^2/W	Parameter for f_{PAR} in Jarvis' model
k_2	0.0016	$1/K^2$	Parameter for f_{T_a} in Jarvis' model
T_{opt}	298	K	Parameter for f_{T_a} in Jarvis' model
D_x	$1250 \cdot 10^{-6}$	MPa	Parameter for f_D in Jarvis' model
ψ_{l0}	-4.5	MPa	Parameter for f_{ψ_l} in Jarvis' model
ψ_{l1}	-0.05	MPa	Parameter for f_{ψ_l} in Jarvis' model
$g_{s\ max}$	$25 \cdot 10^{-3}$	m/s	Maximum stomatal conductance

Table 10: Atmospheric parameters

Parameter	Value	Units	Description
p_0	$101325 \cdot 10^{-6}$	MPa	Atmospheric pressure
c_a	$350 \cdot 10^{-6}$	mol/mol	Atmospheric CO2 concentration
c_i	$0.3 \cdot c_a \cdot 10^6$	mol/mol	CO2 concentration in the stomatal pore
U_w	2	m/s	Wind speed
d	$\cong 0.75\ h_c$	m	Displacement height
ε	$\cong d/10$	m	Momentum roughness height
ε_q	$\cong 0.2 \cdot \varepsilon$	m	Water vapor roughness height
g_a	$= \dots f(U_w, k, h_c, d, \varepsilon, \varepsilon_q)$	MPa	Aerodynamic conductance
t_0	$6 \cdot 3600$	s	Parameter for leaf available energy
δ	$12 \cdot 3600$	s	Day length

Table 11: Carbon assimilation parameters

Parameter	Value	Units	Description
H_{kc}	59430	J/mol	Activation energy for K_c
H_{k0}	36000	J/mol	Activation energy for K_0
H_{vv}	116300	J/mol	Activation energy for $V_{c\,max}$
H_{dv}	202900	J/mol	Deactivation energy for $V_{c\,max}$
H_{vj}	79500	J/mol	Activation energy for J_{max}
H_{dj}	201000	J/mol	Deactivation energy for J_{max}
J_{max0}	75	$\mu mol_{elecrons}/(m^2 s)$	CO2 assimilation parameter
K_{c0}	302	$\mu mol/mol$	Michaelis constant for CO2 at T_0
K_{o0}	256	$\mu mol/mol$	Michaelis constant for O2 at T_0
o_i	0.209	mol/mol	Oxygen concentration
T_0	293.2	K	Reference temperature
S_v	650	J/mol	Entropy term
$V_{c\,max0}$	50	$\mu mol/(m^2 s)$	CO2 assimilation parameter
γ_0	34.6	$\mu mol/mol$	CO2 compensation point at T_0
γ_1	0.0451	$1/K$	Parameter compensation point
γ_2	0.000347	$1/K^2$	Parameter compensation point
$\bar{\lambda}$	$550 \cdot 10^{-9}$	m	Average wavelength of PAR
κ_2	0.20	$mol_{elecrons}/mol_{photons}$	Quantum yield of photosynthesis

4.2 Initialize the variables for the temporal loop

Once all the parameters are provided, you need to initialize the variables that will enter the temporal loop. Make sure to adjust the units to those requested in the Matlab code!

Temporal evolution:

- Total duration of 10 days
- $\Delta t = 1200$ seconds.

Create a time vector that represents time intervals of Δt over a 10-day period. Each day contains time intervals starting at Δt and incrementing by Δt, up to 86400 seconds (i.e. total number of seconds in a day). The vector should repeat this daily pattern for 10 consecutive days. Use the function `repmat`, as in the example below:

```
% Create the time vector
days_tot = 10;                     % days   Length of the dry-down
Deltat   = 1200;                   % s      Timestep (20 min)
day_1    = Deltat:Deltat:86400;    % s      Single day
```

```
time     = repmat(day_1, 1, days_tot);  % s
```

The initial conditions are listed below:

- The initial soil moisture is $s_{t=1} = 0.5$
- The initial height of the atmospheric boundary layer is $h_{ABL} = 50$ m
- The initial temperature of the ABL is $t_{ABL} = 283\ K$
- The initial humidity of the ABL is $q_{ABL} = 0.0008$ kg/kg

as well as in the template script: template_SPAC_02_partB.

