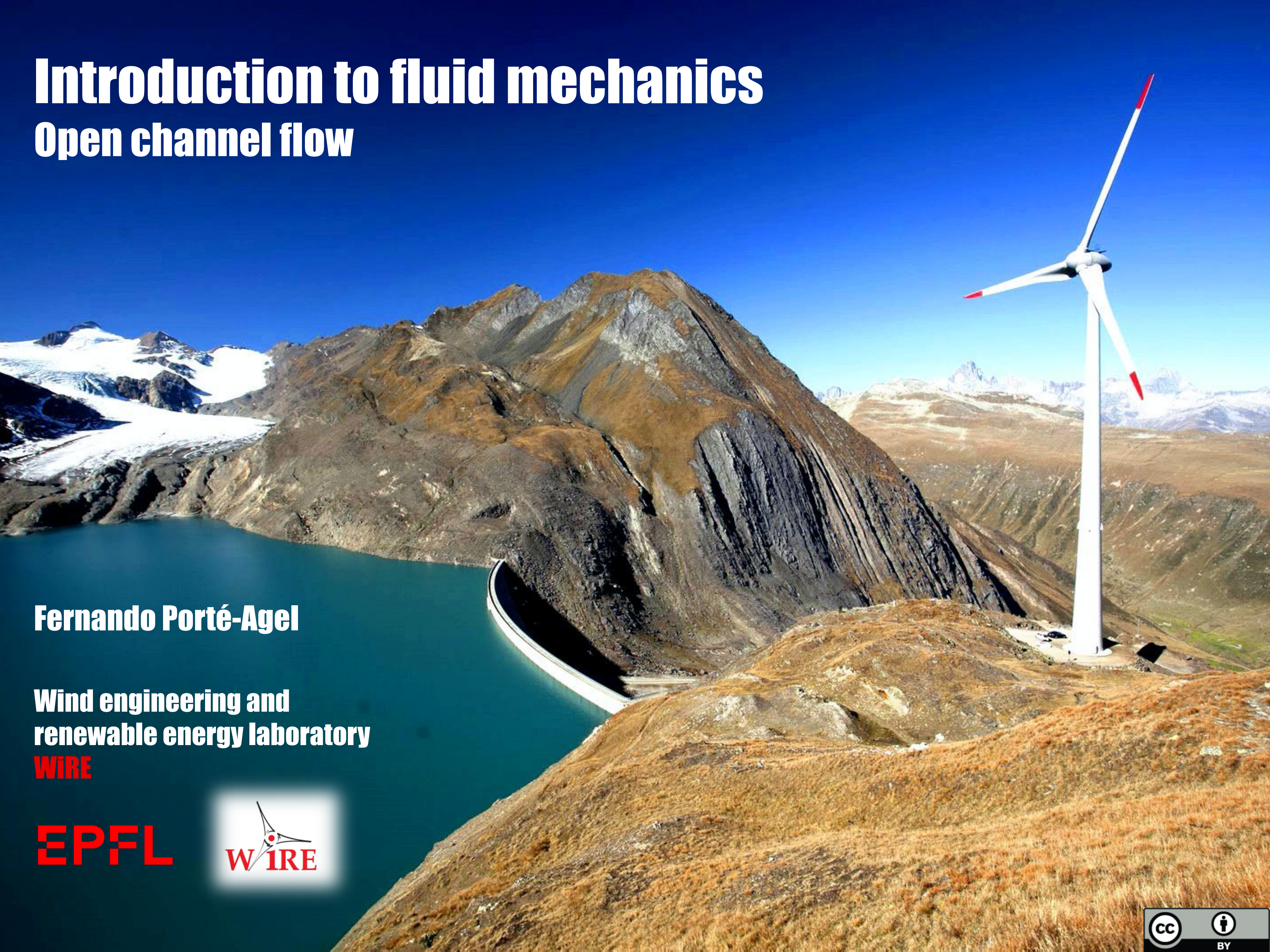
Introduction to fluid mechanics
Open channel flow



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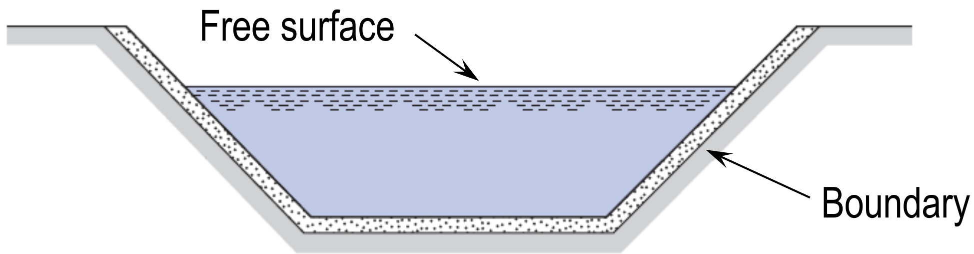
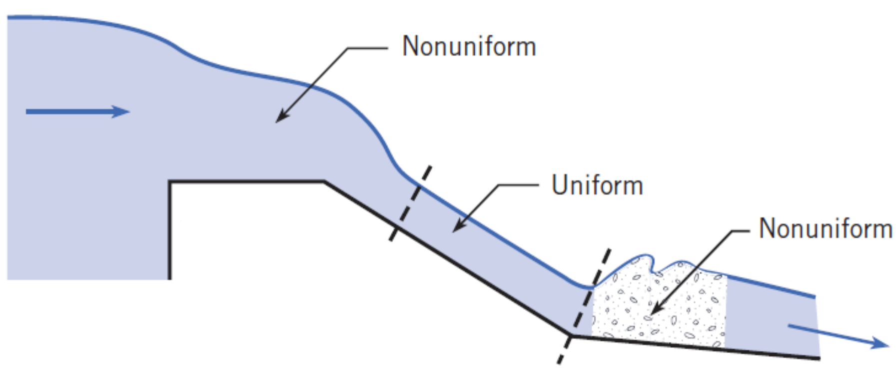
Introduction

Introduction

- Open channel flow is one in which a liquid flows with a *free surface* (not in contact with a boundary)



- If the flow characteristics (height, cross section, etc.) do not change from one section to another along the channel it is called uniform flow. Otherwise it is called non-uniform.



Dimensional analysis

- The movement of the fluid is characterized by two main effects: gravity forces and friction forces.

