Introduction to fluid mechanics

Flow in conduits

Fernando Porté-Agel

Wind engineering and renewable energy laboratory
WiRE

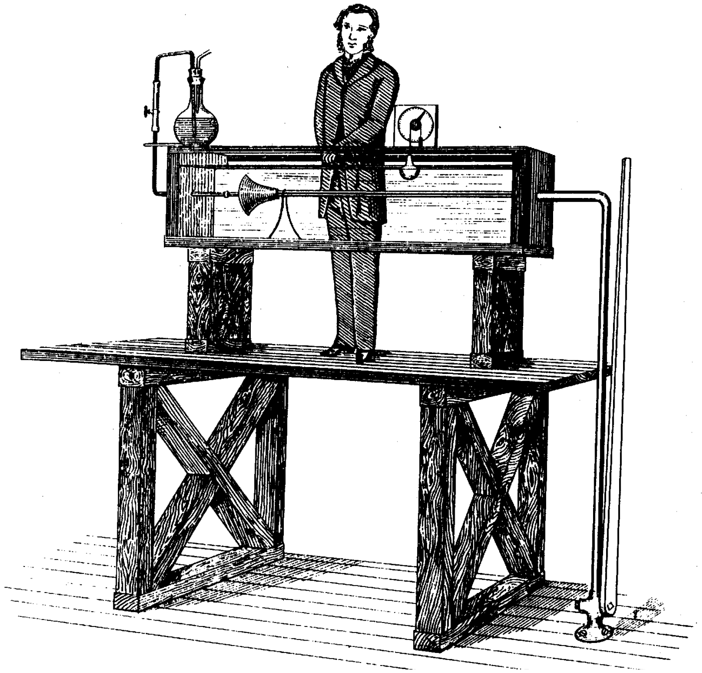
Chapter 10: Flow in Conduits

- **Conduit:** Any pipe, tube or duct completely filled with a flowing fluid.
- Analyzing **flow in conduits:** Great importance in various engineering applications.

Main Goal of this chapter: Predict head loss in pipelines.

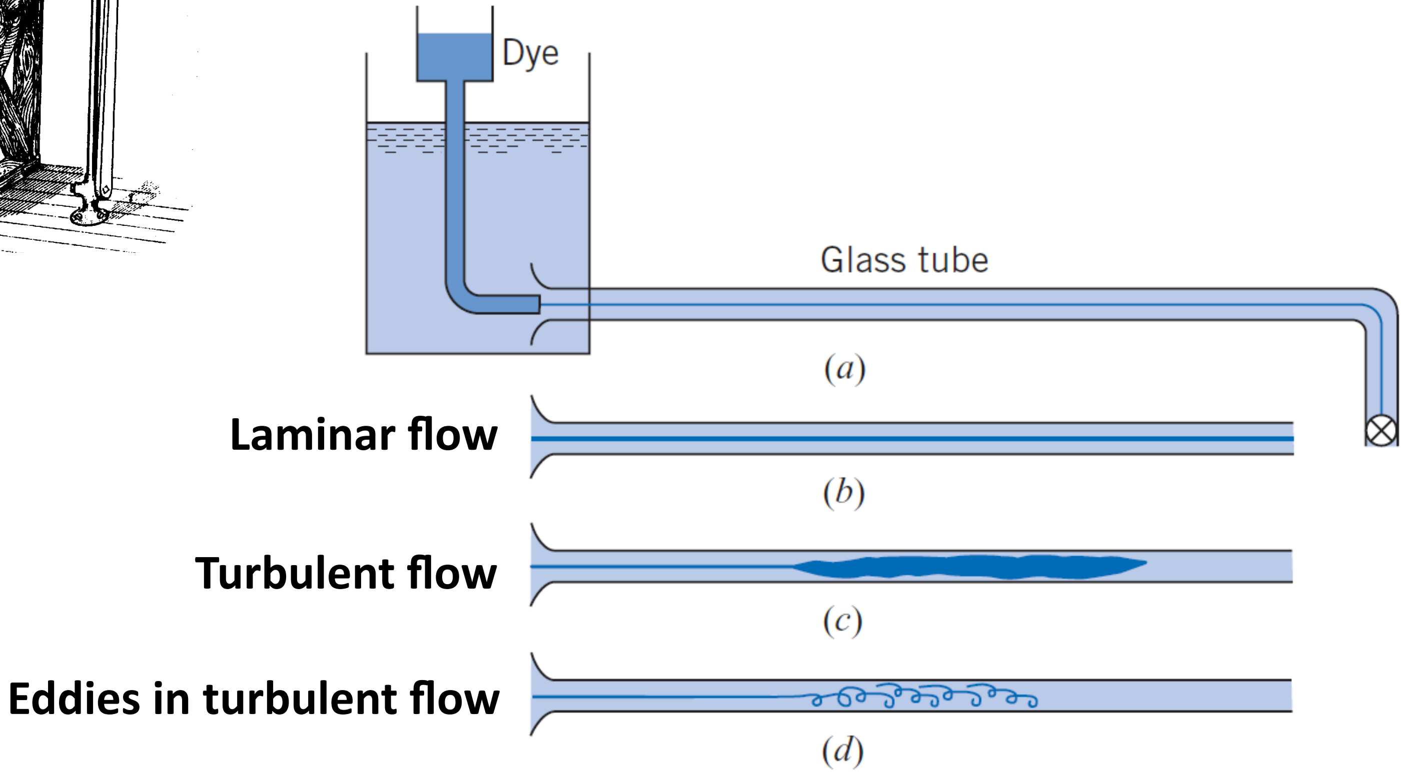
Laminar and Turbulent Flow

Reynolds' experiment

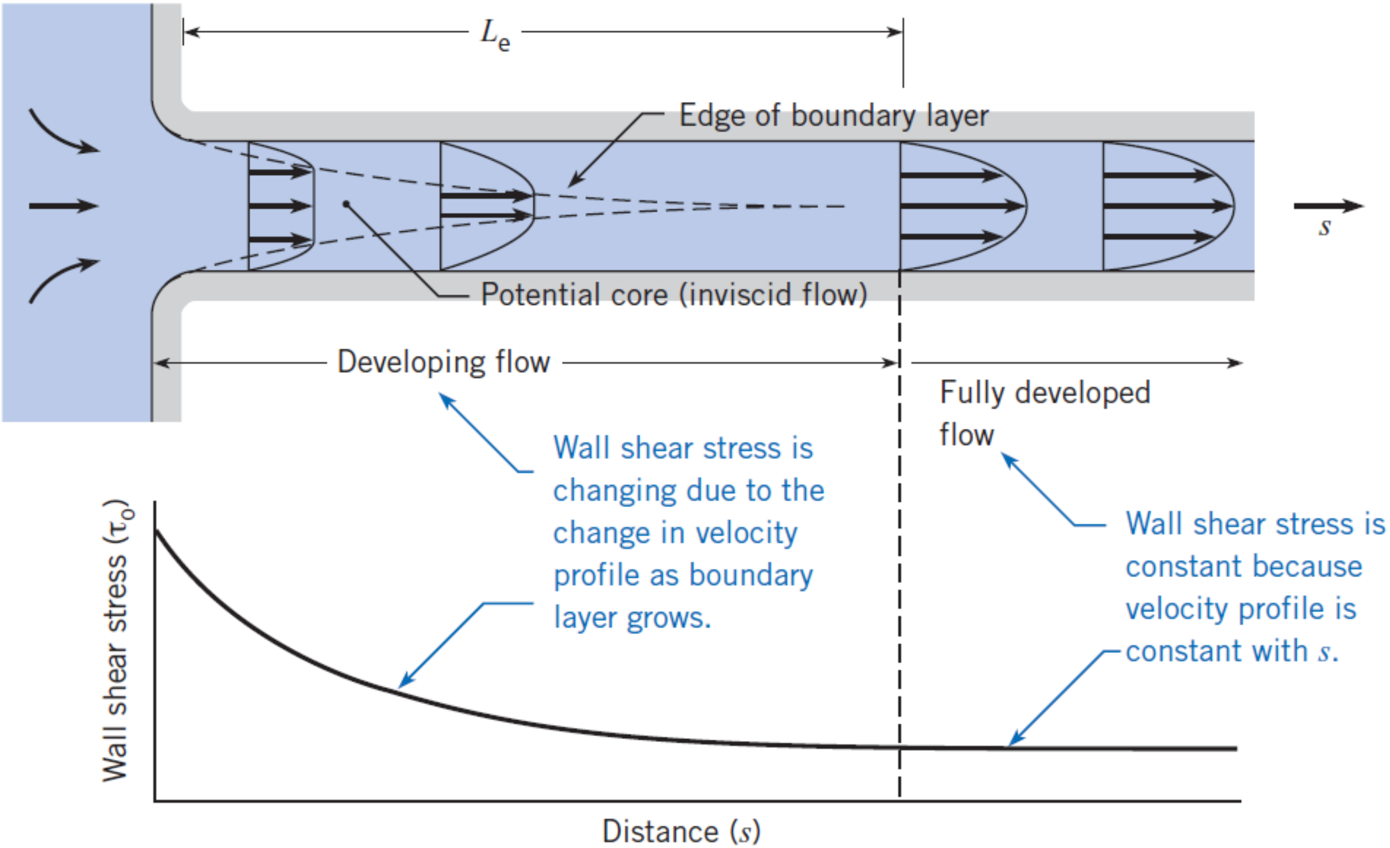


Reynolds Number: $\mathrm{Re} = \dfrac{VD}{\nu} = \dfrac{\rho VD}{\mu}$

$\mathrm{Re} \le 2000$	laminar flow
$2000 \le \mathrm{Re} \le 3000$	unpredictable
$\mathrm{Re} \ge 3000$	turbulent flow



Developing & Fully Developed Flow



- **Entrance Length (L_e):** The distance at which the shear stress reaches to within 2% of the fully developed value.
- For flow entering a **circular** pipe

$$\begin{cases} \dfrac{L_e}{D} = 0.05\text{Re} & (\text{laminar flow: } \text{Re} \leq 2000) \\ \dfrac{L_e}{D} = 50 & (\text{turbulent flow: } \text{Re} \geq 3000) \end{cases}$$

Example: Classifying Flow in Conduits

Problem: - Classify the flow as laminar or turbulent.