Since you are dealing with daily dynamics, you need to distinguish between daytime and nighttime. Consider the daytime between 6am and 6pm, as in the example below.

```
%% Temporal evolution
for i=2:length(time)

    if (time(i)<=6*3600) || (time(i)>=18*3600) %<6h or >18h, i.e., nighttime
        ...
    else % daytime
        ...
    end
    ...
end
```

Note that during the night:

- the conductances and the atmospheric variables are set equal to their initial conditions
- the carbon assimilation parameters and the photosynthetically active radiation (PAR) are equal to zero
- $E,\ s$ are set equal to their value at the previous time step.

4.3 Complete the code with the missing equations and run it

Once all variables are initialized, the code is almost ready to be executed. In the template script template_SPAC_02_partB you will find a skeleton of the code, that you need to complete with the equations for the above-mentioned variables.

If an equation is not explicitly written in this document, you will find it in the course material (Lectures 03, 04, 05).

Note that:

- Indexed variables should be consistent across the code. For example, if the soil moisture s changes across time ($s = s(t)$), all the variables that include it should be indexed in the same way. In the template, you will find a couple of provided equations (for $g_{sr}(i)$ and $VPD(i)$), as an example.
- All the units should be consistent, so make sure to convert, for example, Pa to MPa, etc.

- The lines of code needed to solve the system of Equations (4) – (5) for ψ_l and T_l are provided. It adopts the function `fsolve`, i.e. a built-in algorithm for nonlinear equations. Right click on `fsolve` in Matlab to open the Help window and carefully read how it works.
- The "balanceDRYDOWN" function (provided in the material) returns the unknowns $\psi_l(t)$ and $T_l(t)$. They will be the inputs needed at the following time step $t+1$.
- Compute the net assimilation A_n according to the Farquhar et al. (1980) model (Lecture material 05 – The Atmosphere).

4.4 Plot the results

Make a plot of the temporal evolution of the following variables, as shown in Figure 2:

- Soil moisture s
- Stomatal conductance g_s
- ψ_l and ψ_S
- EV
- A_n, A_q, and A_C
- h_{ABL}

4.5 Run the model for different soil textures

Let's now consider several soil types characterized by the parameters listed in Table 5. Run the model by adopting these parameters.

Make a plot in which you compare the shifts with respect to the case of loam. You are free to decide how to better show the changes, e.g., by overlapping the plots. Make sure that the plot is clear (legend well stated, not too many lines together, etc.). One suggestion is to plot each variable in a single figure (not "subplot") and use different colors for each soil texture.

Which variable among those listed at point 4.4 is most sensitive to variations in soil texture? Justify your answer.

4.6 Run the model for a different vegetation type

Let's now consider a soil type of your choice (Table 3 or Table 5) but with a different vegetation cover (Table 6).

Make a plot to graphically assess the changes in the variables. Again, you are free to display the results in the way that you like the most. Justify the observed changes.

Part C: Stochastic soil moisture balance

So far, we have focused on soil moisture evolution in the absence of rainfall variability. However, this highly unpredictable forcing is a fundamental driver of soil-plant dynamics. To explicitly account for such hydroclimatic forcing and its highly intermittent nature, we will model it as Poisson process (see lecture 6).

At the end of this section, you will construct a simple soil moisture balance model (e.g., Figure 5) driven by stochastic rainfall dynamics, where canopy interception and runoff are neglected for simplicity.