- Two important non-dimensional numbers:

  - **Froude number**:

$$Fr^2 = \frac{inertial\ forces}{gravity\ forces} = \frac{\rho\, L^2\, V^2}{\gamma\, L^3} = \frac{V^2}{L\, \gamma/\rho} \qquad\qquad Fr = \frac{V}{\sqrt{gL}}$$

  - **Reynolds number**:

$$Re = \frac{inertial\ forces}{friction\ forces} = \boxed{\frac{V\, R_h}{\nu}}$$

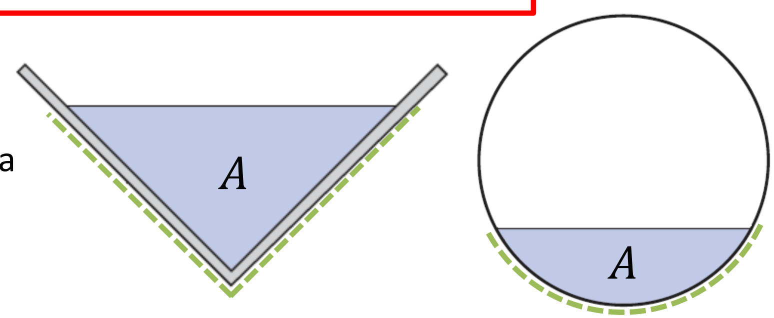
with this definition turbulence occurs for $Re > 500$

Hydraulic radius:

$$R_h = A/P$$

A: cross-sectional area

P: wetted perimeter

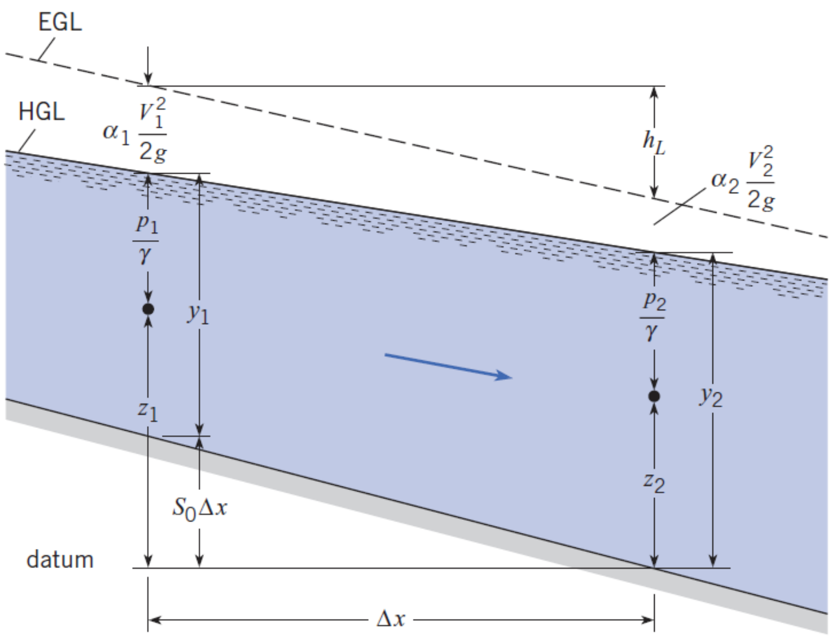


Energy equation for channel flow

- Reminder of the energy equation for pipe flow and let the pump head and turbine head equal zero:

$$\left(\frac{p_1}{\gamma}+\alpha_1\frac{\overline{V}_1^2}{2g}+z_1\right)+h_p=\left(\frac{p_2}{\gamma}+\alpha_2\frac{\overline{V}_2^2}{2g}+z_2\right)+h_t+h_L$$

  – Steady
  – Uniform cross-section
  – Turbulent flow ($\alpha_1 = \alpha_2 \approx 1.0$)



- From the sketch:

$$\frac{p_1}{\gamma}+z_1=y_1+S_0\Delta x \quad \text{and} \quad \frac{p_2}{\gamma}+z_2=y_2$$

  – S_0 : slope of the channel

- It yields:

$$y_1+\frac{V_1^2}{2g}+S_0\Delta x=y_2+\frac{V_2^2}{2g}+h_L$$

* Valid for uniform and nonuniform flows!

Steady uniform flow

- Uniform flow implies no velocity and height change along the slope:
  - Then, the energy equation becomes: $y_1 + \frac{V_1^2}{2g} + S_0 \Delta x = y_2 + \frac{V_2^2}{2g} + h_L$
  - As a result, the slopes of HGL and EGL are the same as the channel's:

$$S_0 = h_f / L$$

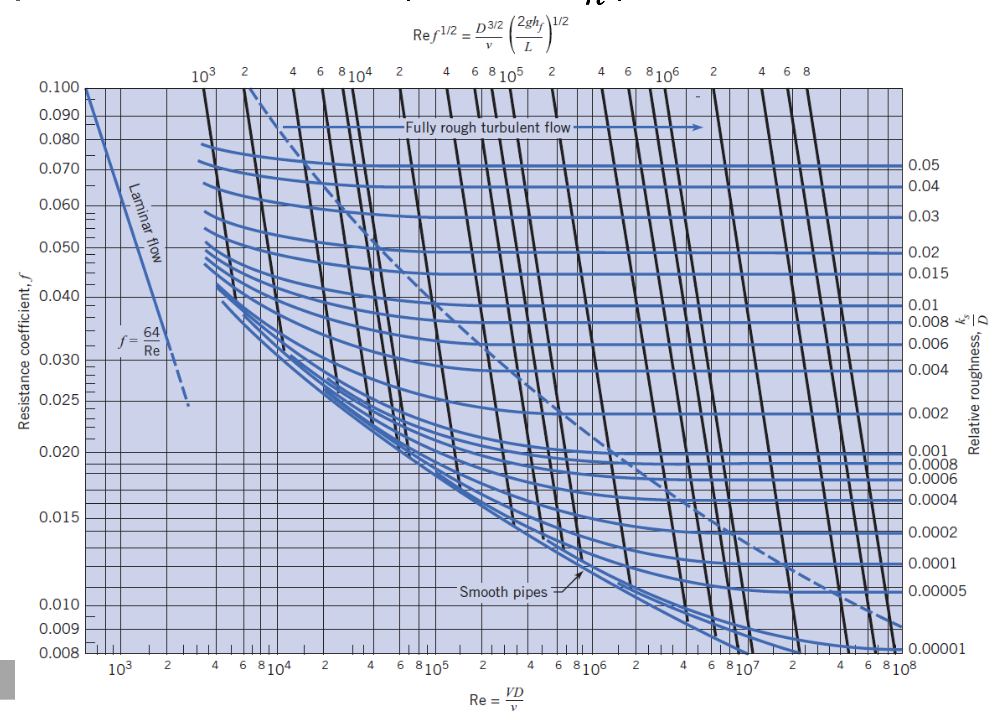
  - Then, the head loss h_f is controlled solely by the slope of the channel
  - The Darcy-Weisbach equation for open channel flow is ($D \rightarrow 4\,R_h$):

$$\frac{h_f}{L} = \frac{f}{4R_h}\frac{V^2}{2g}$$

  - Then, the velocity is:

$$V = \sqrt{\frac{8g}{f} R_h S_0}$$

    - f: friction factor
    - Obtained from Moody diagram



Friction Factor in Turbulent Flows (pipe and channel)

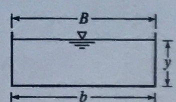
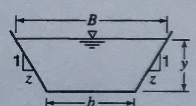
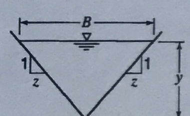
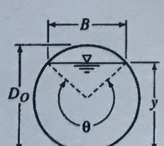
Equivalent sand roughness k_s for different pipe/channel materials

Boundary Material	k_s, Millimeters	k_s, Inches
Glass, plastic	Smooth	Smooth
Copper or brass tubing	0.0015	6×10^{-5}
Wrought iron, steel	0.046	0.002
Asphalted cast iron	0.12	0.005
Galvanized iron	0.15	0.006
Cast iron	0.26	0.010
Concrete	0.3 to 3.0	0.012–0.12
Riveted steel	0.9–9	0.035–0.35
Rubber pipe (straight)	0.025	0.001