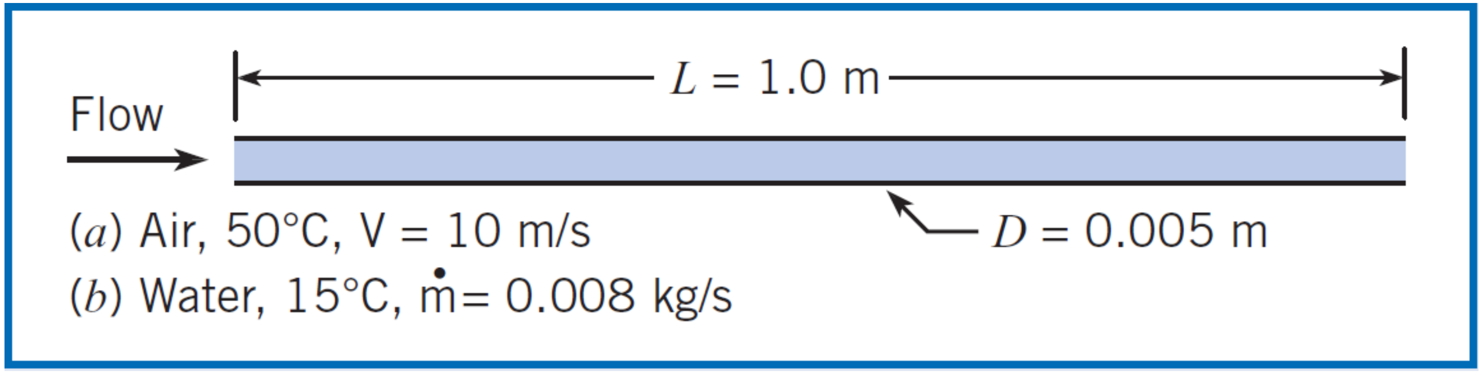
- Calculate the entrance length.

Flowing Fluids :

1. Air (50 $^{\circ}C$) with a speed of 12 m/s
2. Water (15 $^{\circ}C$) with a mass flow rate of 8 g/s

Properties:

1. Air (50°C), Table A.3, $\nu = 1.79 \times 10^{-5}\ m^2/s$.
2. Water (15°C), Table A.5, $\mu = 1.14 \times 10^{-3}\ N \cdot s/m^2$.



Example: Classifying Flow in Conduits

Solution:

a. Air

$$\mathrm{Re} = \frac{VD}{\nu} = \frac{(12\ \mathrm{m/s})(0.005\mathrm{m})}{1.79 \times 10^{-5}\ \mathrm{m^2/s}} = 3350$$

Since $\mathrm{Re} > 3000$, the **flow is turbulent.**

$$L_e = 50D = 50(0.005\ \mathrm{m}) = \boxed{0.25\ \mathrm{m}}$$

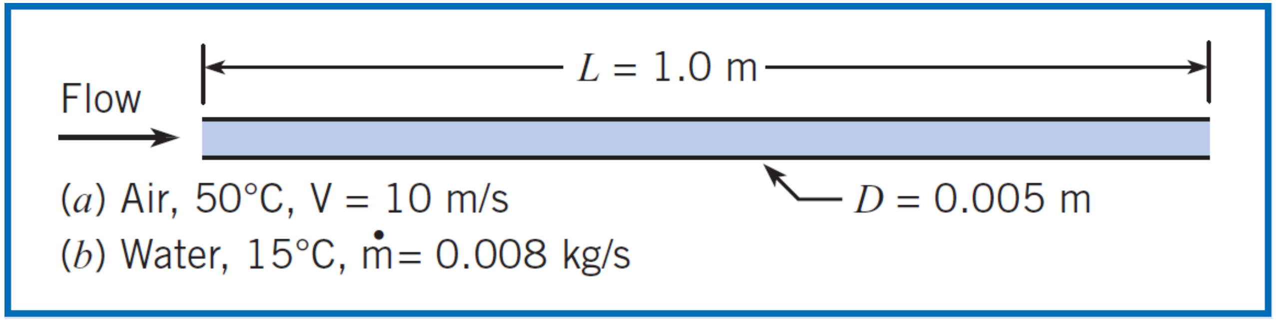
b. Water

$$\mathrm{Re} = \frac{4\dot{m}}{\pi D \mu} = \frac{4(0.008\ \mathrm{kg/s})}{\pi(0.005\mathrm{m})(1.14 \times 10^{-3}\ \mathrm{N \cdot s/m^2})}$$

$$= 1787$$

Since $\mathrm{Re} < 2000$, the **flow is laminar.**

$$L_e = 0.05\mathrm{Re}D = 0.05(1787)(0.005\mathrm{m}) = \boxed{0.447\ \mathrm{m}}$$



Combined (Total) Head Loss

From energy chapter:

$$\frac{P_1}{\gamma} + \alpha_1 \frac{V_1^2}{2g} + z_1 + h_p = \frac{P_1}{\gamma} + \alpha_1 \frac{V_1^2}{2g} + z_1 + h_t + h_L$$

$$(\text{Total head loss}) = (\text{Pipe head loss}) + (\text{Component head loss})$$

- **Pipe head loss:**
  - Related to **fully developed** flow in conduits.
  - Caused by **shear stresses** that act on the flowing fluid.
- **Component head loss:**
  - Related to flow through **valves**, **bends** and etc.

✓ **Pipe head loss** is predicted with the **Darcy-Weisbach** equation.

Darcy-Weisbach Equation

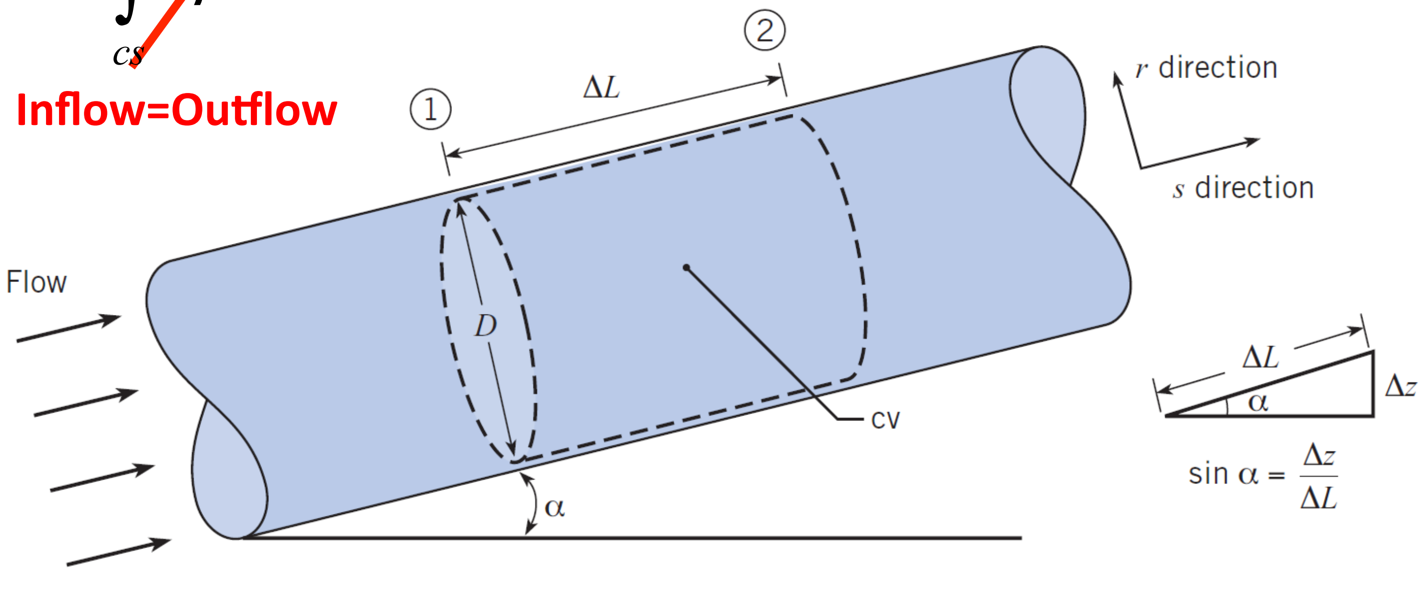
- Assume **fully-developed** and **steady** flow in a round tube.
- Consider the below cylindrical **control volume**.
- Momentum Equation in the **streamwise** direction:

$$\sum \vec{F} = \frac{d}{dt}\int_{cv} \vec{v}\rho d\forall + \int_{cs} \vec{v}\rho \vec{V}.d\vec{A}$$

= 0 **= 0**

Steady **Inflow=Outflow**

$$\sum \vec{F} = 0$$



Darcy-Weisbach Equation

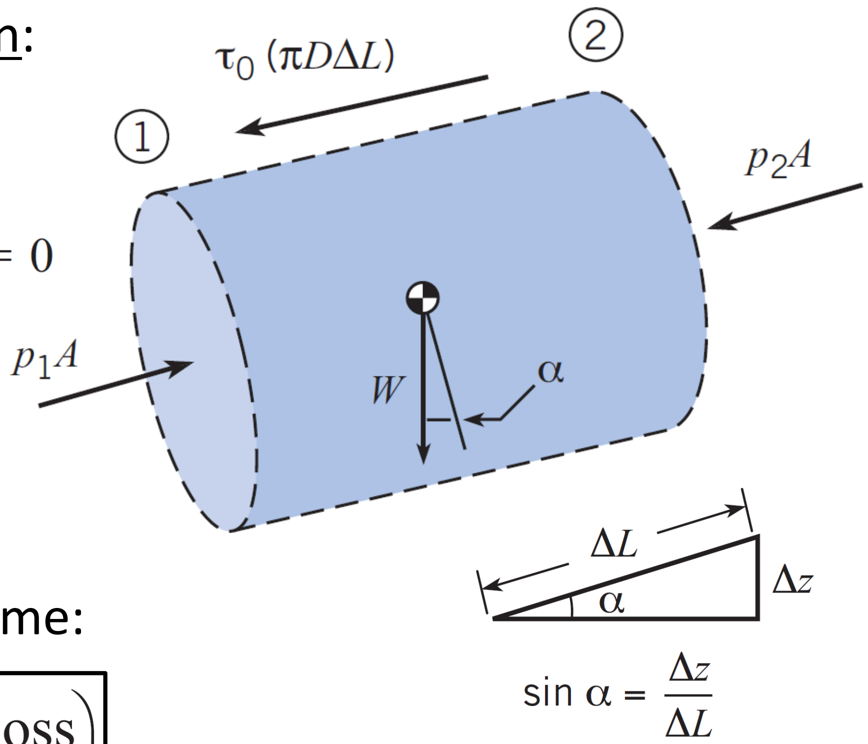
- Summing of **forces** in the streamwise direction:

$$F_{\text{pressure}} + F_{\text{shear}} + F_{\text{weight}} = 0$$

$$(p_1 - p_2)\left(\frac{\pi D^2}{4}\right) - \tau_0(\pi D \Delta L) - \gamma\left[\left(\frac{\pi D^2}{4}\right)\Delta L\right]\sin\alpha = 0$$

