4
The Electromagnetics of Circuits

4.1 INTRODUCTION

Much of the engineering design and analysis of electromagnetic interactions are done through the mechanism of lumped-element circuits. In these, the energy-storage elements (inductors and capacitors) and the dissipative elements (resistors) are connected to each other and to sources or active elements within the circuit by conducting paths of negligible impedance. There may be mutual couplings, either electrical or magnetic, but in the ideal circuit these couplings are planned and optimized. The advantage of this approach is that functions are well separated and cause-and-effect relationships readily understandable. Powerful methods of synthesis, analysis, and computer optimization of such circuits have consequently been developed.

Most of the individual elements in an electrical circuit are small compared with wavelength so that fields of the elements are *quasistatic*; that is, although varying with time, the electric or magnetic fields have the spatial forms of static field distributions. There are important distributed effects in many real circuits, but often they can be represented by a few properly chosen lumped coupling elements. But in some circuits, of which the transmission lines are primary examples, the distributed effects are the major ones and must be considered from the beginning. In some cases in which the lumped idealizations described above do not strictly apply, lumped-element *models* can nevertheless be deduced and are useful for analysis because of the powerful circuit methods that have been developed.

We have introduced the lumped-circuit concepts, inductance and capacitance, in our studies of static fields. We have also seen how the skin effect phenomenon in conductors changes both resistance and inductance at high frequencies. We now wish to examine circuits and circuit elements more carefully from the point of view of electromagnetics. It is easy to see the idealizations required to derive Kirchhoff's laws from Maxwell's equations. It is also possible to make certain extensions of the concepts when the simplest idealizations do not apply. In particular, introduction of the retardation concepts shows that circuits may radiate energy when comparable in size with wavelength. The amount of radiated power may be estimated from these extended circuit ideas for some configurations. But for certain classes of circuits it becomes impossible to make the extensions without a true field analysis. We shall look at both types of circuits in this chapter.

The Idealizations in Classical Circuit Theory

4.2 Kirchhoff's Voltage Law

Kirchhoff's two laws provide the basis for classical circuit theory. We begin with the voltage law as a way of reviewing the basic element values of lumped-circuit theory. The law states that for any closed loop of a circuit, the algebraic sum of the voltages for the individual branches of the loop is zero:

$$\sum_i V_i = 0 \tag{1}$$

The basis for this law is Faraday's law for a closed path, written as

$$-\oint \mathbf{E} \cdot \mathbf{dl} = \frac{\partial}{\partial t} \int_S \mathbf{B} \cdot \mathbf{dS} \tag{2}$$

and the definition of voltages between two reference points of the loop,

$$V_{ba} = -\int_a^b \mathbf{E} \cdot \mathbf{dl} \tag{3}$$

To illustrate the relation between the circuit expression (1) and the field expressions (2) and (3), consider first a single loop with applied voltage $V_0(t)$ and passive resistance, inductance, and capacitance elements in series (Fig 4.2*a*). A convention for positive voltage at the source is selected as shown by the plus and minus signs on the voltage generator, which means by (3) that field of the source is directed from b to a when V_0 is positive. A convention for positive current is also chosen, as shown by the arrow on $I(t)$. The interpretation of (1) by circuit theory for this basic circuit is then known to be

$$V_0(t) - RI(t) - L\frac{dI(t)}{dt} - \frac{1}{C}\int I(t)\,dt = 0 \tag{4}$$

To compare, we break the closed line integral of (2) into its contributions over the several elements:

$$-\int_a^b \mathbf{E} \cdot \mathbf{dl} - \int_b^c \mathbf{E} \cdot \mathbf{dl} - \int_c^d \mathbf{E} \cdot \mathbf{dl} - \int_d^a \mathbf{E} \cdot \mathbf{dl} = \frac{\partial}{\partial t} \int_S \mathbf{B} \cdot \mathbf{dS} \tag{5}$$