Geometric elements of open channel cross-sections

From: "WATER RESOURCES ENGINEERING" by L.W. MAYS (2011) [BOOK]

Table 15.3.2 Geometric Elements of Channel Cross-Sections

Cross-section	Area A	Wetted perimeter P	Hydraulic radius R	Top width B (T)
 Rectangle	by	$b+2y$	$\frac{by}{b+2y}$	b
 Trapezoid	$(b+zy)y$	$b+2y\sqrt{1+z^2}$	$\frac{(b+zy)y}{b+2y\sqrt{1+z^2}}$	$b+2zy$
 Triangle	zy^2	$2y\sqrt{1+z^2}$	$\frac{zy}{2\sqrt{1+z^2}}$	$2zy$
 Circle	$\frac{1}{8}(\theta-\sin\theta)D_0^2$	$\frac{1}{2}\theta D_0$	$\frac{1}{4}\left(1-\frac{\sin\theta}{\theta}\right)D_0$	$(\sin{}^1/{}_2\theta)D_0$ or $2\sqrt{y(D_0-y)}$

Steady uniform flow – Chezy equation

- Traditional method of calculating the discharge

- Remember last equation: $V = \sqrt{\frac{8g}{f}} \cdot \sqrt{R_h S_0}$

- Defining the coefficient $C = \sqrt{\frac{8g}{f}}$, the velocity becomes:

$$V = C\sqrt{R_h S_0}$$

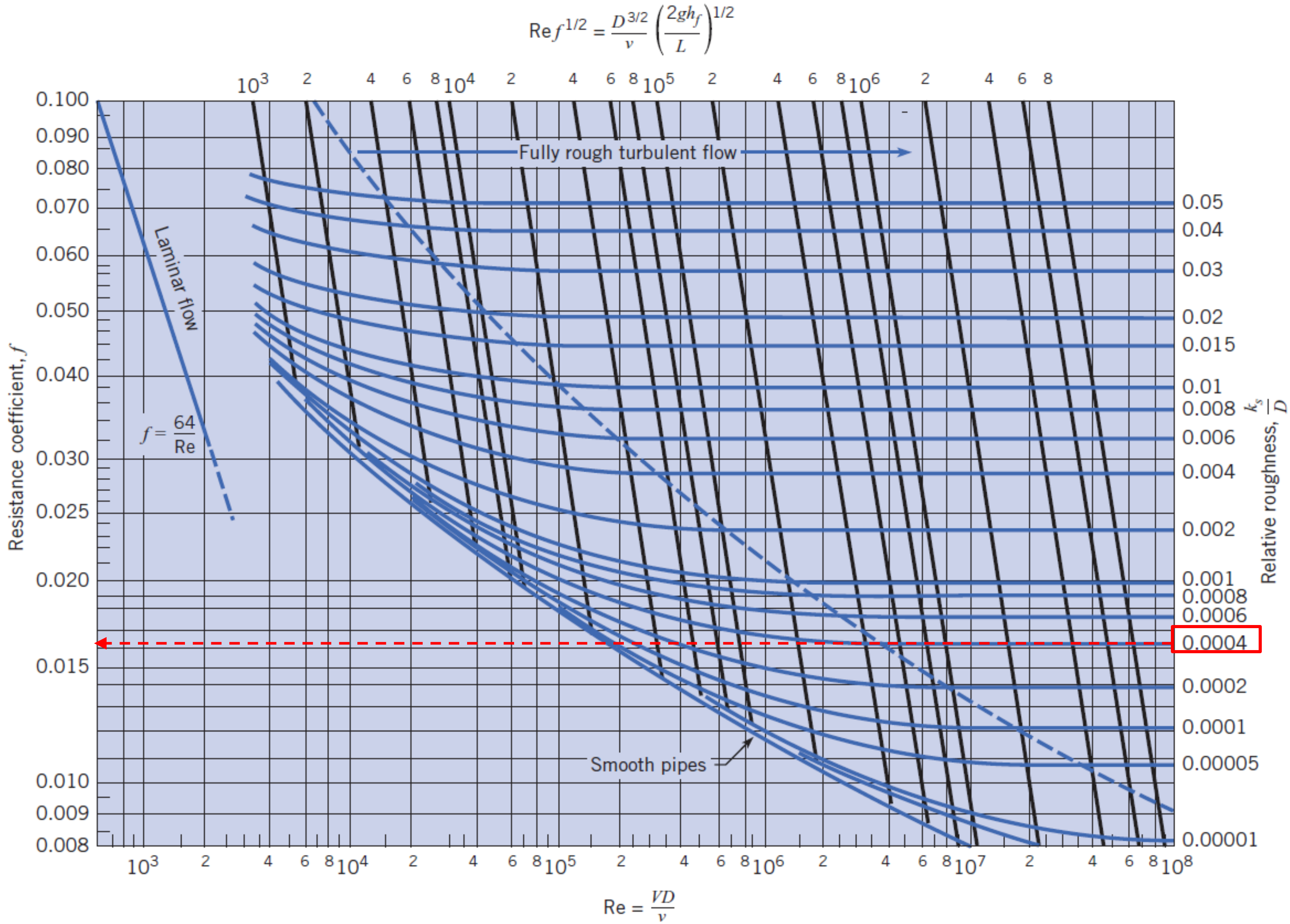
- And the discharge is given by:

Chezy equation

$$Q = VA = CA\sqrt{R_h S_0}$$

* Needs an acceptable guess of f

Steady uniform flow - Exercise



Steady uniform flow - Exercise

EXAMPLE 15.2 ESTIMATING *Q* FOR UNIFORM FLOW USING DARCY-WEISBACH EQUATION

Estimate the discharge of water that a concrete channel 3 m wide can carry if the depth of flow is 2 m and the slope of the channel is 0.0016.

Problem Definition

Situation: Uniform flow, concrete surface.

Find: Discharge in m^3/s.

Properties: Concrete, Table 10.4: $k_s = 0.3$ to 3 mm, or 0.0003 to 0.003 m.

Plan

1. Find channel velocity by relating channel slope to h_f/L with Eq. (15.9).
   - Use the Moody diagram to find f.
   - Assume a roughness for first estimate of $k_s/4R_h$ to use with Reynolds number.
   - Select a first estimate of f, which is opposite $k_s/4R_h$ on the Moody diagram
   - Solve for V, first iteration.
   - Calculate new Reynolds number with this value of V; check f against reasonable convergence criterion.
2. Calculate $Q = VA$.

Solution

1. For Eq. (15.9), $V = \sqrt{\frac{8g}{f} R_h S_0}$; need to get a value for f.

2a. Assume $k_s = 0.0015$ m. Relative roughness is

$$\frac{k_s}{4R_h} = \frac{0.0015\ \text{m}}{4(6\ \text{m}^2/7\ \text{m})} = \frac{0.0015\ \text{m}}{4(0.86\ \text{m})} = 0.00044$$

2b. Use value of $k_s/4R_h = 0.00044$ as a guide to estimate $f = 0.016$.