$\sin\alpha = (\Delta z / \Delta L)$

$$(p_1 + \gamma z_1) - (p_2 + \gamma z_2) = \frac{4\Delta L \tau_0}{D} \quad \textbf{(I)}$$



- Apply the energy equation to the control volume:

$$\frac{P_1}{\gamma} + z_1 = \frac{P_2}{\gamma} + z_2 + h_f$$

$$h_f = \left(\begin{array}{c}\text{head loss}\\ \text{in a pipe}\end{array}\right)$$

$$\Longrightarrow (p_1 + \gamma z_1) - (p_2 + \gamma z_2) = \gamma h_f \quad \textbf{(II)}$$

From **(I)** and **(II)**

Darcy-Weisbach Equation

$$h_f = \begin{pmatrix} \text{head loss} \\ \text{in a pipe} \end{pmatrix} = \frac{4L\tau_0}{D\gamma}$$

Rearranging the right side of the equation:

$$h_f = \left(\frac{L}{D}\right)\left\{\frac{4\tau_0}{\rho V^2/2}\right\}\left\{\frac{\rho V^2/2}{\gamma}\right\} = \left\{\frac{4\tau_0}{\rho V^2/2}\right\}\left(\frac{L}{D}\right)\left\{\frac{V^2}{2g}\right\}$$

friction factor: $f \equiv \frac{(4 \cdot \tau_0)}{(\rho V^2/2)} \approx \frac{\text{shear stress acting at the wall}}{\text{kinetic pressure}}$

Darcy-Weisbach Equation: $h_f = f \frac{L}{D} \frac{V^2}{2g}$

- Flow should be **fully developed** and **steady**.
- This equation can be used for both **laminar** and **turbulent** flows.
- However, **friction factor** should be determined for each type of flows.

Laminar flow in Pipes

- Laminar flows occur when $\mathbf{Re} \leq 2000$
- In general, laminar flows are important where **viscous forces** are <u>dominant</u> (e.g., microchannels, lubricant flows).
- Laminar flows in a round tube are called ***Poiseuille*** or ***Hagen-Poiseuille*** flows due to their pioneering studies in low speed flows.

J. Poiseuille(1797-1869)

G. Hagen(1797-1884)

- Using **Momentum** and **Energy** Eqs., it can be shown that for laminar flows:

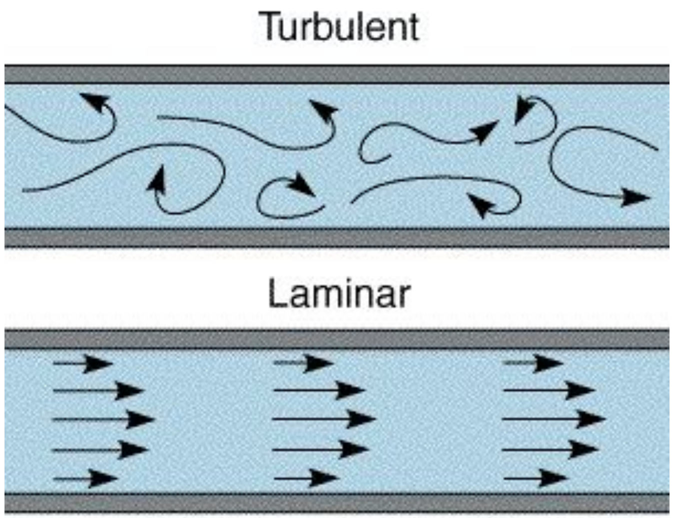
$$f = \frac{64}{\mathbf{Re}_D}$$

Key assumptions to derive the above equation:

- **Laminar** flow - **Fully developed** flow - **Steady** flow - **Newtonian** fluid

Turbulent flow in Pipes

- A flow regime in which fluid motion is **chaotic, eddying** and **unsteady**.
- Turbulent flows occur when $\mathbf{Re} \geq 3000$
- Most flows in conduits are **turbulent**.



- In general, the time-average velocity profile in **turbulent flows** can be described using two approaches:

1- Empirical **power law** formula

2- Logarithmic velocity profile *similar to the boundary layer over a flat plate*

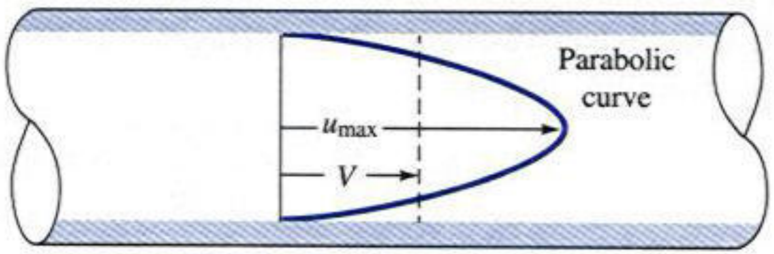
Turbulent Velocity Profile: Power Law Profile

$$\frac{u(r)}{u_{max}} = \left(\frac{r_0 - r}{r_0}\right)^m$$

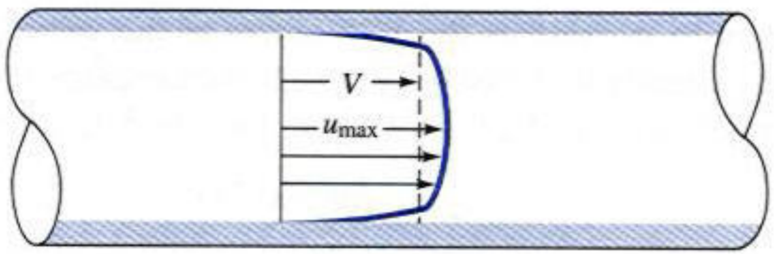
Re	4×10^3	2.3×10^4	1.1×10^5	1.1×10^6	3.2×10^6
m	$\frac{1}{6.0}$	$\frac{1}{6.6}$	$\frac{1}{7.0}$	$\frac{1}{8.8}$	$\frac{1}{10.0}$
u_{max}/V	1.26	1.24	1.22	1.18	1.16

m** decreases with the increase of **Re

Laminar flow



Turbulent flow



Turbulent Velocity Profile: Logarithmic Profile

$$\frac{u(r)}{u_*} = 2.44 \ \ln \frac{u_*(r_0 - r)}{\nu} + 5.56$$

$u_* = \sqrt{\tau_0 / \rho}$: shear velocity

τ_0 : wall shear stress

- After some algebra and using logarithmic profile for the velocity distribution, **friction factor, *f*** for turbulent flows can be described as:

$$\frac{1}{\sqrt{f}} = 2.0\log_{10}(\mathrm{Re}\sqrt{f}\,) - 0.8$$

L. Prandtl (1875-1953)

First derived by **L. Prandtl** in 1935

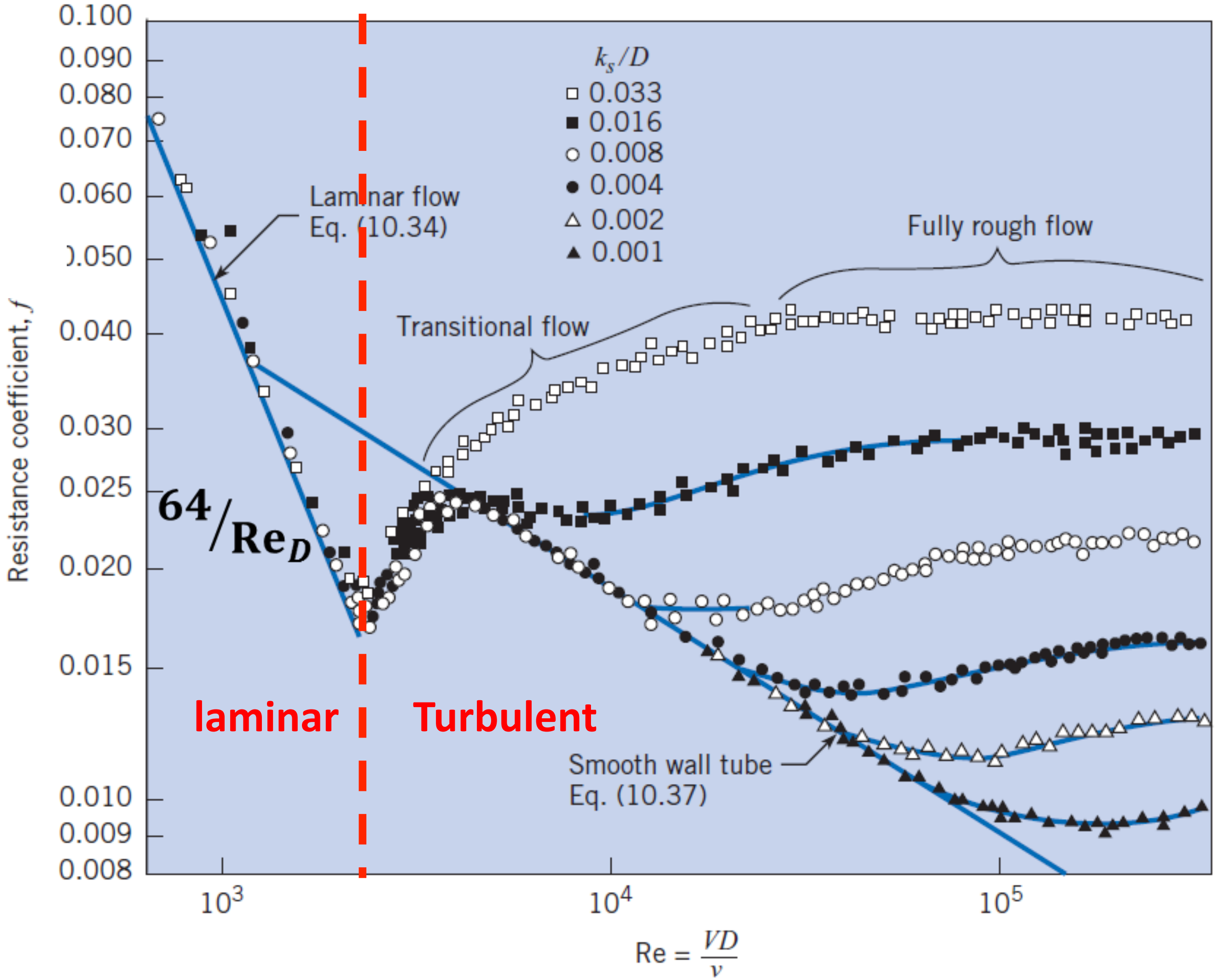
⚠ This Eq. is only **valid** for **smooth wall tubes**.

