Linear System Theory

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Lecture III

Invariant Polynomials

Motivation: Looking at a linear system with transfer matrix $G(s)$ with m inputs and p outputs has many representations in state-space form

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

We have seen that system similarity preserve the transfer function i.e.

$$\bar{A} = PAP^{-1} \qquad \bar{B} = PB \qquad \bar{C} = CP^{-1} \qquad D$$

$$G(s) = C(sI-A)^{-1}B + D = \bar{C}(sI-\bar{A})^{-1}\bar{B} + \bar{D}$$

input-output

How do properties of $G(s)$ linked to input-output behaviors connect with invariant properties of the A matrix? ?

stabilization: Stabilization is linked to the possibility of changing the structure of the A matrix, changing its invariants.

First goal: determine the invariant structure of the A matrix.

Side goal:

From a theoretical level, what is the algebraic setting for establishing the invariants, generalizations?

- Annihilating polynomial
- coprime polynomials.
- Polynomial map.
- Minimal annihilating polynomial (MAP)

Theorem 1 There always exist a vector having same MAP as the whole space.

Theorem 2 Two subspaces are direc summands

$\Longleftrightarrow$

Their MAP are coprime

Theorem 3 Every space splits as a direct sum of cyclic supspaces

main Theorem of lecture

$$V = I_1 \oplus I_2 \oplus \cdots I_r$$

where $\psi(I_1)$ is the MAP of I_1
and $\psi(I_2)$ is the MAP of I_2

$\psi(I_2)$ divides $\psi(I_1)$.

etc. $\psi(I_{j+1})$ divides $\psi(I_j)$. $r > j \geq 2$.

$$A = I$$

$$|\lambda I - I|$$

$$= \begin{vmatrix} \lambda-1 & & 0 \\ & \lambda-1 & \\ 0 & \ddots & \lambda-1 \end{vmatrix}$$

$$= (\lambda-1)^n$$

$$(\lambda-1)^n \Big|_{\lambda=A=I}$$

$$(I-I)^n = 0$$

$\Rightarrow$ $\lambda(A)$ is not the minimal annihilating polynomial of the A matrix

MAP (minimal annihilating polynomial) is

$$\underbrace{(\lambda - I)}_{\text{invariant polynomial}}$$

$$A \in \mathbb{R}^{n\times n}$$

invariant $\mathrm{char}(A)$

$$\lambda(A) \triangleq |\lambda I - A|$$

$\longrightarrow$ eigenvalues of A

and PAP^{-1} has the same eigenvalues as A.

$$|\lambda PP^{-1} - PAP^{-1}|$$

$$|P|\,|\lambda I - A|\,|P|^{-1} = |\lambda I - A|$$

Cayley Hamilton theorem

$$\lambda(A)\Big|_{\lambda=A} = 0 \in \mathbb{R}^{n\times n}$$

$$\forall A \in \mathbb{R}^{n\times n}$$

quotients relative to invariant subspaces
relative annihilating polynomial.

Lemma Polynomial maps commute although matrices generally don't

$$\psi_1(A)\,\psi_2(A) = \psi_2(A)\,\psi_1(A).$$

Lemma RMAP always divide MAP.

divisors and invariant polynomials

Result without proof

Set $|\lambda I - A| = D_n(\lambda)$

Compute the greatest common divisors of all
minors of order $n-r$
this defines $D_{n-r}(\lambda)$

define $\dfrac{D_n(\lambda)}{D_{n-1}(\lambda)} = i_1(\lambda) \qquad \dfrac{D_{n-1}(\lambda)}{D_{n-2}(\lambda)} = i_2(\lambda)$

we get the invariant polynomials.

Proof: linked to finding the companion
matrix linked to each cyclic subspace.

Polynomial maps are elements of a torsion module over a principal ideal domain.

$\boxed{\text{Vector space + endomorphism} \quad \cong \quad \text{torsion module over } \underbrace{F(\lambda)}_{\text{a principal ideal domain}}}$

Choose a basis of V, say $e_1, e_2, \dots, e_n$

denote by $\varphi_1(\lambda), \varphi_2(\lambda), \dots, \varphi_n(\lambda)$

the minimal polynomials of these vectors. $\dim\{e_1, Ae_1, A^2e_1, \dots\} = p$

Let $\psi(\lambda)$ least common multiple of these polynomials

Then $\psi(\lambda)$ is MAP of the whole space.

