Diffusion equation : point source + infinite domain :

$t=0$ mass M uniform over A (section)

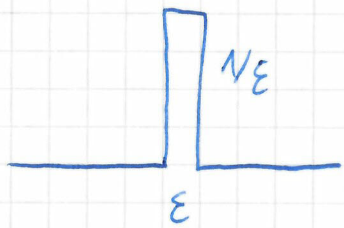
$m = M/A$

$$\frac{\partial C}{\partial t}(x,t) = D\frac{\partial^2 C}{\partial x^2} \qquad (*)$$

$$C(x=\pm\infty, t) = 0 \quad \forall t$$

$$C(x,0) = M/A\ \delta(x) = m\,\delta(x)$$

$\delta(x)$ Dirac delta "function" $= \begin{cases} \infty & x=0 \\ 0 & x\neq 0 \end{cases}$



↳ Put a mass in volume going to zero ⇒ infinite concentration

Properties of $\delta(x)$: 1) $\int_{-\infty}^{\infty} \delta(x)\,dx = 1$

2) $\int_{-\infty}^{\infty} \delta(x-x_0) f(x)\,dx = f(x_0)$

$$\Rightarrow A\cdot\int_{-\infty}^{\infty} C(x,0)\,dx = A\cdot m \int_{-\infty}^{\infty} \delta(x) f(x)\,dx = M \qquad ⚠\ m = M\,L^{-2}\,!$$

Dimensional analysis

C, x, t, D, m relevant variables (dimensional) $n=5$

M, L, T fundamental $k=3$

$n-k=2$ Dimensionless quantities

$$\Pi_1 = \frac{C\sqrt{Dt}}{m} \qquad \Pi_2 = \frac{x}{\sqrt{Dt}}$$

$$\Pi_1 = \phi(\Pi_2) \Rightarrow C(x,t) \overset{(**)}{=} \frac{m}{\sqrt{Dt}}\,\phi\left(\frac{x}{\sqrt{Dt}}\right)$$

Recall that shape does not change in time (simulation)

Inserting (**) in (*) we get

$$\frac{\partial^2 C}{\partial x^2} = \frac{m}{(Dt)^{3/2}} \frac{d^2 \phi(\pi_2)}{d\pi_2^2}$$

$$\frac{\partial C}{\partial t} = -\frac{1}{2}\frac{m}{\sqrt{Dt}\,t}\, \phi(\pi_2) - \frac{1}{2}\frac{m}{\sqrt{Dt}\,t} \overset{\pi_2}{\frac{x}{\sqrt{Dt}}} \frac{d\phi}{d\pi_2}$$

$$\Rightarrow \frac{d^2\phi}{d\pi_2^2} + \frac{\pi_2}{2}\frac{d\phi}{d\pi_2} + \frac{1}{2}\phi(\pi_2) = 0 \quad (ODE) \qquad (***)$$

↳ simpler to solve!

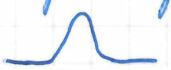
Boundary conditions

1) $\lim\limits_{x \to \pm\infty} C(x,t) = 0 \Rightarrow \lim\limits_{\pi_2 \to \pm\infty} \phi(\pi_2) = 0$

2) $C(x,0) = m\,\delta(x) \Rightarrow \underbrace{\frac{m}{\sqrt{Dt}}}_{\to\infty} \phi\Big(\underbrace{\frac{x}{\sqrt{Dt}}}_{?}\Big) \overset{t\to 0}{=} m\,\delta(x)$

$$(***) \Rightarrow \frac{d}{d\pi_2}\left[\frac{d\phi}{d\pi_2} + \frac{\pi_2}{2}\phi\right] = 0 \Rightarrow \frac{d\phi}{d\pi_2} + \frac{\pi_2}{2}\phi = K_1$$

$\pi_2 = 0$ we have $\frac{d\phi}{d\pi_2} = 0$ (symmetry) $\Rightarrow K_1 = 0$



Thus we have $\frac{d\phi}{d\pi_2} + \frac{\pi_2}{2}\phi = 0 \Rightarrow \frac{d\phi}{\phi} = -\frac{\pi_2}{2} d\pi_2$

$$\Rightarrow \ln(\phi) = -\frac{\pi_2^2}{4} + K_2 \Rightarrow \phi(\pi_2) = e^{K_2 - \pi_2^2/4} = K_3\, e^{-\pi_2^2/4}$$

$$C(x,t) = \frac{M}{A\sqrt{Dt}} \cdot K_3\, e^{-\frac{1}{4}\frac{x^2}{Dt}} \qquad \sigma^2 = 2Dt$$

$C(x=0, t\to 0) = \infty$
↳ B.C. (2)

Property of the dirac function (1) + B.C. (2) $\Rightarrow A \cdot \int_{-\infty}^{\infty} C(x,0) = M$

$\Rightarrow$ (exercise set 3, problem 2) $K_3 = \frac{1}{2\sqrt{\pi}}$

Check that B.C. (1) ist satisfied (trivial).