Friction Factor in Turbulent Flows

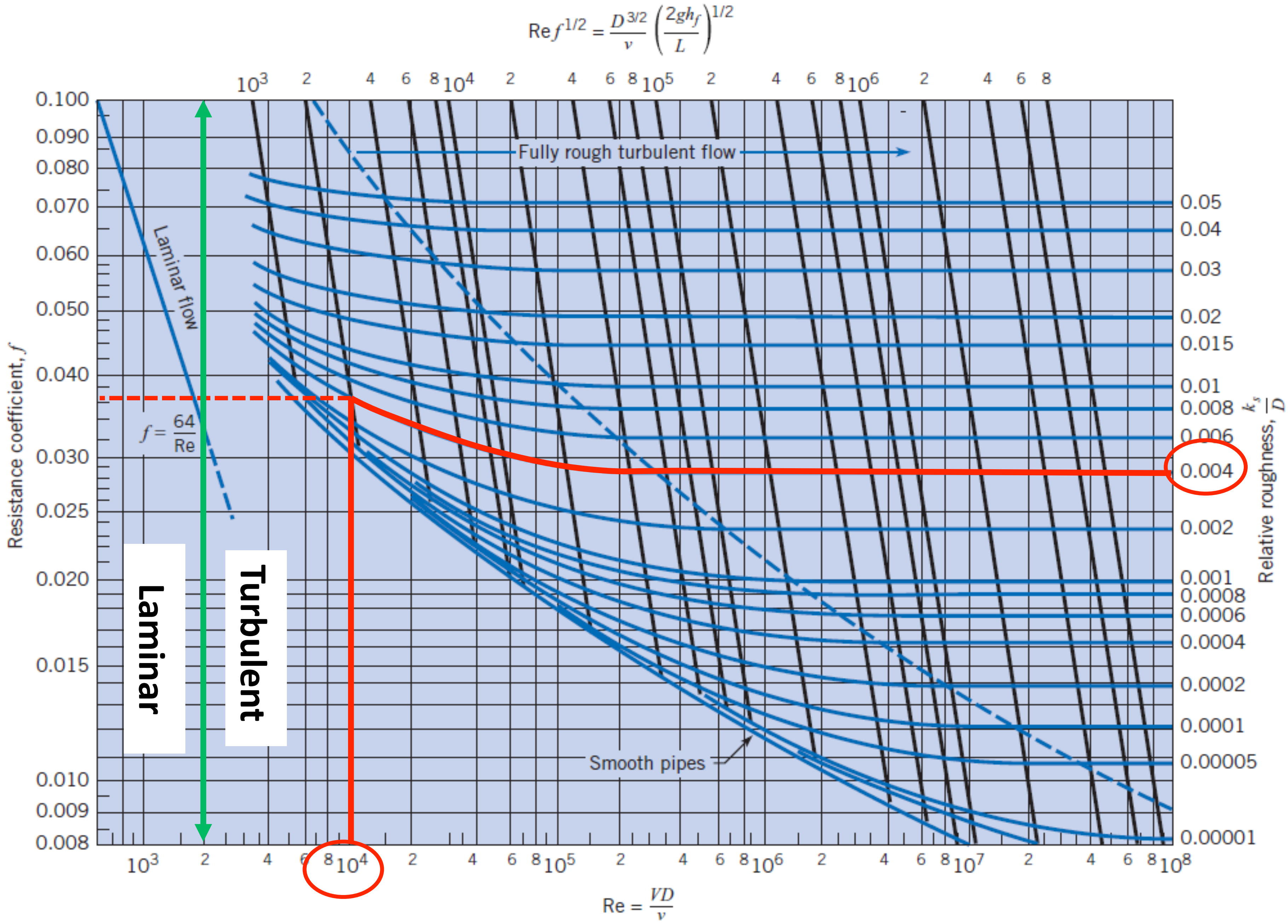
- Prandtl's student, namely **Nikuradse**, investigated the **effect of surface roughness** on the **friction factor, *f***.

k_s : sand roughness height

k_s / D : relative roughness



Moody Diagram



Friction Factor in Turbulent Flows

Type of Flow	Parameter Ranges		Influence of Parameters on f
Laminar Flow	$\text{Re} < 2000$	NA	f depends on Reynolds number f is independent of wall roughness (k_s/D)
Turbulent Flow, Smooth Tube	$\text{Re} > 3000$	$\left(\frac{k_s}{D}\right)\text{Re} < 10$	f depends on Reynolds number f is independent of wall roughness (k_s/D)
Transitional Turbulent Flow	$\text{Re} > 3000$	$10 < \left(\frac{k_s}{D}\right)\text{Re} < 1000$	f depends on Reynolds number f depends on wall roughness (k_s/D)
Fully Rough Turbulent Flow	$\text{Re} > 3000$	$\left(\frac{k_s}{D}\right)\text{Re} > 1000$	f is independent of Reynolds number f depends on wall roughness (k_s/D)

Equivalent sand roughness k_s for different pipe materials

Boundary Material	k_s, Millimeters	k_s, Inches
Glass, plastic	Smooth	Smooth
Copper or brass tubing	0.0015	6×10^{-5}
Wrought iron, steel	0.046	0.002
Asphalted cast iron	0.12	0.005
Galvanized iron	0.15	0.006
Cast iron	0.26	0.010
Concrete	0.3 to 3.0	0.012–0.12
Riveted steel	0.9–9	0.035–0.35
Rubber pipe (straight)	0.025	0.001

Solving Turbulent Pipe Flow Problems

1. Moody diagram

Case 1

	Knowns	Unknowns	
	• Pipe length • Pipe diameter • Flow rate	• **Head loss**	**Straightforward Approach**

2. Alternative solution

Swamee & Jain formula

$$f = \frac{0.25}{\left[\log_{10}\left(\frac{k_s}{3.7D} + \frac{5.74}{\mathrm{Re}^{0.9}}\right)\right]^2}$$

(instead of Moody diagram)

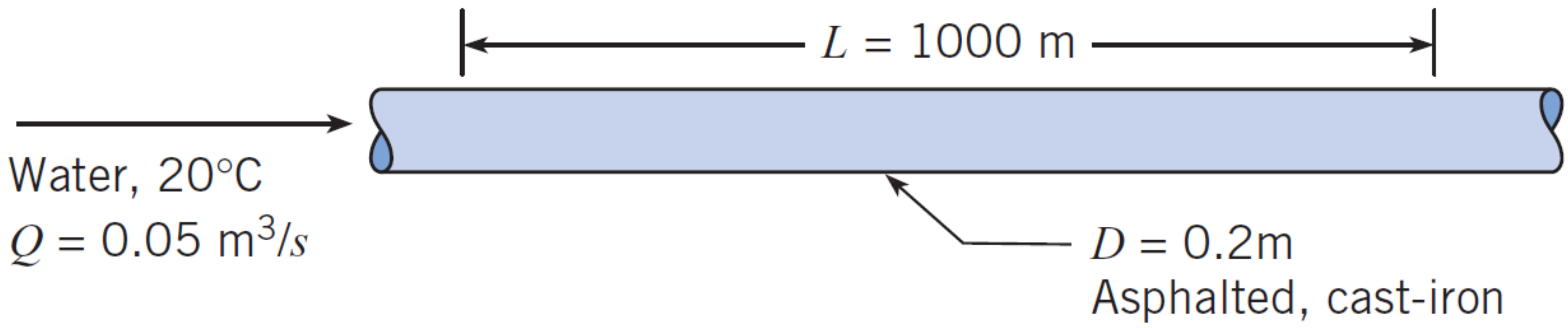
Example: Head loss in a pipe (case 1)

Problem

Water ($T = 20°C$) flows at a rate of 0.05 m^3/s in a 20 cm (diameter) asphalted cast-iron pipe. What is the head loss per kilometer of pipe?