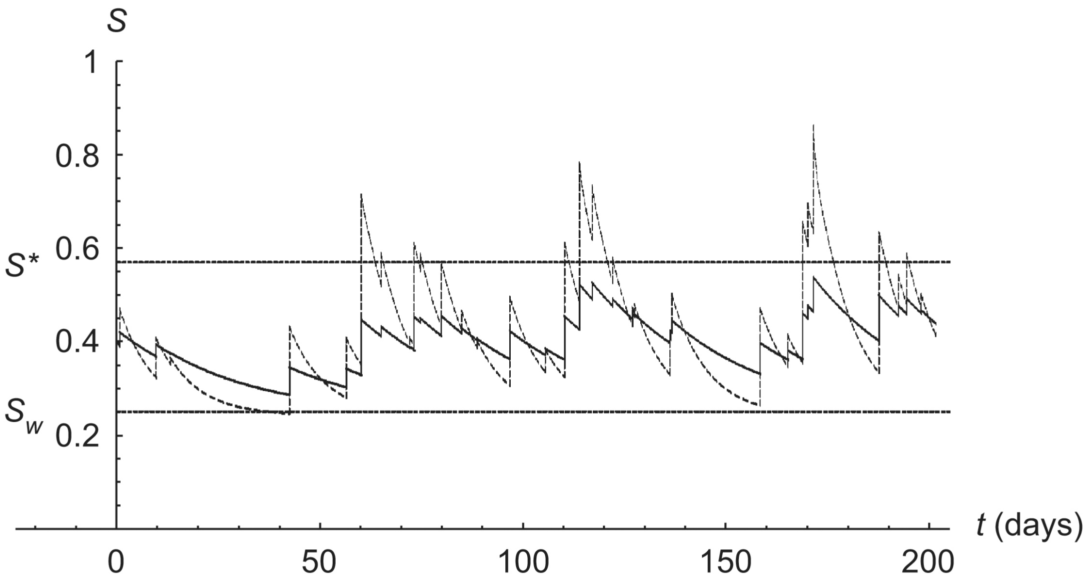

Figure 5: Example of soil moisture evolution for the same rainfall sequence in a loamy soil, for $Z_r = 90$ cm (solid line) and $Z_r = 30$ cm (dashed line). Figure from Porporato and Yin (2022).

1. Overview and governing equations

We consider a simplified soil water balance equation at the daily timescale:

$$n\, Z_r \frac{ds}{dt} = IN - OUT$$

$$n\, Z_r \frac{ds}{dt} = R - (EV + L) \tag{1}$$

Where s is the vertically averaged soil moisture, n is the soil porosity, R is the rainfall rate, EV is the evapotranspiration and L the leakage. The latter two terms of Eq. (1) represent the water losses of the system, namely $\chi(s) = EV(s) + L(s)$ (see Figure 6).

Note that in this formulation we are neglecting the contributions of canopy interception C and runoff Q for simplicity.

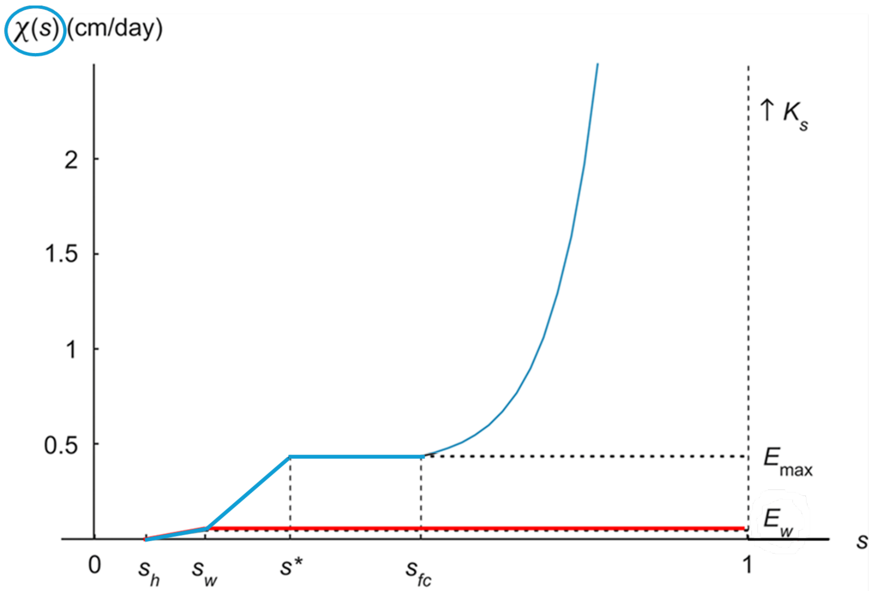

Figure 6: Soil water losses (evapotranspiration and leakage), $\chi(s)$, as a function of relative soil moisture (light blue line).