(since every vector is representable in the form

$$V = \sum_{i=1}^{n} \alpha_i e_i .$$
)

Definition and notation (direct sum)

$V = V_1 \oplus V_2$, $\Longleftrightarrow$ no vector in common between V_1 and V_2 except the nul vector.

Definition: (invariant subspace)

A subspace $V' \subset V$ is called invariant with respect to the operator A, if $AV' \subset V'$

if $x \in V'$ implies $Ax \in V'$.

$\dim \left(\operatorname{span} \{ e_1, Ae_1, A^2 e_1, \dots \} \right) = p$

$A^p e_1 = \left(-\alpha_1 A^{p-1} - \alpha_2 A^{p-2} \dots \right) e_1$

$\Rightarrow \left(\lambda^p + \alpha_1 \lambda^{p-1} + \alpha_2 \lambda^{p-2} + \dots + \alpha_p \right)$

is a MAP of e_1

Theorem: Given A as operator, $A: V \mapsto V$

Let the minimal polynomial over the field F (field associated with the vector space V) be

$\psi(\lambda)$ and suppose that $\psi(\lambda)$

as $\psi(\lambda) = \psi_1(\lambda)\,\psi_2(\lambda)$

if $\psi_1(\lambda)$ and $\psi_2(\lambda)$ are coprime then

$\exists V_1$ and $\exists V_2$ invariant subspaces

such that $V = V_1 \oplus V_2$ and

$\psi_1(\lambda)$ is MAP of V_1

$\psi_2(\lambda)$ is MAP of V_2.

Proof: coprimeness implies $1 = \psi_1 \chi_1 + \psi_2 \chi_2$

$\exists \chi_1$ and χ_2

let $x \in V \Rightarrow$

$$x = \underbrace{\psi_1(A)\chi_1(A)x}_{x''} + \underbrace{\psi_2(A)\chi_2(A)x}_{x'}$$

Let $V_1 \triangleq \{x \in V \mid \psi_1(A)x = 0\}$

$V_2 \triangleq \{x \in V \mid \psi_2(A)x = 0\}$

$x' \in V_1$ since $\psi_1(A)x' = \psi_1 \psi_2 \chi_2 x$

$= \psi \chi_2 x = \chi_2 \psi x$

$= 0$

$x'' \in V_2$ since $\psi_2(A)x''$

$= \psi_2 \psi_1 \chi_1 x = \chi_1 \psi x = \chi_1 0 = 0.$

[redacted] and V_2 have only 0 as common vector.

for if $x_0 \in V_1$ & $x_0 \in V_2$

$\psi_1 x_0 = 0$ and $\psi_2 x_0 = 0$

using the coprime identity $1 = \psi_1 \chi_1 + \psi_2 \chi_2$

$x_0 = \psi_1 \chi_1 x_0 + \psi_2 \chi_2 x_0 = 0 + 0 = 0.$

a contradiction

$\Longrightarrow \quad V = V_1 \oplus V_2$

V_1 is invariant : $x \in V_1 \Longrightarrow \psi_1 x = 0$ (by definition)

$0 = A \psi_1 x = \psi_1 A x \Longrightarrow Ax \in V_1$

$\psi_1(\lambda)$ is the MAP.

suppose $\tilde{\psi}_1(\lambda)$ is a map of V_1

$\tilde{\psi}_1 \psi_2 x = \psi_2 \tilde{\psi}_1 x' + \tilde{\psi}_1 \psi_2 x''$

$= 0 + 0 = 0$

Since x is arbitrary $\Rightarrow$ $\psi_1 \cdot \psi_2$ is MAP of V which contradicts that ψ is MAP of U.

Theorem in a vector space there always exist a vector with MAP equal to the one of the whole space V

Case 1. MAP of V is a power of a polynomial (important for feedback.)

$$\psi(\lambda) = (\varphi(\lambda))^{\ell}$$

in V choose a basis $e_1, e_2, \dots, e_n$.