Assumptions

- Fully developed flow
- $v = 1\times10^{-6}\ m^2/s$



Example: Head loss in a pipe (case 1)

Solution

1. Mean velocity

$$V = \frac{Q}{A} = \frac{0.05\ \text{m}^3/\text{s}}{(\pi/4)(0.2\ \text{m})^2} = 1.59\ \text{m/s}$$

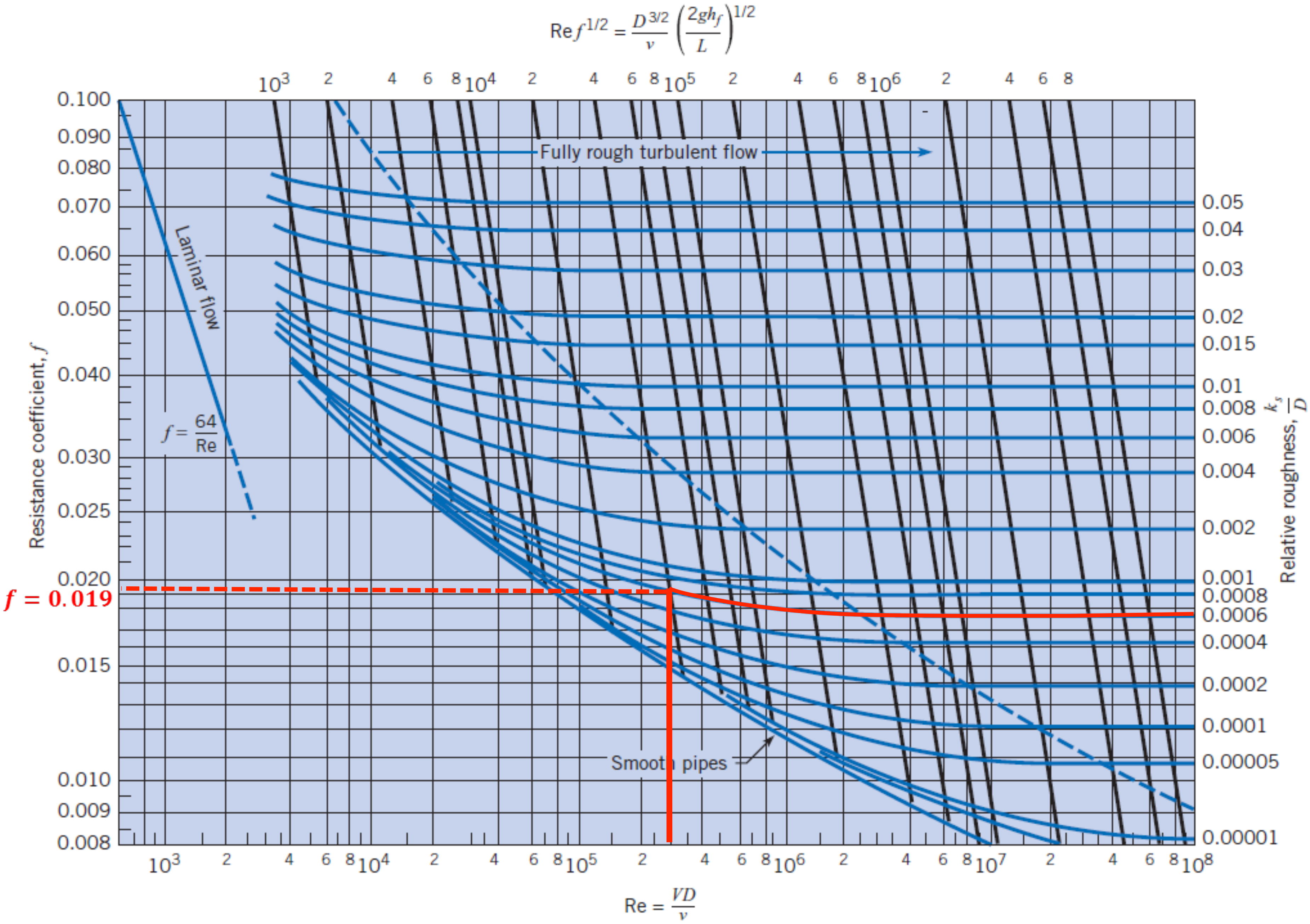
2. Reynolds number

$$\text{Re} = \frac{VD}{\nu} = \frac{(1.59\ \text{m/s})(0.20\ \text{m})}{10^{-6}\ \text{m}^2/\text{s}} = 3.18 \times 10^5$$

3. Resistance coefficient

- Equivalent sand roughness (Table 10.4): $k_s = 0.12$ mm
- Relative roughness:
  $k_s/D = (0.00012\ \text{m})/(0.2\ \text{m}) = 0.0006$

Moody Diagram



Example: Head loss in a pipe (case 1)

Solution

4. Darcy-Weisbach equation

$$h_f = f\left(\frac{L}{D}\right)\left(\frac{V^2}{2g}\right) = 0.019\left(\frac{1000\ \text{m}}{0.20\ \text{m}}\right)\left(\frac{1.59^2\ \text{m}^2/\text{s}^2}{2(9.81\ \text{m}/\text{s}^2)}\right)$$

$$= \boxed{12.2\ \text{m}}$$

Solving Turbulent Pipe Flow Problems

1. Moody diagram

Case 2

	Knowns	Unknowns	
	• (Head loss) • Pipe length • Pipe diameter	• **Flow rate** • **(Head loss)**	**Iterative Approach** *(needed if head loss is not known)*

2. Alternative solution *(only if head loss is known)*

Swamee & Jain formula

$$Q = -2.22 D^{5/2} \sqrt{g h_f / L} \log\left(\frac{k_s}{3.7D} + \frac{1.78\nu}{D^{3/2}\sqrt{g h_f / L}} \right)$$

Example: Flow Rate in a pipe (case 2)

Problem

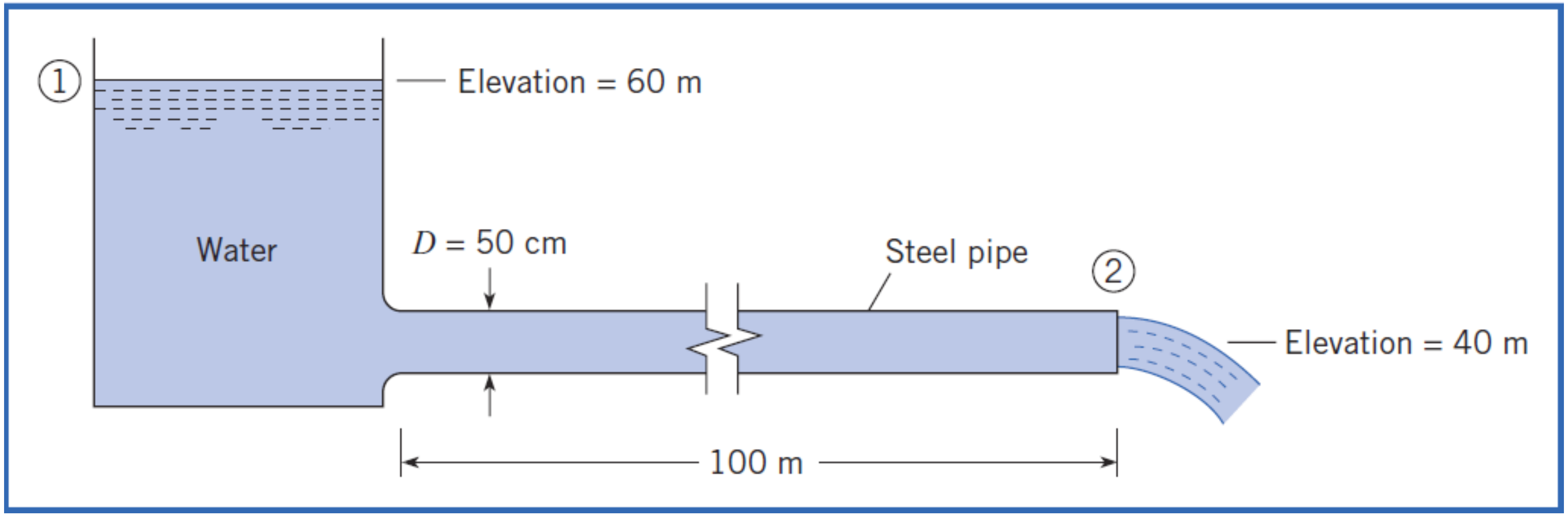
Water ($T = 20°C$) flows from a tank through a 50 cm diameter steel pipe.

Determine the discharge of water.

Assumptions

- Fully developed flow
- Include only the head loss in the pipe

1. Water (20°C), Table A.5: $\nu = 1 \times 10^{-6}\ m^2/s$.
2. Steel pipe, Table 10.4, equivalent sand roughness: $k_s = 0.046$ mm. Relative roughness (k_s/D) is 9.2×10^{-5}.



Example: Flow Rate in a pipe (case 2)

1. Energy equation (reservoir surface to outlet)

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2 + h_L$$

$$0 + 0 + 60 = 0 + \frac{V_2^2}{2g} + 40 + f\frac{L}{D}\frac{V_2^2}{2g}$$

or

$$V = \left(\frac{2g \times 20}{1 + 200f}\right)^{1/2} \qquad (1)$$

2. First trial (iteration 1)
   - Guess a value of $f = 0.020$.
   - Use eq. (1) to calculate $V = 8.86$ m/s.
   - Use $V = 8.86$ m/s to calculate $\mathrm{Re} = 4.43 \times 10^6$.
   - Use $\mathrm{Re} = 4.43 \times 10^6$ and $k_s/D = 9.2 \times 10^{-5}$ on the Moody diagram to find that $f = 0.012$.
   - Use eq. (1) with $f = 0.012$ to calculate $V = 10.7$ m/s.

Example: Flow Rate in a pipe (case 2)

3. Second trial (iteration 2)
   - Use $V = 10.7$ m/s to calculate $\text{Re} = 5.35 \times 10^6$.
   - Use $\text{Re} = 5.35 \times 10^6$ and $k_s/D = 9.2 \times 10^{-5}$ on the Moody diagram to find that $f = 0.012$.