2c. First iteration for V gives

$$V = \sqrt{\frac{8(9.81\ \text{m/s}^2)(0.86\ \text{m})(0.0016)}{0.016}} = 2.59\ \text{m/s}$$

2d. Recalculate Reynolds number.

$$\text{Re} = V\frac{4R_h}{\nu} = \frac{2.59\ \text{m/s}(3.44\ \text{m})}{1.14 \times 10^{-6}\ \text{m}^2/\text{s}} = 7.8 \times 10^6$$

Using this new value of Re, and with $k_s/4R_h = 0.00044$, read f as 0.016. This value of f is the same as previous estimate—meets reasonable convergence criterion.

$$V = \boxed{2.59\ \text{m/s}}$$

3. Compute Q.

$$Q = VA = 2.59\ \text{m/s}(6\ \text{m}^2) = \boxed{15.54\ \text{m}^3/\text{s}}$$

Steady uniform flow – Manning equation

$$Q = \frac{1}{n} A\, R_h^{2/3} S_0^{1/2}$$

- A common *engineering* approach to the C coefficient is:

$$C = \frac{{R_h}^{1/6}}{n}$$

- Then the discharge can be calculated with:

Manning's equation

$$Q = \frac{1.0}{n} A\, R_h^{2/3}\, S_0^{1/2}$$

- n depends on the roughness
- Only valid for SI units!

* Dimensionally "incorrect"
'1.0' is in fact a dimensional coefficient

Table 15.1 TYPICAL VALUES OF ROUGHNESS COEFFICIENT, MANNING'S *n*

Lined Canals	*n*
Cement plaster	0.011
Untreated gunite	0.016
Wood, planed	0.012
Wood, unplaned	0.013
Concrete, troweled	0.012
Concrete, wood forms, unfinished	0.015
Rubble in cement	0.020
Asphalt, smooth	0.013
Asphalt, rough	0.016
Corrugated metal	0.024
Unlined Canals	
Earth, straight and uniform	0.023
Earth, winding and weedy banks	0.035
Cut in rock, straight and uniform	0.030
Cut in rock, jagged and irregular	0.045
Natural Channels	
Gravel beds, straight	0.025
Gravel beds plus large boulders	0.040
Earth, straight, with some grass	0.026
Earth, winding, no vegetation	0.030
Earth, winding, weedy banks	0.050
Earth, very weedy and overgrown	0.080

Steady uniform flow

Problem Definition

EXAMPLE 15.4 CALCULATING DISCHARGE AND MANNING'S *n* USING CHEZY EQUATION

If a channel with boulders has a slope of 0.0030, is 30 m wide, has an average depth of 1.3 m, and is known to have a friction factor of 0.130, what is the discharge in the channel and what is the numerical value of Manning's n for this channel?

Problem Definition

Situation: Uniform flow, channel with known f.

Find:

1. Discharge, Q, in m^3/s.
2. Numerical value of Manning's n for this channel with boulders.

$$V = \left[\sqrt{\frac{(8)(9.81\ m/s^2)}{0.130}}\right]\left[\sqrt{(1.3\ m)(0.0030)}\right] = 1.53\ m/s$$

2. Discharge

$$Q = VA = (1.53)(30 \times 1.3) = \boxed{60\ m^3/s}$$

3. Manning's n, using the SI unit equation (Eq. 15.16).

$$n = \frac{1.0}{Q} A R_h^{2/3} S_0^{1/2}$$

$$n = \left(\frac{1.0}{60\ m^3/s}\right)(30 \times 1.3\ m^2)(1.3\ m)^{2/3}(0.003)^{1/2}$$

$$n = \boxed{0.0424}$$

Review

Note: This calculated value of n is within the range of typical values in Table 15.1 under the category of "Unlined Canals, Cut in rock."

Note: This example and Example 15.3 show that f in the Darcy-Weisbach equation can be related to Manning's n for uniform-flow conditions.

Steady uniform flow - EXAMPLES

Best hydraulic cross-section

Recall Manning's equation: $Q = \frac{1}{n} A\, R_h^{2/3} S_0^{1/2}$

- Channel geometry that yields a minimum wetted perimeter for a given cross-sectional area.
- This minimizes the friction losses in the channel and therefore maximizes the discharge.
- Section factor: $AR_h^{2/3} = A(A/P)^{2/3}$ (from Manning's equation)

EXAMPLE 15.6 BEST HYDRAULIC SECTION FOR A RECTANGULAR CHANNEL

Determine the best hydraulic section for a rectangular channel with depth y and width B.

Problem Definition

Situation: Rectangular channel with depth y and width B.

Find: Best hydraulic section.

Plan

1. Set $A = By$ and $P = B + 2y$ so that both are a function of y.
2. Let A be constant, and minimize P.
   - Differentiate P with respect to y and set the derivative equal to zero.
   - Express the result of minimizing P as a relation between y and B.

Solution

1. Relate A and P in terms of y.

$$P = \frac{A}{y} + 2y$$

2a. Minimize P.

$$\frac{dP}{dy} = \frac{-A}{y^2} + 2 = 0$$

$$\frac{A}{y^2} = 2$$

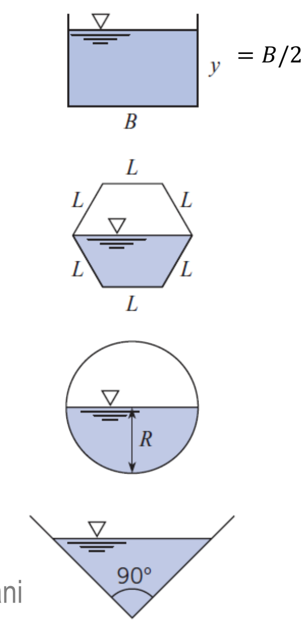
2b. Express result in terms of y and B.

$$A = By, \text{ so}$$

$$\frac{By}{y^2} = 2 \quad \text{or} \quad \boxed{y = \frac{1}{2}B}$$

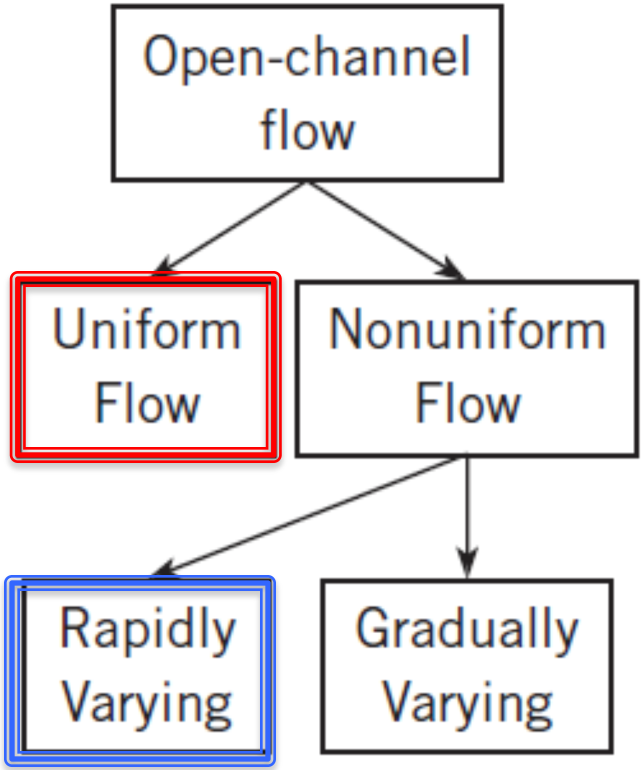
The best hydraulic section for a rectangular channel occurs when the depth is one-half the width of the channel, see Fig. 15.4.

