Linear System Theory EDOC	**Project VI**: Finite Fields Dr. Ph. Müllhaupt

The purpose of this project is to study pole placement when the field is finite. One has to implement all computations over the field $\mathbb{F} := \mathbb{Z}/p\mathbb{Z}$ where p is some prime number that will be set at the beginning of the computations. The codes should allow changing the prime number p easily. Implement in Matlab or Mathematica or any numerical and/or computer algebra package of your choice.

- Read paper [1] carefully.
- Implement simulating the following discrete-time system

$$x_{k+1} = Ax_k + Bu_k$$

  where all vectors and matrices contain elements of the above finite fields $\mathbb{F}$, i.e. $A \in \mathbb{F}^{n\times n}$, $B \in \mathbb{F}^{n\times 1}$, and $x_k \in \mathbb{F}^{n\times 1}$ and $u_k \in \mathbb{F}$, for all $k \in \mathbb{N}$.
- Design a procedure that assigns the closed-loop characteristic polynomial to any desired characteristic polynomial by suitably computing a feedback $u_k = Kx_k$, with $K \in \mathbb{F}^{1\times n}$.
- Now consider the classical field $\mathbb{R}$ and choose any four by four matrix A that is unstable (at least one eigenvalue outside the unit circle in the complex plane) together with a B matrix that leads to a controllable system. Design a feedback $u_k = Kx_k$ so that the closed-loop system has all eigenvalues inside the unit circle (for example choose two damped oscillatory modes). Simulate the system when the field is $\mathbb{R}$.
- Use finite fields with increasing p so as to approximate numbers in $\mathbb{R}$ by those in $\mathbb{F}$. This means that real numbers normally represented in floating point numbers (when considering $\mathbb{R}$) are now represented by fixed point numbers (when considering $\mathbb{F}$) with p indicating the number of bits used to represent the numbers. This also means that one has to choose the position of the decimal position in the fixed-point representation.
- For each successive p, find a feedback row matrix K so that the closed-loop characteristic polynomial is as close as possible as the one computed when the field is $\mathbb{R}$.
- Simulate and compare the results as the prime number p increases. The purpose is that the more bits are used to represent floating point real numbers, the closer the results obtained in the finite field case (fixed point approximation) should be.
- Discuss the position of the closed-loop eigenvalues chosen and the minimal precision required to achieve satisfactory simulations (one tries to lower as much as possible the prime number p while maintaining results as close as possible as those in the real case $\mathbb{R}$).
- Then increase the size n of the state variables x_k and the matrices A and B up to $n = 10$ thereby assigning more modes. Repeat the steps as in the case of $n = 4$ described above.

References

[1] S. K. Mitter and R. Foulkes. Controllability and Pole Assignment for Discrete Time Linear Systems Defined over Arbitray Fields. *SIAM J. Control*, 9(1), 1971.