4. Convergence. The value of $f = 0.012$ is unchanged between the first and second trials. Therefore, there is no need for more iterations.
5. Flow rate

$$Q = VA = (10.7\ \text{m/s}) \times (\pi/4) \times (0.50)^2\ \text{m}^2 = \boxed{2.10\ \text{m}^3/\text{s}}$$

Solving Turbulent Pipe Flow Problems

1. Moody diagram

Case 3

Knowns	Unknowns
• (Head loss)	• **Pipe diameter**
• Pipe length	• **(Head loss)**
• Flow rate	

Iterative Approach
(needed always if head loss unknown)

2. Alternative solution *(only if head loss is known)*

Streeter & Wylie formula

$$D = 0.66\left[k_s^{1.25}\left(\frac{LQ^2}{gh_f}\right)^{4.75} + \nu Q^{9.4}\left(\frac{L}{gh_f}\right)^{5.2}\right]^{0.04}$$

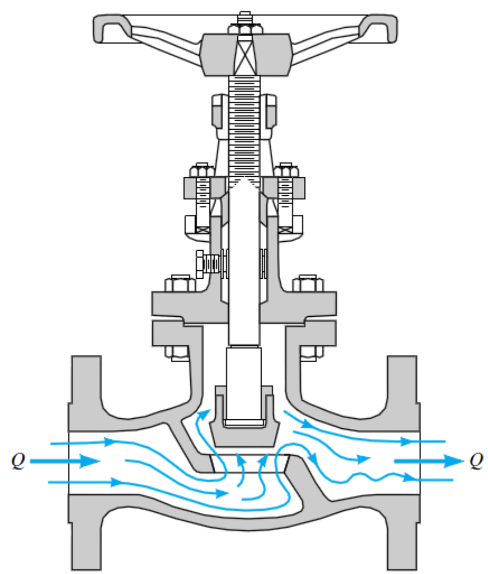
(Total head loss) = (Pipe head loss) + (Component head loss)

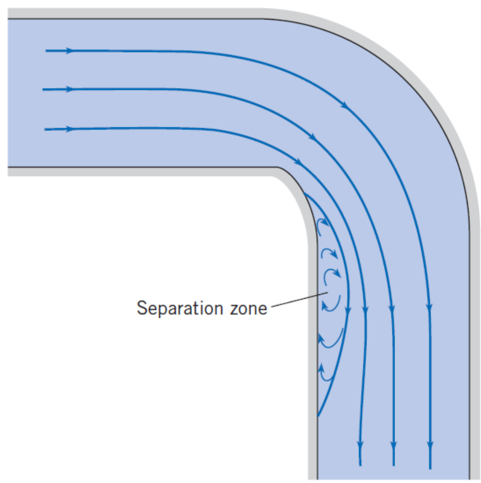
Pipe entrance

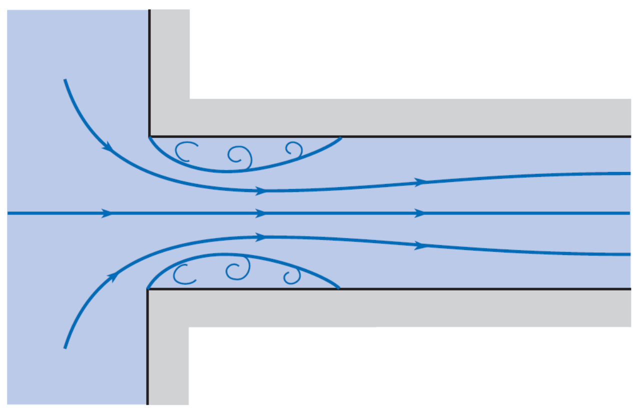
Valves

Bends

Expansion, Contraction





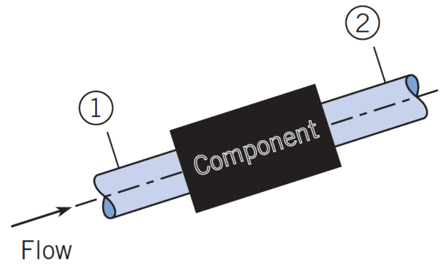


The **head loss** in components is mainly due to the **separation** created at these regions.

Component Head Loss

- **Minor Loss Coefficient, *K***

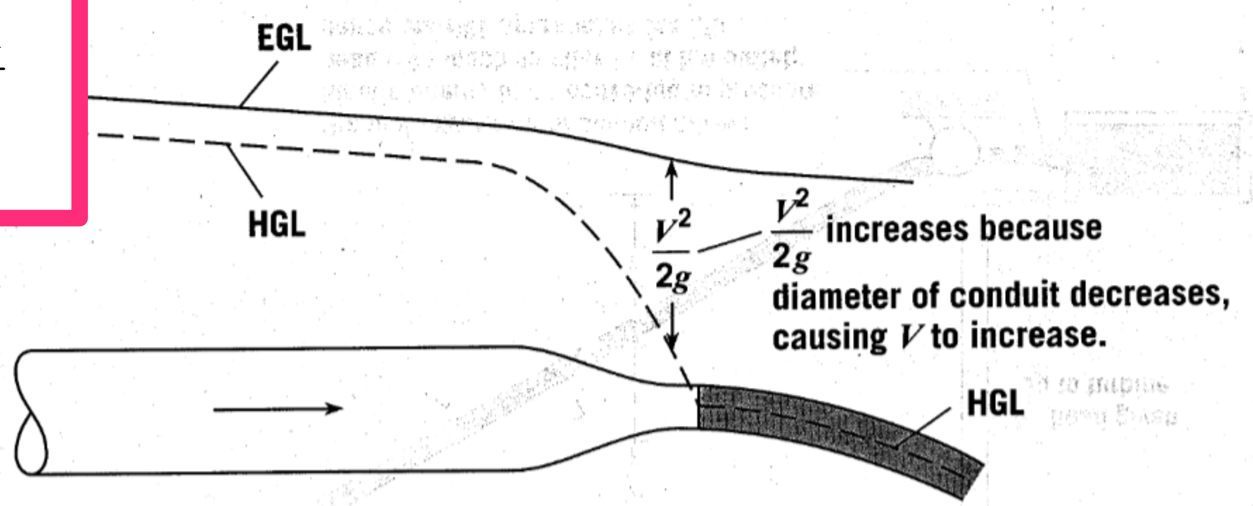
$$K = \frac{\text{drop in total head across component}}{\text{velocity head}} = \frac{\Delta h_{total}}{V^2/2g}$$



True ☐ **False** ☑

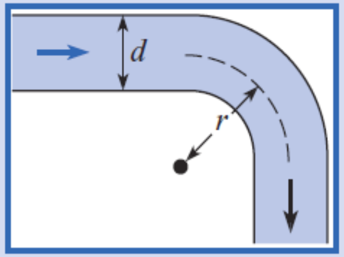
In your book (10th ed, page 380)

$$K = \frac{\text{drop in piezometric head}}{\text{velocity head}}$$



Minor Loss Coefficient

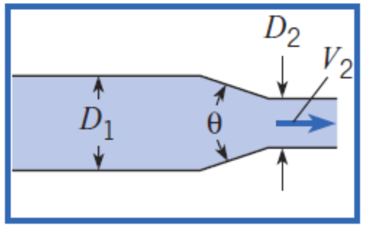
90° smooth bend



r/d		(16) and (9)
1	K_b = 0.35	
2	0.19	
4	0.16	
6	0.21	
8	0.28	
10	0.32	

Contraction

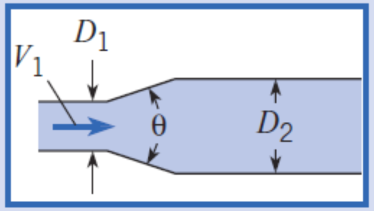
$h_L = K_C V_2^2/2g$



D_2/D_1	K_C $\theta = 60°$	K_C $\theta = 180°$	(10)
0.00	0.08	0.50	
0.20	0.08	0.49	
0.40	0.07	0.42	
0.60	0.06	0.27	
0.80	0.06	0.20	
0.90	0.06	0.10	

Expansion

$h_L = K_E V_1^2/2g$



D_1/D_2	K_E $\theta = 20°$	K_E $\theta = 180°$	(9)
0.00		1.00	
0.20	0.30	0.87	
0.40	0.25	0.70	
0.60	0.15	0.41	
0.80	0.10	0.15	

Combined Head Loss

$$\{\text{Total head loss}\} = \{\text{Pipe head loss}\} + \{\text{Component head loss}\}$$

$$h_L = \sum_{\text{pipes}} f\frac{L}{D}\frac{V^2}{2g} + \sum_{\text{components}} K\frac{V^2}{2g} = \frac{V^2}{2g}\left[\sum_{\text{pipes}} f\frac{L}{D} + \sum_{\text{components}} K\right]$$

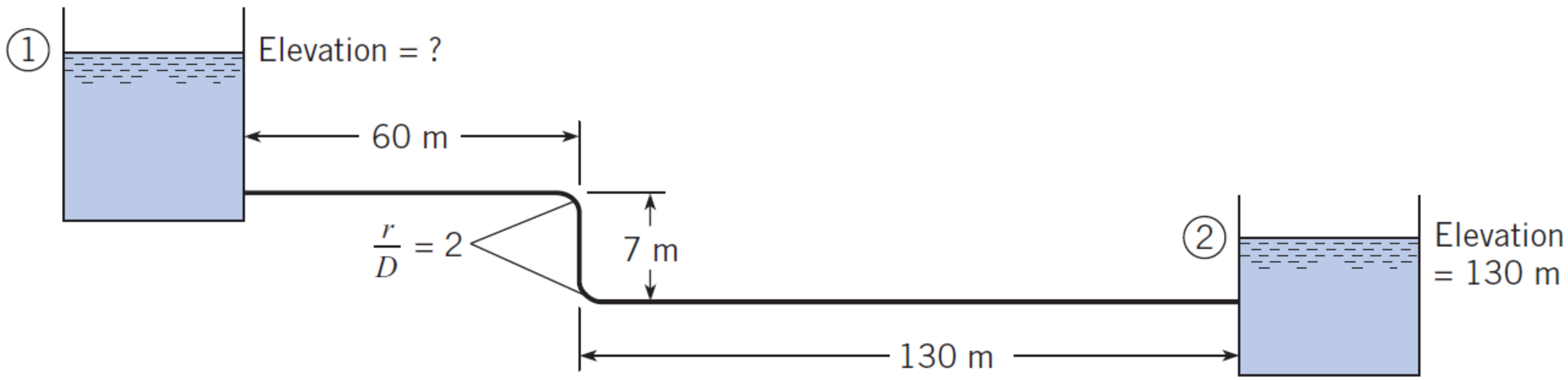
Example: Pipe System

Problem

If oil ($v = 1\times10^{-6}\ m^2/s,\ \text{S=0.9}$) flows from the upper to the lower reservoir at a rate of 0.028 $m^3 s$ in the 15 cm smooth pipe, what is the elevation of the oil surface in the upper reservoir?