The MAP of e_1 is a divisor of $\psi(\lambda)$

hence $\psi_1 = (\varphi(\lambda))^{\ell_1}$

$$e_j \Rightarrow \psi_j = (\varphi(\lambda))^{\ell_i}, \quad j=1,\dots,n.$$

but MAP of V is the least common multiple of the ψ_j, $j=1,\dots,n$.

$\psi(\lambda)$ coincides with the MAP of one of the basis vectors. $\exists k$, $\psi = \psi_k$.

$\Rightarrow e_k$ has same MAP as V.

Lemma: $e' \longrightarrow \psi_1$ MAP, i.e $\psi_1 e' = 0$

$e'' \longrightarrow \psi_2$ MAP, i.e $\psi_2 e'' = 0$

(in full $\psi_1(A)e' = \psi_2(A)e'' = 0$).

if ψ_1 and ψ_2 are coprime then

MAP of $e'+e''$ is $\psi_1\psi_2 = \psi$

Proof: Let χ an arbitrary AP of $e'+e'' \triangleq e$

$$\chi e = 0$$

first multiply by ψ_2: $\psi_2 \chi e = 0$

$$\psi_2 \chi (e'+e'') = 0$$

$$\psi_2 \chi e' + \psi_2 \chi e'' = 0$$

$$\psi_2 \chi e' + \chi \underbrace{\psi_2 e''}_{0} = 0$$

$$\psi_2 \chi e' = 0$$

$\Longrightarrow$ $\psi_2 \chi$ is AP of e'

hence $\psi_2\chi$ is divisible by ψ_1

and coprimeness imply χ is divisible by ψ_1.

$\Longrightarrow$ similarly χ is divisible by ψ_2.

$\Longrightarrow$ χ is divisible by ψ

Hence every AP of e is divisible by $\psi_1 \psi_2$

$\Longrightarrow$ $\psi_1 \psi_2$ is MAP of $e = e' + e''$.

Proof in the general case (existence of x sharing same MAP as the whole space).

Let us decompose the MAP of $V = \psi(\lambda)$ into irreducible factors over F

$$\psi = \varphi_1^{c_1} \varphi_2^{c_2} \dots \varphi_s^{c_s}$$

$\varphi_1, \dots, \varphi_s$ are distinct

$$V = I_1 \oplus I_2 \oplus \dots \oplus I_s$$

MAP of I_i's are powers of irreducible polynomials, we know that

$\forall I_i$, $\exists e_i$ having same MAP $(\varphi_i^{c_i})$

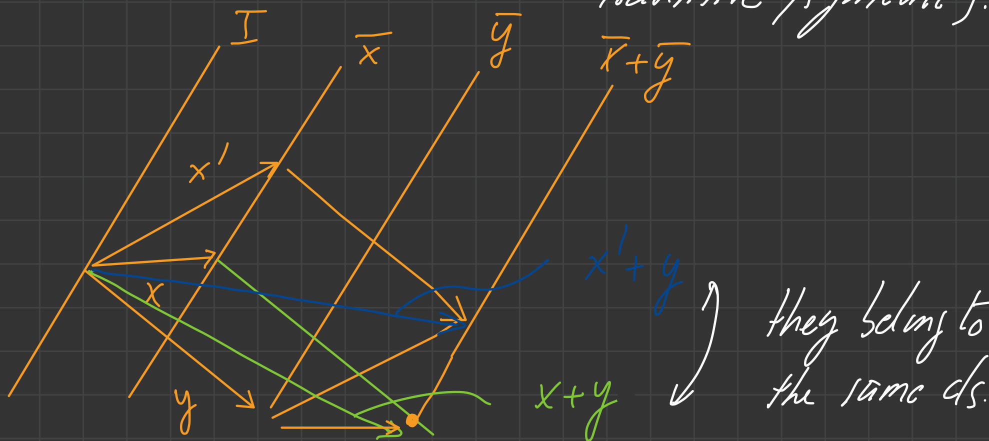
Since these MAPs are coprime, the vector $e = \sum_{i=1}^{s} e_i$ will have MAP equal to ψ by the above lemma.

Quotient vector space

$$x \equiv y \pmod{I}$$

$$\iff \quad y - x \in I$$

it is an equivalence class (reflexive, transitive, symmetric).