Some examples:



Non-uniform flow

- **Steady Uniform** flow:
  - Constant velocity along a streamline
  - **Manning** and **Chezy** equations are valid
- **Steady non-uniform** flow:
  - Velocity change along a streamline
  - No change with time
  - **Two types**:
    - Rapidly varying
      - Changes take place over a short distance
      - Neglect the resistance of the channel walls and bottom
    - **Gradually varying**
      - Changes take place more gradually, longer distances
      - Surface resistance is a significant variable



Rapidly varied flow - Example

Water flows in a rectangular channel with a velocity of 3 m/s and a depth of 3 m. What is the change in water depth and water surface elevation produced by a gradual change in bottom elevation of 30 cm?

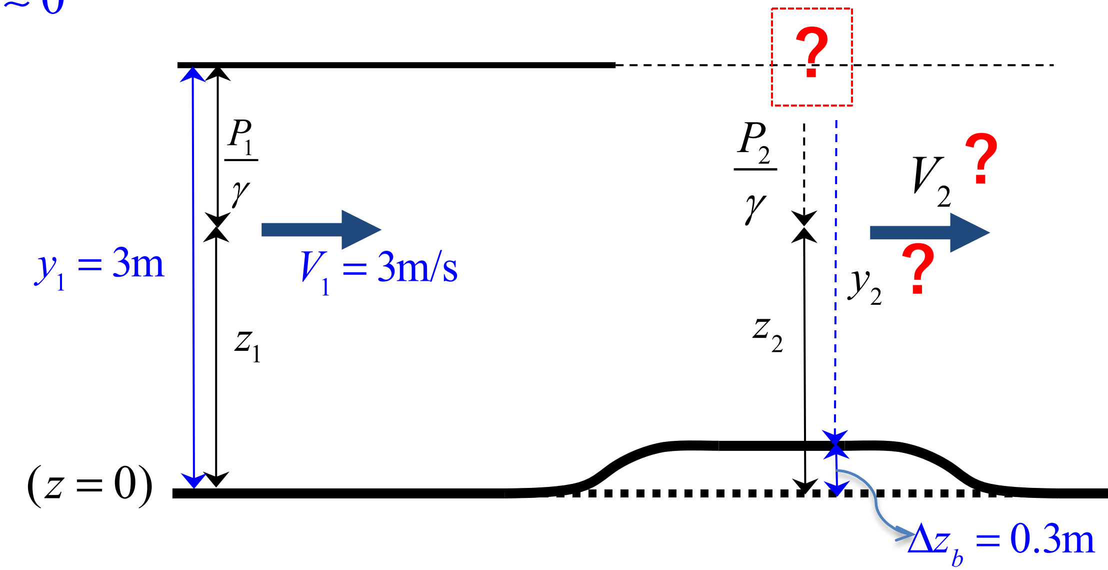
$$\frac{p_1}{\gamma} + z_1 + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + z_2 + \frac{V_2^2}{2g} + h_L$$

$h_L \approx 0$

In this case, we can write *(see sketch)*:

$$y_1 + \frac{V_1^2}{2g} = y_2 + \Delta z_b + \frac{V_2^2}{2g}$$

Note: 2 unknowns: y_2 and V_2



From continuity: $V_1 A_1 = V_2 A_2 \rightarrow V_1 b y_1 = V_2 b y_2 \rightarrow V_2 = V_1 \frac{y_1}{y_2}$

$$y_1 + \frac{V_1^2}{2g} = y_2 + \Delta z_b + \frac{V_1^2}{2g}\frac{y_1^2}{y_2^2}$$

Note: now only one unknown: y_2 → $y_2 = 2.5$ m

Water surface elevation **drops 0.2 m**.

Rapidly varied flow – Cases when S_0 and h_L can be neglected

- Energy equation:
  - Horizontal channel ($S_0 = 0$)
  - Neglect head loss ($h_L = 0$)

$$y_1 + \frac{V_1^2}{2g} + \cancel{S_0 \Delta x} = y_2 + \frac{V_2^2}{2g} + \cancel{h_L}$$

$$y_1 + \frac{V_1^2}{2g} = y_2 + \frac{V_2^2}{2g}$$

- Specific energy definition:

Length units!

$$E = y + \frac{V^2}{2g} \quad ; \quad E_1 = E_2$$

- Introducing continuity:

$$A_1 V_1 = A_2 V_2 = Q$$

- Energy equation can be rewritten as:
  - A_1 and A_2 are functions of the depth y_1 and y_2
  - The specific energy depends only on the depth!

$$y_1 + \frac{Q^2}{2gA_1^2} = y_2 + \frac{Q^2}{2gA_2^2}$$

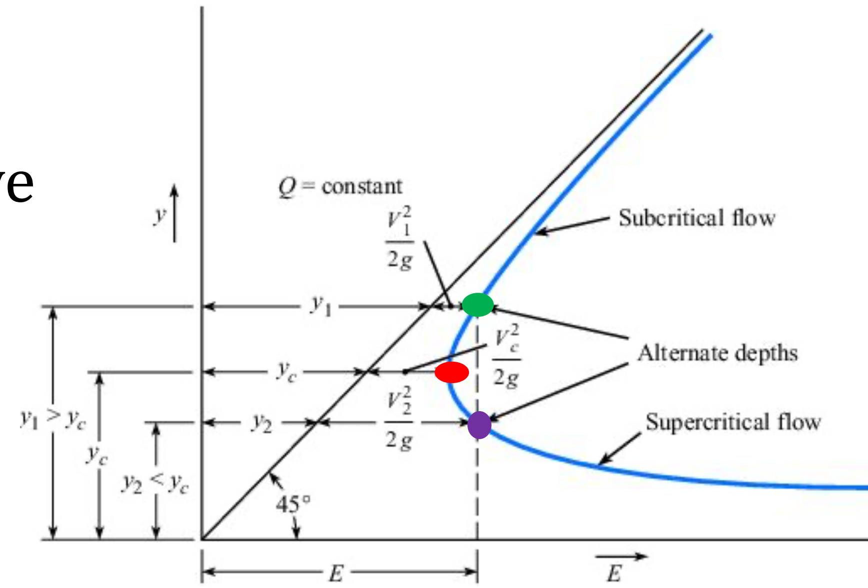
Rapidly varied flow

$$E = y + \frac{Q^2}{2gA^2}$$

- The energy of a flow will always be in the forms of:
  - Potential energy
  - Kinetic energy

- For a given Q and specific energy E, we normally have two possible solutions:
  - Subcritical flow ●
    - Most of the energy in potential form
    - Slow and deep
  - Supercritical flow ●
    - Most of the energy in kinetic form
    - Fast and shallow