Properties:

1. Oil: $\nu = 4\times10^{-5}\ \text{m}^2/\text{s},\ S = 0.9$.
2. Minor head loss coefficients, Table 10.5; entrance $= K_e = 0.5$; bend $= K_b = 0.19$; outlet $= K_E = 1.0$.



Example: Pipe System

Solution

1. Energy equation and term-by-term analysis

$$\frac{p_1}{\gamma} + \alpha_1 \frac{\overline{V}_1^2}{2g} + z_1 + h_p = \frac{p_2}{\gamma} + \alpha_2 \frac{\overline{V}_2^2}{2g} + z_2 + h_t + h_L$$

$$0 + 0 + z_1 + 0 = 0 + 0 + z_2 + 0 + h_L$$

$$z_1 = z_2 + h_L$$

Interpretation: Change in elevation head is balanced by the total head loss.

2. Combined head loss equation

$$h_L = \sum_{\text{pipes}} f \frac{L}{D} \frac{V^2}{2g} + \sum_{\text{components}} K \frac{V^2}{2g}$$

$$h_L = f \frac{L}{D} \frac{V^2}{2g} + \left(2K_b \frac{V^2}{2g} + K_e \frac{V^2}{2g} + K_E \frac{V^2}{2g} \right)$$

$$= \frac{V^2}{2g} \left(f \frac{L}{D} + 2K_b + K_e + K_E \right)$$

3. Combine eqs. (1) and (2).

$$z_1 = z_2 + \frac{V^2}{2g} \left(f \frac{L}{D} + 2K_b + K_e + K_E \right)$$

Example: Pipe System

Solution

4. Resistance coefficient

- Flow rate equation (5.8)

$$V = \frac{Q}{A} = \frac{(0.028\ \mathrm{m^3/s})}{(\pi/4)(0.15\ \mathrm{m})^2} = 1.58\ \mathrm{m/s}$$

- Reynolds number

$$\mathrm{Re} = \frac{VD}{\nu} = \frac{1.58\ \mathrm{m/s}(0.15\ \mathrm{m})}{4 \times 10^{-5}\ \mathrm{m^2/s}} = 5.93 \times 10^3$$

  Thus, flow is turbulent.

- Swamee-Jain equation (10.39)

$$f = \frac{0.25}{\left[\log_{10}\left(\frac{k_s}{3.7D} + \frac{5.74}{\mathrm{Re}^{0.9}}\right)\right]^2} = \frac{0.25}{\left[\log_{10}\left(0 + \frac{5.74}{5930^{0.9}}\right)\right]^2} = 0.036$$

Example: Pipe System

Solution

5. Calculate z_1 using (3):

$$z_1 = (130\ \text{m}) + \frac{(1.58\ \text{m/s})^2}{2(9.81)\text{m/s}^2}$$

$$\left(0.036\frac{(197\ \text{m})}{(0.15\ \text{m})} + 2(0.19) + 0.5 + 1.0\right)$$

$$\boxed{z_1 = 136\ \text{m}}$$

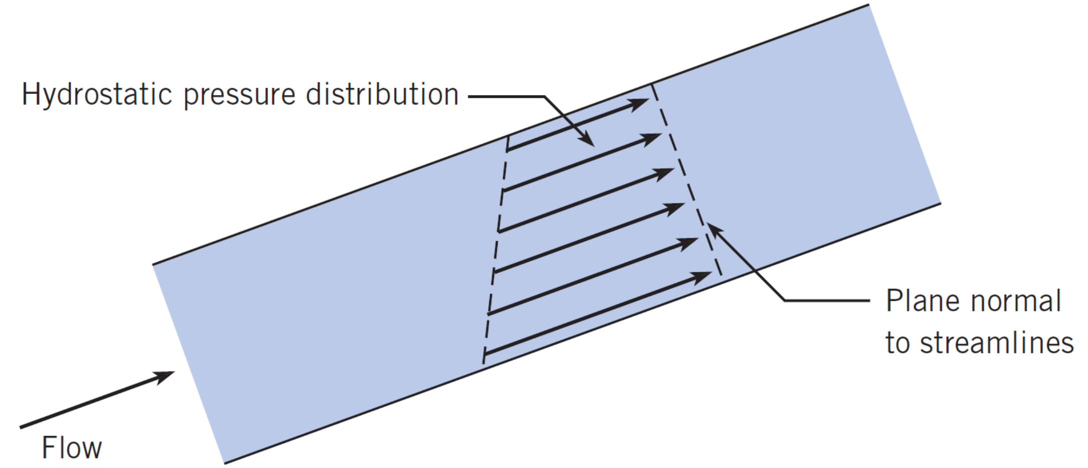
Optional reading

Derivation of the friction Coefficient for Laminar flows

Stress Distribution in Pipe Flow

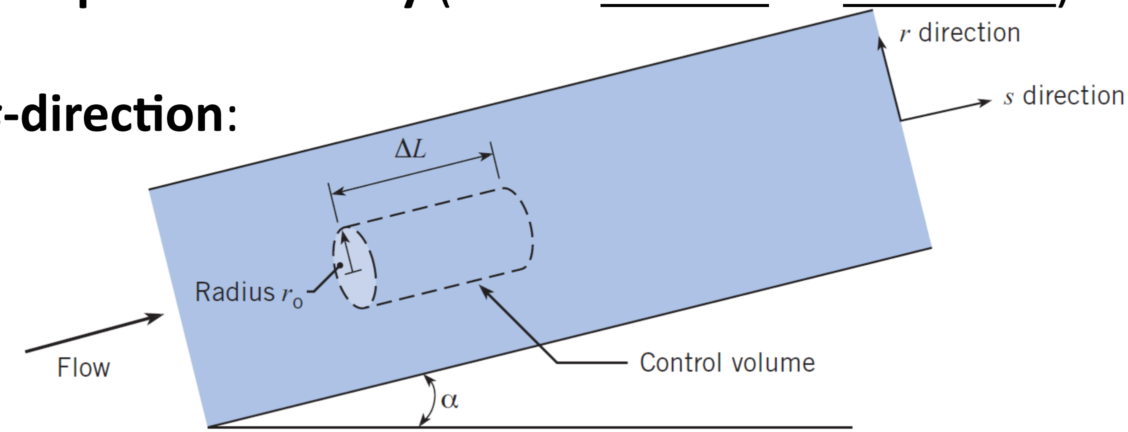
- In pipe flow, the pressure acting on the plane normal to the flow direction is hydrostatic.

> Why is the pressure distribution **hydrostatic**?



- Consider the below control volume for a **Newtonian** fluid in a round tube.
- Assume that the flow is **fully developed** and **steady** (either laminar or turbulent).
- Applying momentum Eq. in the ***s*-direction**:

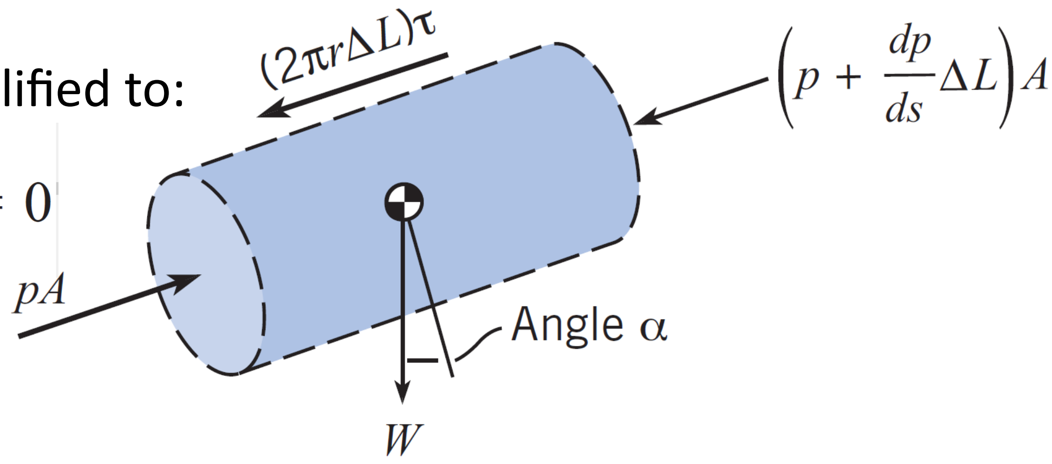
$$\sum F_s = 0$$



Stress Distribution in Pipe Flow

- The momentum Eq. is therefore simplified to:

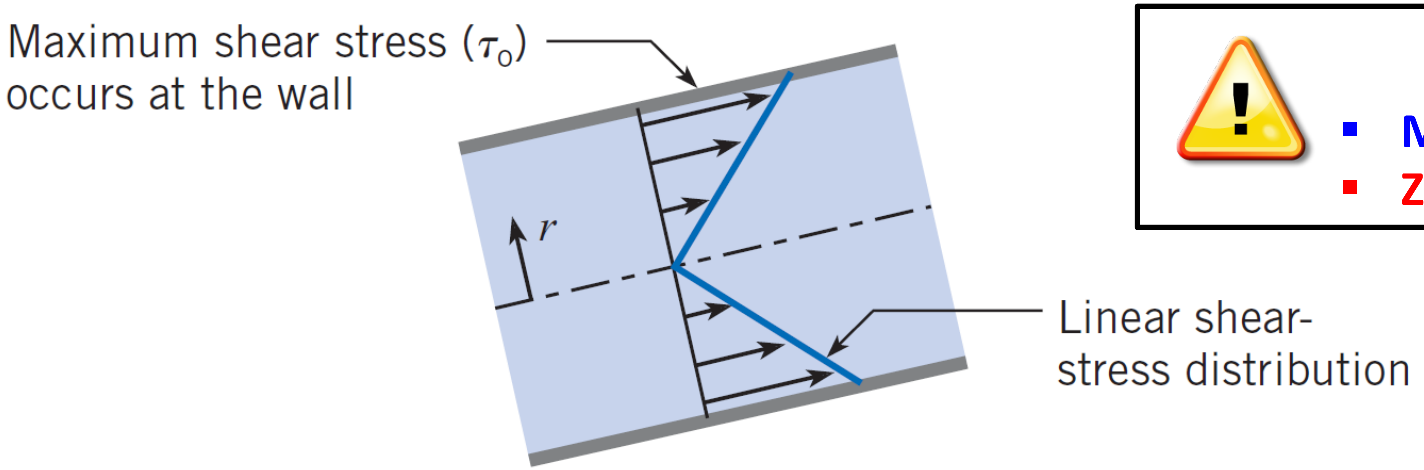
$$\sum F_s = F_{\text{pressure}} + F_{\text{weight}} + F_{\text{shear}} = 0$$



$$\Rightarrow \quad pA - \left(p + \frac{dp}{ds}\Delta L\right)A - W\sin\alpha - \tau(2\pi r)\Delta L = 0$$