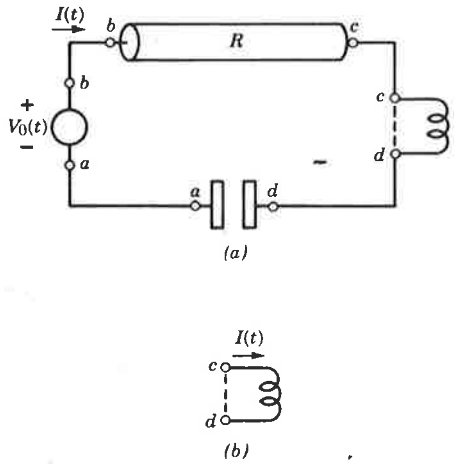


FIG. 4.2 (*a*) Series circuit with resistor, inductor, and capacitor. (*b*) Detail of inductor.

or

$$V_0(t) + V_{cb} + V_{dc} + V_{ad} = \frac{\partial}{\partial t}\int_S \mathbf{B} \cdot d\mathbf{S} \tag{6}$$

The right side of (6) is not zero as is the right side of (4), but we recognize it as the contribution to emf generated by any rate of change of magnetic flux within the path defined as the circuit. If not entirely negligible, it can be considered as arising from an inductance of the loop which can be added to the lumped element L, or a mutually induced coupling if the flux is from an external source. Thus we will from here on consider it as negligible or included in L so that the right side of (6) is zero. (Mutual effects are added later.) We now examine separately the three voltage terms related to the passive components R, L, and C.

Resistance Element The field expression to be applied to the resistive material is the differential form of Ohm's law,

$$\mathbf{J} = \sigma \mathbf{E} \tag{7}$$

so that the voltage V_{cb} is

$$V_{cb} = -\int_b^c \mathbf{E} \cdot d\mathbf{l} = -\int_b^c \frac{\mathbf{J}}{\sigma} \cdot d\mathbf{l} \tag{8}$$

where the path is taken along some current flow path of the conductor. Conductivity σ may vary along this path. At dc or low frequencies, current I is uniformly distributed over the cross section A of the conductor, which can also vary with position. Thus

$$V_{cb} = -\int_b^c \frac{I\,dl}{\sigma A} = -IR \tag{9}$$

where

$$R = \int_b^c \frac{dl}{\sigma A} \tag{10}$$

This last is the usual dc or low-frequency resistance. The situation is more complicated at higher frequencies because of the effect of the changing magnetic fields on currents within the conductor. Current distribution over the cross section is then nonuniform, and the particular path along the conductor must be specified. In the plane skin effect analysis of Chapter 3, current was related to electric field at the surface to define a surface impedance. We shall return to this concept later in the chapter for conductors of circular cross section.

Inductance Element The voltage across the terminals of the inductive element comes from the time rate of change of magnetic flux within the inductor, shown in the figure as a coil. Assuming first that resistance of the conductor of the coil is negligible, let us take a closed line integral of electric field along the conductor of the coil, returning by the path across the terminals (Fig. 4.2*b*). Since the contribution along the part of the path which follows the conductor is zero, all the voltage appears across the terminals:

$$-\oint \mathbf{E} \cdot \mathbf{dl} = -\int_{c(\text{cond.})}^{d} \mathbf{E} \cdot \mathbf{dl} - \int_{d(\text{term.})}^{c} \mathbf{E} \cdot \mathbf{dl} = -\int_{d(\text{term.})}^{c} \mathbf{E} \cdot \mathbf{dl} \tag{11}$$

By Faraday's law, this is the time rate of change of magnetic flux enclosed:

$$-\int_{d(\text{term.})}^{c} \mathbf{E} \cdot \mathbf{dl} = -V_{dc} = \frac{\partial}{\partial t}\int_S \mathbf{B} \cdot \mathbf{dS} \tag{12}$$

Inductance L is defined as the magnetic flux linkage per unit of current (Sec. 2.5)

$$L = \left[\int \mathbf{B} \cdot \mathbf{dS}\right] \Big/ I \tag{13}$$

so the voltage contributed by this term, assuming L independent of time, is

$$V_{cd} = \frac{\partial}{\partial t}(LI) = L\frac{dI}{dt} \tag{14}$$

Note that in computing flux enclosed by the path, we add a contribution each time we follow another turn around the flux. Thus for N turns, the contribution to induced voltage is just N times that of one turn, provided the same flux links each turn. This enters into the calculation of L and will be seen specifically when we find inductance of a coil.