Definition Relative minimal annihilating polynomial / RMAP

MAP: of x (w.r.t. A) $\quad \sigma(A)x = 0$

RMAP (mod I) $\quad \sigma(A)x \equiv 0 \pmod{I}$.

Decomposition of a Space into cyclic subspaces

Let $\sigma(\lambda) = \lambda^p + \alpha_1 \lambda^{p-1} + \cdots + \alpha_p$

MAP of a vector v.

$v, Av, \dots A^{p-1} v$ are linearly independent.

but

$$A^p v = -\alpha_p v - \alpha_{p-1} Av - \dots - \alpha_1 A^{p-1} v$$

$$\boxed{\text{span}\{v, Av, \dots, A^{p-1} v\} \triangleq I}$$

is invariant and p-dimensional

Definition: such an invariant space is called cyclic since it is generated by a single vector v.

Lemma: every vector $x \in I$ is representable as a linear combination of the generating vectors, i.e. $\exists$ polynomial $\chi(\lambda)$ of degree $p-1$ such that

$$\boxed{x = \chi(A) v}$$

Remark: MAP of I = MAP of v

Main result of the lecture — Main Theorem

Let $\psi_1 = \psi = \lambda^m + \alpha_1 \lambda^{m-1} + \dots + \alpha_m$ be the MAP of the whole space V.

Then $\exists\, v \in V$ having the same MAP (previous theorem!). Let I_1 denote the .

$$\boxed{I_1 \triangleq \operatorname{span}\{v, Av, \dots, A^{m-1}v\}}$$

if $n = m$, the $V = I_1$ and the theorem is proved.

So let $m < n$, and let us compute the

RMAP of $V \pmod{I_1}$ i.e.

$$\psi_2(\lambda) = \lambda^p + \beta_1 \lambda^{p-1} + \dots + \beta_p$$

Remark: ψ_2 is a divisor of ψ_1, i.e,

$$\exists\, \varphi \quad \boxed{\psi_1 = \psi_2 \varphi}$$

this is easy since if σ_1 is the RMAP of $x \pmod{I}$ and σ is MAP of V, then $\sigma x = 0$

But $\sigma x = 0$ implies $\sigma x \equiv 0 \pmod{I}$

so that σ is a RAP of x (mod I)
therefore since it is a RAP it is divisible by the minimal one which is the RMAP (mod I), i.e. σ_1.

in V, $\exists v^*$ with RMAP (mod I_1) equal to ψ_2. (same constructions as with the MAP)

meaning that

$$\psi_2 v^* \equiv 0 \pmod{I_1}$$

i.e $\exists \chi$

$$\boxed{\psi_2 v^* = \chi v}$$

let us apply φ to both sides

$$\varphi \psi_2 v^* = \varphi \chi v$$

$$\psi_1 v^* = \varphi \chi v$$

But ψ_1 is MAP hence $\psi_1 v^* = 0$

$$0 = \varphi \chi v$$

$\Longmapsto$ $\varphi \chi$ is AP of v it is divisible by

$$\varphi \chi = \psi_1 \mu = \psi_2 \varphi \mu$$

$$\Rightarrow \quad \chi = \psi_2 \mu$$

$$\boxed{\psi_2 v^* = \chi v}$$ becomes $\psi_2 v^* = \psi_2 \mu v$

$$\psi_2 (v^* - \mu v) = 0$$

let us define $\boxed{w \triangleq v^* - \mu v}$

so that $\psi_2 w = 0$

ψ_2 is AP of w, and therefore

divisible by the RMAP of v^* (mod I_1)

since $w \equiv v^*$ (mod I_1)

Since ψ_2 is the RMAP, ψ_2 is MAP.

it follows then that

span $\{w, Aw, \ldots A^{p-1}w\}$ is cyclic and invariant.

$\Longrightarrow$ $\{v, Av, \ldots, A^{m-1}v, w, Aw, \ldots, A^{p-1}w\}$

are lin. indep. and a basis of

if $n = m+p$ $\Longrightarrow$ $V = I_1 \oplus I_2$

and the proof is finished.

If not $n > m+p$ iterate mod $(I_1 \oplus I_2)$