$\sin\alpha = \Delta z / \Delta L$ $\quad W = \gamma A \Delta L$ $\Rightarrow$

$$\tau = \frac{r}{2}\left[-\frac{d}{ds}(p + \gamma z)\right]$$



Shear Stress

- **Maximum** at the **wall**.
- **Zero** at the **centerline**.

Laminar flow in Pipes

- A flow regime in which fluid motion is **smooth**.
- In general, laminar flow is important where **viscous forces** are <u>dominant</u> (e.g., microchannels, lubricant flows).
- Laminar flows occur when $\mathbf{Re} \leq 2000$
- Laminar flow in a round tube is called ***Poiseuille*** or ***Hagen-Poiseuille*** due to their pioneering studies in low speed flows.

J. Poiseuille(1797-1869)

G. Hagen(1797-1884)

Laminar Flow: Velocity Profile

- For a **Newtonian** fluid, the stress is related to rate-of-strain by:

$$\tau = \mu \frac{dV}{dy} \qquad \textbf{(I)}$$

y : the distance from the pipe wall

r : radial coordinate

r_0 : pipe radius

- Replacing y with $r - r_0$:

$$\mu\left(\frac{dV}{dy}\right) = \mu\left(\frac{dV}{dr}\right)\left(\frac{dr}{dy}\right) = -\left(\mu\frac{dV}{dr}\right) \qquad \textbf{(II)}$$

- The stress distribution was previously derived as:

$$\tau = \frac{r}{2}\left[-\frac{d}{ds}(p + \gamma z)\right] \qquad \textbf{(III)}$$

(I) (II) (III) →

$$\underbrace{-\left(\frac{2\mu}{r}\right)\left(\frac{dV}{dr}\right)}_{f(r)} = \underbrace{\frac{d}{ds}(p + \gamma z)}_{g(s)}$$

→ The above Eq. is true **if and only if** each side is equal to a **constant**.

Laminar Flow: Velocity Profile

- This results in:

$$\text{constant} = \frac{d}{ds}(p + \gamma z) = \left(\frac{\Delta(p + \gamma z)}{\Delta L}\right) = \left(\frac{\gamma \Delta h}{\Delta L}\right)$$

Δh: change in piezometric head in ΔL

- Inserting the above Eq. in the last Eq. of the previous page yields:

$$\frac{dV}{dr} = -\left(\frac{r}{2\mu}\right)\left(\frac{\gamma \Delta h}{\Delta L}\right)$$

- Integrating the above Eq. with respect to *r* yields:

$$V = -\left(\frac{r^2}{4\mu}\right)\left(\frac{\gamma \Delta h}{\Delta L}\right) + C \quad \textbf{(I)}$$

$$V(r = r_0) = 0 \xrightarrow{\textbf{(I)}} 0 = -\frac{r_0^2}{4\mu}\left(\frac{\gamma \Delta h}{\Delta L}\right) + C \quad \textbf{(II)}$$

$$\xrightarrow{\textbf{(I) (II)}} \boxed{V = \frac{r_0^2 - r^2}{4\mu}\left[-\frac{d}{ds}(p + \gamma z)\right] = -\left(\frac{r_0^2 - r^2}{4\mu}\right)\left(\frac{\gamma \Delta h}{\Delta L}\right)}$$

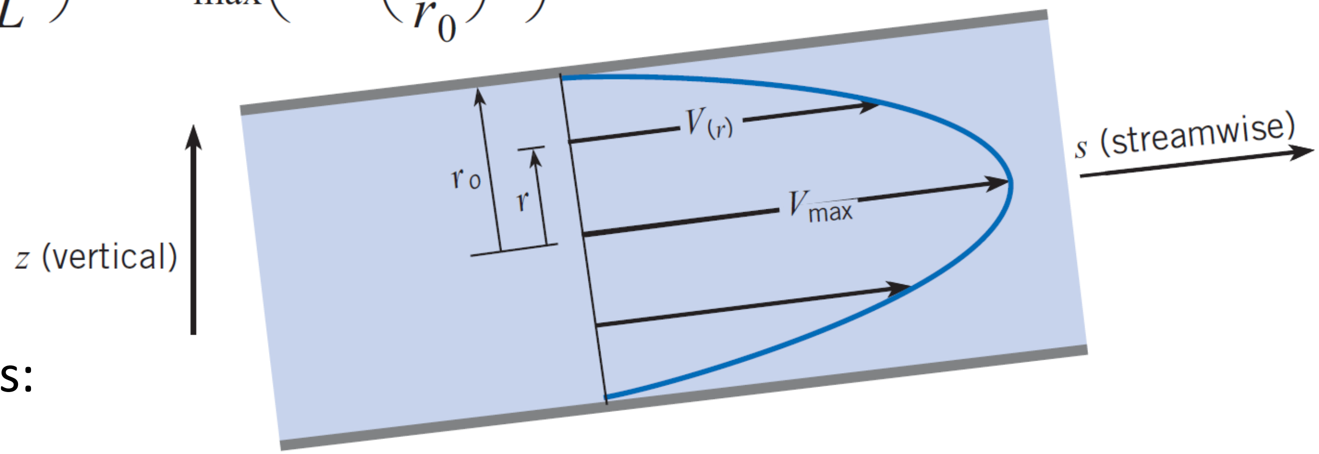
Laminar Flow: Velocity and Discharge

- **Maximum velocity** occurs at $r = r_0$:

$$V_{max} = -\left(\frac{r_0^2}{4\mu}\right)\left(\frac{\gamma \Delta h}{\Delta L}\right)$$

- Finally, the velocity can be described as:

$$V(r) = -\left(\frac{r_0^2 - r^2}{4\mu}\right)\left(\frac{\gamma \Delta h}{\Delta L}\right) = V_{max}\left(1 - \left(\frac{r}{r_0}\right)^2\right)$$



- **Discharge** of the pipe is:

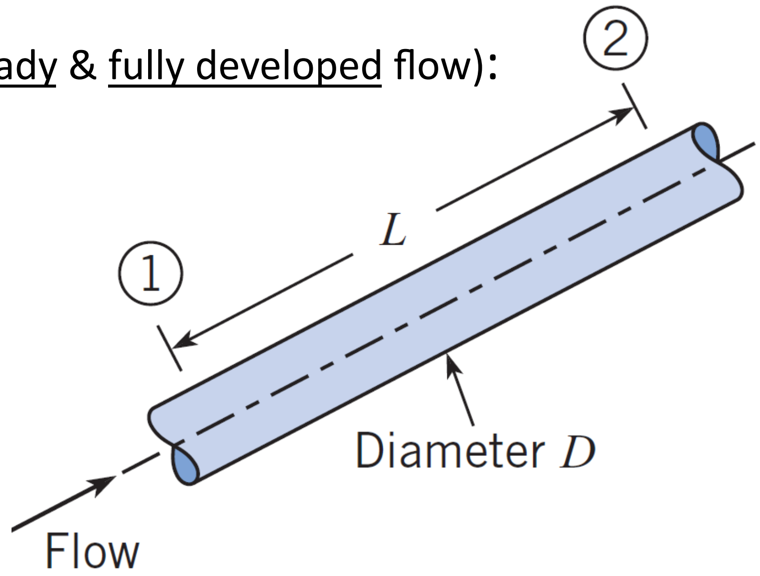
$$Q = \int V\,dA$$

$$= -\int_0^{r_0} \frac{(r_0^2 - r^2)}{4\mu}\left(\frac{\gamma \Delta h}{\Delta L}\right)(2\pi r\,dr) = -\left(\frac{\pi r_0^4}{8\mu}\right)\left(\frac{\gamma \Delta h}{\Delta L}\right)$$

Laminar Flow: Head Loss

- Apply the **energy Eq.** from section **1** to **2** (steady & fully developed flow):

$$\left(\frac{p_1}{\gamma} + z_1\right) = \left(\frac{p_2}{\gamma} + z_2\right) + h_f \qquad \textbf{(I)}$$



- **Mean velocity** for **laminar flow** is:

$$\overline{V} = \frac{Q}{A} = -\left(\frac{D^2}{32\mu}\right)\left(\frac{\gamma \Delta h}{\Delta L}\right)$$

$$\Delta h = \left(\frac{p_1}{\gamma} - z_1\right) - \left(\frac{p_2}{\gamma} - z_2\right)$$

$$\left(\frac{p_1}{\gamma} + z_1\right) = \left(\frac{p_2}{\gamma} + z_2\right) + \frac{32\mu \overline{V} L}{D^2} \qquad \textbf{(II)}$$

(I) (II) ⇒

$$h_f = \frac{32\mu L \overline{V}}{\gamma D^2}$$

$$h_f = f\frac{L}{D}\frac{V^2}{2g}$$

$$f = \frac{64}{\text{Re}}$$

Key assumptions to derive the above equation:

- **Laminar** flow - **Fully developed** flow - **Steady** flow - **Newtonian** fluid