Upon normalization with respect to the active soil depth ($n\,Z_r$) the complete form of the losses reads:

$$\frac{\chi(s)}{n\,Z_r}=\frac{EV(s)+L(s)}{n\,Z_r}=\begin{cases}0 & 0<s\le s_h\\ \eta_w\dfrac{s-s_h}{s_w-s_h} & s_h<s\le s_w\\ \eta_w+(\eta-\eta_w)\dfrac{s-s_w}{s^*-s_w} & s_w<s\le s^*\\ \eta & s^*<s\le s_{fc}\\ \eta+m\left[e^{\beta(s-s_{fc})}-1\right] & s_{fc}<s\le 1\end{cases}$$

where:

$\eta_w=\frac{E_w}{nZ_r}$; $\eta=\frac{E_{max}}{nZ_r}$; $m=\frac{K_s}{nZ_r\left[e^{\beta(1-s_{fc})}-1\right]}$

The rainfall at the daily timescale is modeled with a marked Poisson process, characterized by the two parameters λ (mean frequency of rainfall events) and α (mean depth per event).

Finally, the soil water balance equation reads:

$$\frac{ds}{dt}=F(t)-\chi(s)$$

where $F(t)$ represents the jumps.

2. Prepare the input data

Open the template code "template_SPAC_03_partC" in Matlab and fill in the parameters for a loam soil listed in Table 12. Consider a Poisson process characterized by $\lambda = 0.2\ day^{-1}$ and $\alpha = 2\ cm$.

Table 12: Soil parameters (loam)

Parameter	Value	Units	Description
n	0.45	-	Soil porosity
Z_r	60	cm	Rooting depth
K_s	20	cm/day	Saturated hydraulic conductivity
β	14.8	-	Parameter depending on soil type
s_h	0.19	-	Hygroscopic point
s_w	0.24	-	Wilting point
s^*	0.57	-	Incipient stomatal closure point
s_{fc}	0.65	-	Field capacity point
$s_{t=1}$	0.75	-	Initial value of soil moisture
E_W	0.01	cm/day	Soil evaporation at s_w
E_{max}	3.5	cm/day	Soil evaporation at s^*

3. Run the model and plot the results

The template script is almost ready to run, once it is completed with the missing equations for the losses.

The jumps of the Poisson process (representing the stochastic rainfall) are randomly extracted from an exponential distribution characterized by mean $\gamma = \frac{\alpha}{n\ Z_r}$, through the Matlab function `exprnd`.

You will obtain several realizations (first for loop in the template). Choose one of them and plot it above the others with a thicker black line.

Make a plot of the probability density function (PDF) of the chosen realization, by excluding the first days of spin-up, which you can assess by looking at your plot (usually around one week). To get the PDF you can use the function `ksdensity` in Matlab, with the option 'pdf', as in the following example:

```
pdf = ksdensity(S,'Function','pdf');
```

where S is the evolution of the soil moisture without the first days of spin-up.
You can also evaluate the PDF with the `histogram` function (always with the option 'pdf').

4. Run the model for a year and vary the stochastic parameters

Re-run the model (just one realization) for Tmax=365 days.

Now play with the parameters of the stochastic process (α and λ): change one at a time first, and then both. *How do the temporal evolution and the PDFs shift?* Make a plot to compare the differences among the PDFs obtained with different parameters and discuss them.

References

Daly, E., Porporato, A., & Rodriguez-Iturbe, I. (2004). Coupled dynamics of photosynthesis, transpiration, and soil water balance. Part I: Upscaling from hourly to daily level. *Journal of Hydrometeorology*, *5*(3), 546-558.

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Jones, H. (1992). Plants and Microclimate: *A Quantitative Approach to Environmental Plant Physiology.* Cambridge University Press.

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