If there is finite resistance in the turns of the coil, the second term of (11) is not zero but is the resistance of the coil, R_L, multiplied by current; therefore (11) becomes

$$-\oint \mathbf{E} \cdot \mathbf{dl} = -R_L I - V_{dc} = \frac{\partial}{\partial t}\int_S \mathbf{B} \cdot \mathbf{dS}$$

or

$$V_{cd} = R_L I + L \frac{dI}{dt} \tag{15}$$

Thus, as expected, we simply add another series resistance to take care of finite conductivity in the conductors of the coil.

Capacitive Element The ideal capacitor is one in which we store only electric energy; magnetic fields are negligible so there is no contribution to voltage from changing magnetic fields but only from the charges on plates of the capacitor. The problem is then quasistatic and voltage is synonymous with potential difference between capacitor plates. So, in contrast to the inductor, we can take any path between the terminals of the capacitor for evaluation of voltage V_{da}, provided it does not stray into regions influenced by magnetic fields from other elements. We also take the definition of capacitance from electrostatics (Sec. 1.9) as the charge on one plate divided by the potential difference:

$$C = \frac{Q}{V} \tag{16}$$

Thus, from continuity,

$$I = \frac{dQ}{dt} = \frac{d}{dt}(CV_{da}) = C\frac{dV_{da}}{dt} \tag{17}$$

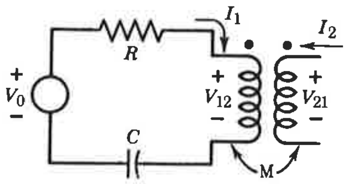
The last term in (17) implies a capacitance which is not changing with time. Integration of (17) with time leads to

$$V_{da} = \frac{1}{C}\int I\,dt \tag{18}$$

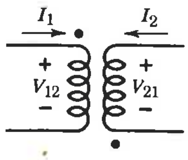
If the dielectric of the capacitor is lossy, there are conduction currents to add to (17), which are represented in the circuit as a conductance $G_C = 1/R_C$ in parallel with C; the value of R_C may be calculated from (10) by using conductivity of the dielectric and area of the capacitor plates.

Induced Voltages from Other Parts of the Circuit In addition to voltages induced by charges and currents of the circuit path being considered, there may be induced voltages from other portions of the circuit. In particular, if the magnetic field from one part of the circuit links another part, an induced voltage is produced through Faraday's law when this magnetic field changes with time. This coupling is represented in the circuit by means of a mutual inductor M, as shown in Fig. 4.2*c*. The value of M is defined as the magnetic flux ψ_{12} linking path 1, divided by the current I_2:

$$M = M_{12} = \frac{\psi_{12}}{I_2} \tag{19}$$



(c)



(d)

FIG. 4.2 (*c*) Circuit with a mutual inductor. (*d*) Designation of mutual coupling with negative M.

The voltage induced in the first path is then

$$V_{12} = \frac{d\psi_{12}}{dt} = M\frac{dI_2}{dt} \tag{20}$$

and the circuit equation (4) is modified to be

$$V_0 - RI_1 - L\frac{dI_1}{dt} - M\frac{dI_2}{dt} - \frac{1}{C}\int I_1\,dt = 0 \tag{21}$$

The mutual inductance M may be either positive or negative depending upon the sense of flux with respect to the defined positive reference for I_2. The sign of M is designated on a circuit diagram by the placing of dots and with sign conventions for currents and voltages as shown; those on Fig. 4.2*c* denote positive M; negative M would be designated as in Fig. 4.2*d*.

Except for certain materials (to be considered in Chapter 13) there is a reciprocal relation showing that the same M gives the voltage induced in circuit 2 by time-varying current in circuit 1:

$$V_{21} = \frac{d\psi_{21}}{dt} = M\frac{dI_1}{dt} \tag{22}$$

All mutual effects to be considered in this chapter have this reciprocal relationship.

In summary, we find that if losses in inductor and capacitor are ignored, the field approach, with understandable approximations, leads to the definitions for the three induced voltage terms for the passive elements used in the circuit approach, Eq. (4). Moreover, the definitions (10), (13), and (16) are the usual quasistatic definitions for these elements. If losses are present, a series resistance is added to L and a shunt conductance to C, again as is commonly done in the circuit approach. Coupling between circuit paths by magnetic flux adds mutual inductance elements. We next examine the Kirchhoff current law and the extension through this to multimesh circuits.