- Critical flow ● occurs when the specific energy is minimum (single solution)



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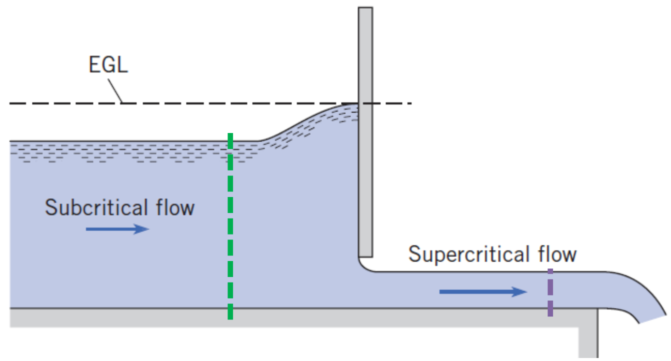
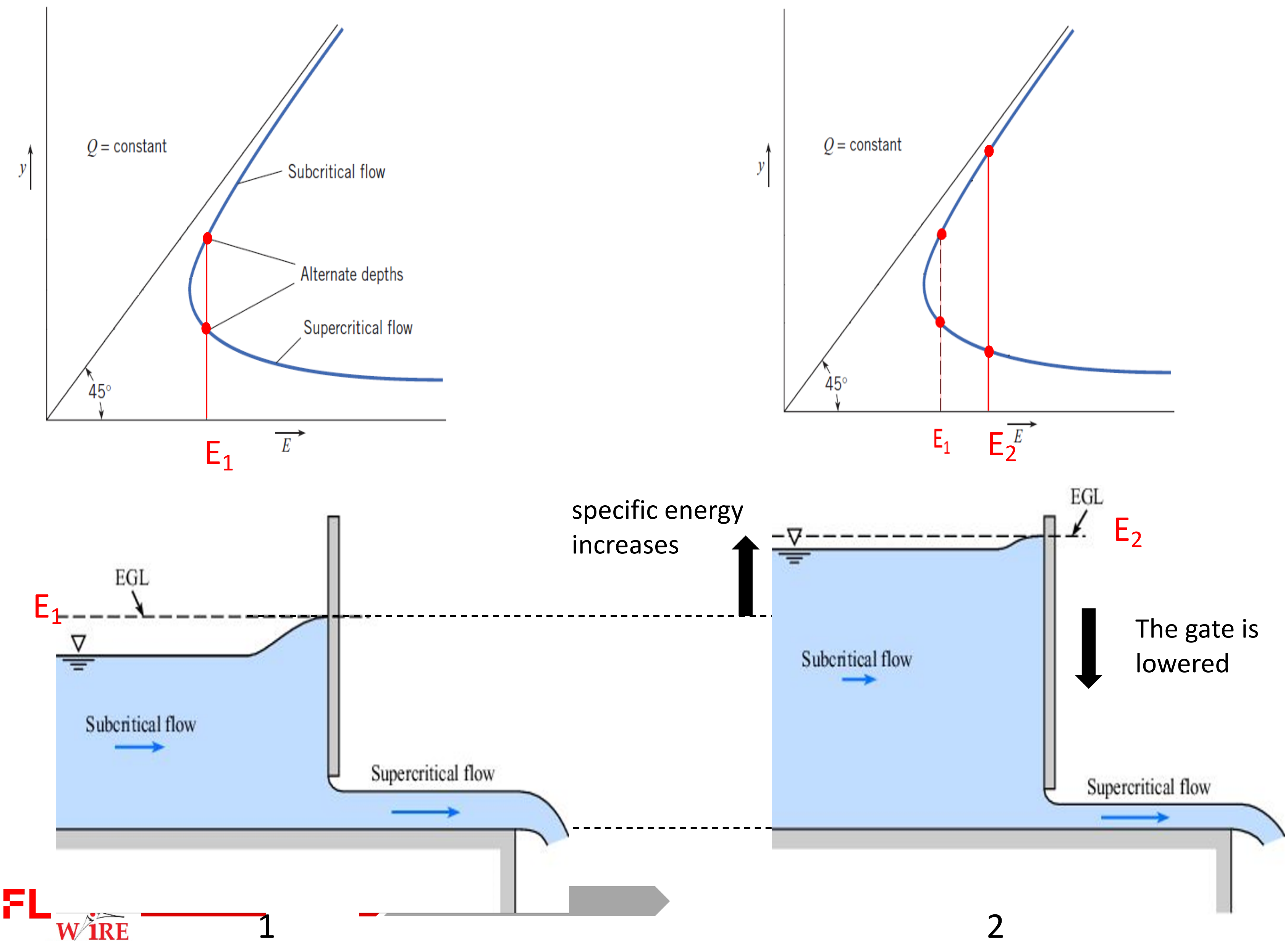


Example - Rapidly varied flow



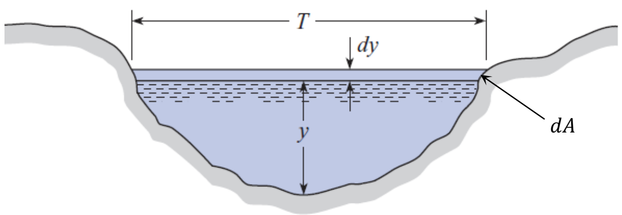
Critical flow

$$E = y + \frac{Q^2}{2gA^2}$$

- Critical flow occurs when **specific energy is minimum.**
  - Derivative of E respect to y equals zero:

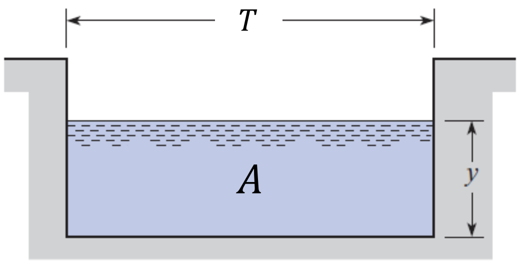
$$\frac{dE}{dy} = 1 - \frac{Q^2}{gA^3} \cdot \frac{dA}{dy} = 0 \quad ; \quad \frac{Q^2 T_c}{gA_c^3} = 1 \quad ; \quad \frac{A_c}{T_c} = \frac{Q^2}{gA_c^2}$$

  - From geometry: $dA = T\,dy$



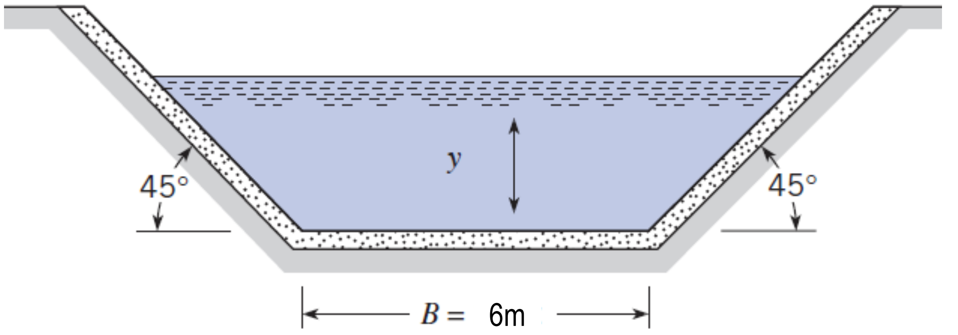
  - Definition of hydraulic depth: $D = \frac{A}{T}$
  - The critical hydraulic depth is: $D_c = \frac{Q^2}{gA_c^2} = \frac{V^2}{g}$
  - Alternatively: $1 = \frac{V}{\sqrt{g\,D_c}} = Fr$
  - For a rectangular channel:
    - $\frac{A}{T} = y$
    - $\frac{Q^2}{A^2} = \frac{q^2}{y^2}$

$$y_c = \left(\frac{q^2}{g}\right)^{1/3}$$



Critical flow - example

- Determine the critical depth for the trapezoidal channel on the image for a discharge of $14m^3/s$



1. Equation for critical flow: $$\frac{A_c}{T_c} = \frac{Q^2}{gA_c^2}$$

2. Knowing the discharge: $$\frac{A_C}{T_C}^3 = \frac{14^2}{9.81} = 20m^2$$

3. Geometry: $A = y(B + y)$ $T = B + 2y$

4. Solve iteratively. Result: $y_C = 0.7m$

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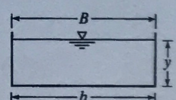
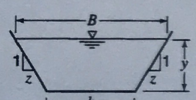
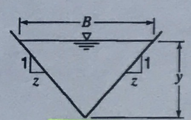
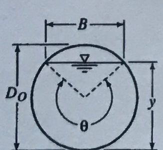
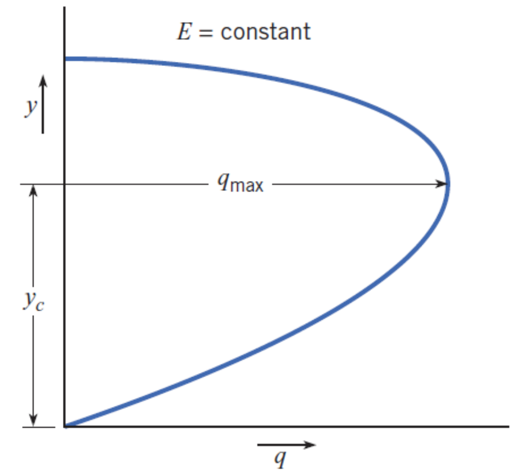
Table 15.3.2 Geometric Elements of Channel Cross-Sections

Cross-section	Area A	Wetted perimeter P	Hydraulic radius R	Top width B (T)
 Rectangle	by	$b + 2y$	$\frac{by}{b+2y}$	b
 Trapezoid	$(b + zy)y$	$b + 2y\sqrt{1+z^2}$	$\frac{(b+zy)y}{b+2y\sqrt{1+z^2}}$	$b + 2zy$
 Triangle	zy^2	$2y\sqrt{1+z^2}$	$\frac{zy}{2\sqrt{1+z^2}}$	$2zy$
 Circle	$\frac{1}{8}(\theta - \sin\theta)D_0^2$	$\frac{1}{2}\theta D_0$	$\frac{1}{4}\left(1 - \frac{\sin\theta}{\theta}\right)D_0$	$(\sin{}^1/{}_2\theta)D_0$ or $2\sqrt{y(D_0 - y)}$

Critical flow

$$E = y + \frac{Q^2}{2gA^2}$$

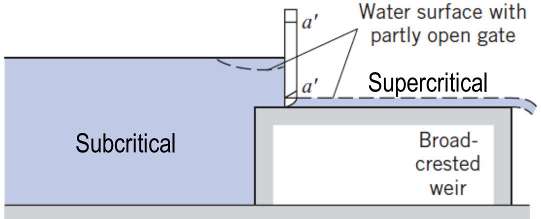
- Critical flow implies maximum channel discharge Q for a given energy E :
  - For a rectangular channel: $E = y + \frac{Q^2}{2gy^2B^2}$
  - Considering a unit width of the channel and letting $q = \frac{Q}{B}$: $E = y + \frac{q^2}{2gy^2}$

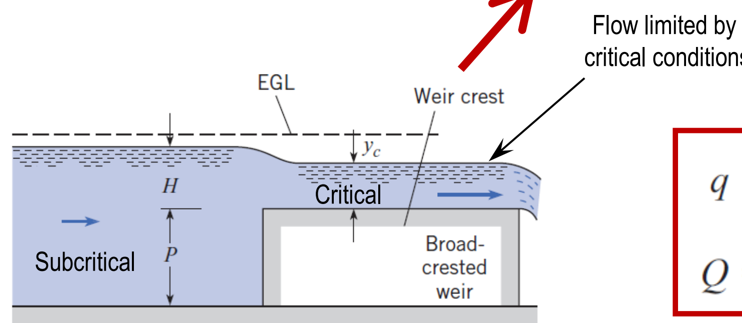


- **Critical flow occurs when:**
  - Specific energy is minimum for a given discharge
  - The discharge is maximum for a given specific energy
  - $\frac{A^3}{T} = \frac{Q^2}{g}$
  - $Fr = 1$
  - For rectangular channels: $y_c = \left(\frac{q^2}{g}\right)^{1/3}$

Critical flow

- Common occurrence of critical flow:
  - Broad-crested weir:



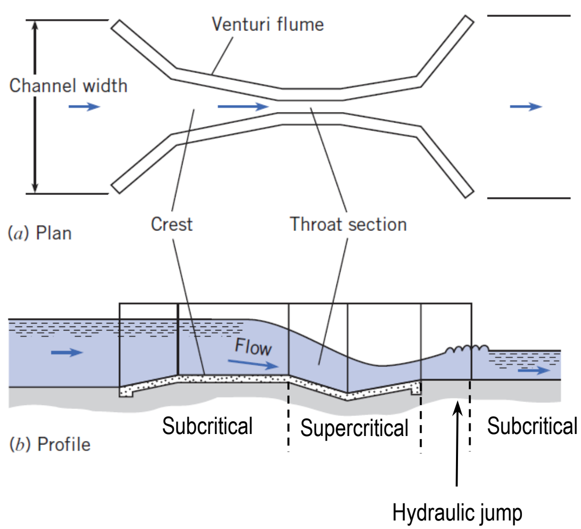


Used to measure discharge in channels

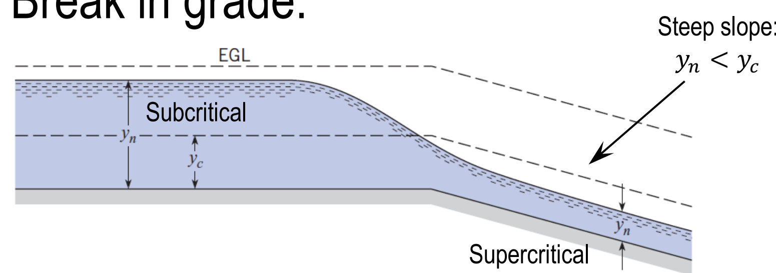
$$q = \sqrt{g y_c^3}$$

$$Q = L\sqrt{g y_c^3}$$

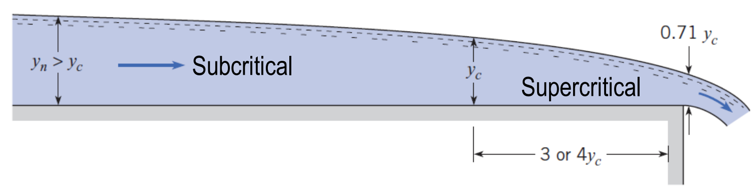
  - Venturi flume:



  - Break in grade:



  - Free overfall:



Hydraulic jump – A case when h_L cannot be neglected **(do as exercise)**

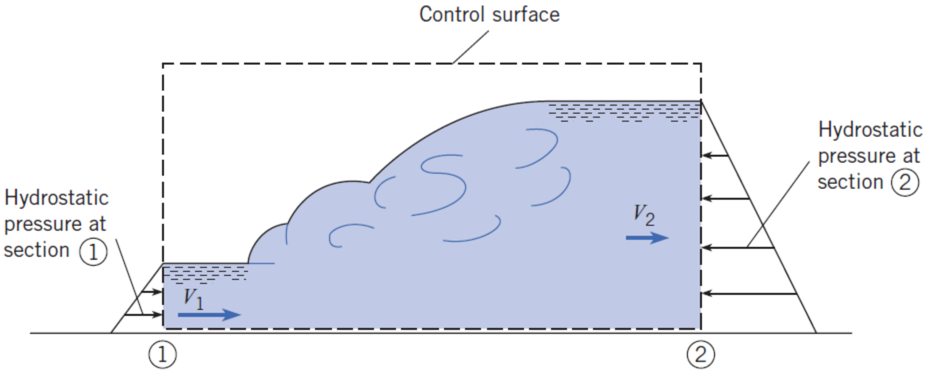
- A hydraulic jump occurs when a supercritical flow is forced to become subcritical. It results in:
  - Abrupt increase in depth
  - **Considerable energy loss**

- The energy loss in the jump is **not known a priori**
  - The **energy equation cannot be used**
  - The **momentum equation** is applied instead:

$$\sum F_x = \dot{m}_2 V_2 - \dot{m}_1 V_1$$

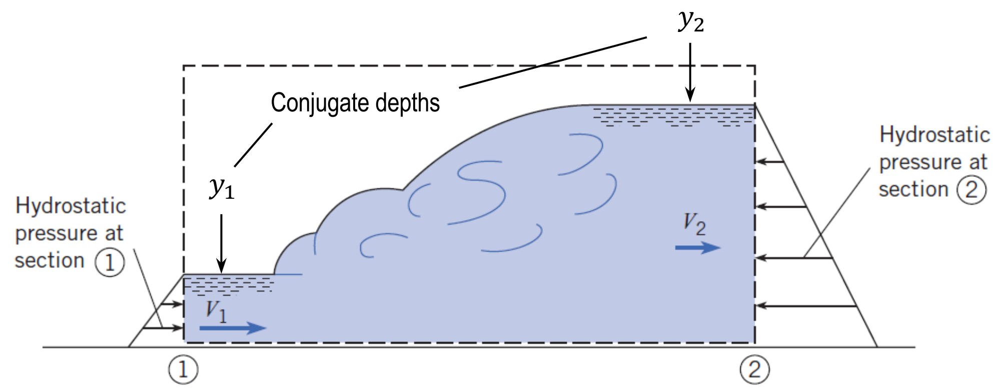
$$\overline{p}_1 A_1 - \overline{p}_2 A_2 = \rho V_2 A_2 V_2 - \rho V_1 A_1 V_1$$

$$\boxed{\overline{p}_1 A_1 + \rho Q V_1 = \overline{p}_2 A_2 + \rho Q V_2}$$



Hydraulic jump (Exercise)

$$\bar{p}_1 A_1 + \rho Q V_1 = \bar{p}_2 A_2 + \rho Q V_2$$



- For a rectangular channel of unit width $\bar{p} = \gamma y_1/2 \quad ; \quad Q = q \quad ; \quad A = y$

  the momentum equation becomes: $$\gamma \frac{y_1^2}{2} + \rho q V_1 = \gamma \frac{y_2^2}{2} + \rho q V_2$$

- Since $q = Vy$: $$\frac{\gamma}{2}(y_1^2 - y_2^2) = \frac{\gamma}{g}(V_2^2 y_2 - V_1^2 y_1)$$

- Rearranging: $$\frac{2V_1^2}{g y_1} = \left(\frac{y_2}{y_1}\right)^2 + \frac{y_2}{y_1}$$

- Introducing the Froude number: $$\left(\frac{y_2}{y_1}\right)^2 + \frac{y_2}{y_1} - 2\mathrm{Fr}_1^2 = 0$$

- Solving the quadratic equation: $$\frac{y_2}{y_1} = \frac{1}{2}\left(\sqrt{1 + 8\mathrm{Fr}_1^2} - 1\right)$$

Hydraulic jump (Exercise)

- The head loss is calculated comparing the specific energy before and after the occurence of the jump. For a rectangular channel:

$$h_L = \frac{(y_2 - y_1)^3}{4y_1y_2}$$

EXAMPLE 15.11 HEAD LOSS IN HYDRAULIC JUMP

Water flows in a rectangular channel at a depth of 30 cm with a velocity of 16 m/s, as shown in the sketch that follows. If a downstream sill (not shown) forces a hydraulic jump, what will be the depth and velocity downstream of the jump? What head loss is produced by the jump?

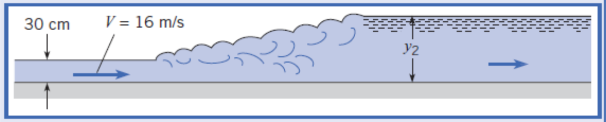
Problem Definition

Situation: Channel is rectangular; upstream depth and velocity known.

Find:

1. Downstream depth and velocity.
2. Head loss produced by the jump.

Sketch:



Plan

1. In order to calculate h_L using Eq. (15.44), must calculate y_2 from the depth ratio equation (Eq. 15.43). This requires Fr_1.
2. Check validity of head loss by comparing to $E_1 - E_2$.

Solution

1. Calculate Fr_1, y_2, V_2, and h_L from Eqs. (Eq. 15.43) and (15.44).

$$\mathrm{Fr}_1 = \frac{V}{\sqrt{gy_1}} = \frac{16}{\sqrt{9.81(0.30)}} = 9.33$$

$$y_2 = \frac{0.30}{2}\left[\sqrt{1 + 8(9.33)^2} - 1\right] = \boxed{3.81\text{ m}}$$

$$V_2 = \frac{q}{y_2} = \frac{(16\text{ m/s})(0.30\text{ m})}{3.81\text{m}} = \boxed{1.26\text{ m/s}}$$

$$h_L = \frac{(3.81 - 0.30)^3}{4(0.30)(3.81)} = \boxed{9.46\text{ m}}$$

2. Compare the head loss to $E_1 - E_2$.

$$h_L = \left(0.30 + \frac{16^2}{2 \times 9.81}\right) - \left(3.81 + \frac{1.26^2}{2 \times 9.81}\right) = 9.46\text{ m}$$

The value is the same, so

validity of h_L equation is verified.