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4.3 Kirchhoff's Current Law and Multimesh Circuits

The current law of Kirchhoff states that the algebraic sum of currents flowing out of a junction is zero. Thus, referring to Fig. 4.3*a*,

$$\sum_{n=1}^{N} I_n(t) = 0 \tag{1}$$

It is evident that the idea behind this law is that of continuity of current, so we refer to the continuity equation implicit in Maxwell's equations, Eq. 3.4(5), or its large-scale equivalent:

$$\oint_S \mathbf{J} \cdot \mathbf{dS} = -\frac{\partial}{\partial t}\int_V \rho\, dV \tag{2}$$

If we apply this to a surface S surrounding the junction, the only conduction current flowing out of the surface is that in the wires, so the left side of (2) becomes just the algebraic sum of the currents flowing out of the wires, as in (1). The right side is the negative time rate of change of charge Q, if any, accumulating at the junction. So (2) may be written

$$\sum_{n=1}^{N} I_n(t) = -\frac{dQ(t)}{dt} \tag{3}$$

A comparison of (1) and (3) shows an apparent difference, but it is only one of interpretation. If Q is nonzero, we know that we take care of this in a circuit problem by adding one or more capacitive branches to yield the capacitive current dQ/dt at the junction. That is, in interpreting (3), the current terms on the left are taken only as convection or conduction currents, whereas in (1) displacement or capacitance currents are included. With this understanding, (1) and (3) are equivalent.

With the two laws, the circuit analysis illustrated in the preceding section can be extended to circuits with several meshes. As a simple example, consider the low-pass filter of Fig. 4.3*b* or 4.3*c*. Although currents and voltages are taken as time-varying,

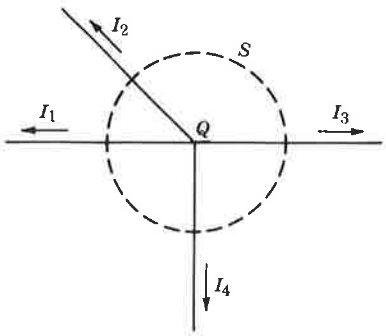
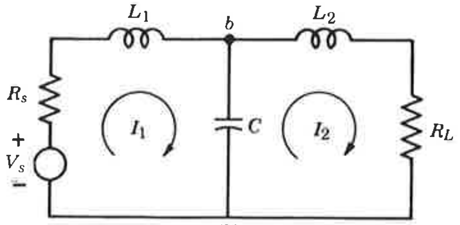
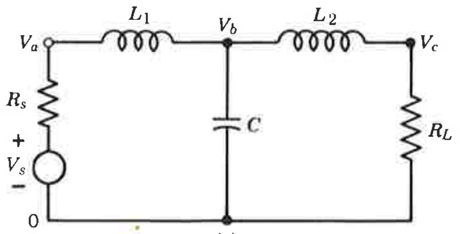


FIG. 4.3*a* Current flow from a junction.



(b)



(c)

FIG. 4.3 Low-pass filter: (*b*) loop current analysis; (*c*) node voltage analysis.

we drop the functional notation for simplicity. Figure 4.3*b* illustrates the standard method utilizing mesh currents I_1 and I_2. Note that the net current through C is $(I_1 - I_2)$, which automatically satisfies the current law at node *b*. The voltage law is then written about each loop as follows:

$$V_s - R_s I_1 - L_1 \frac{dI_1}{dt} - \frac{1}{C}\int (I_1 - I_2)\,dt = 0 \tag{4}$$

$$-\frac{1}{C}\int (I_2 - I_1)\,dt - L_2 \frac{dI_2}{dt} - R_L I_2 = 0 \tag{5}$$

The two equations are then solved by appropriate means to give I_1 and I_2 for a given V_s.

A second standard method of circuit analysis uses node voltages V_a, V_b, and V_c as shown in Fig. 4.3*c*. These are defined with respect to some reference, here taken as the lower terminal of the voltage generator, denoted 0. Then Kirchhoff 's voltage law is automatically satisfied, for if we add voltages around the first loop we have

$$V_s + (V_a - V_s) + (V_b - V_a) + (0 - V_b) \equiv 0 \tag{6}$$

Kirchhoff 's current law is then applied at each of the three nodes as follows:

$$\text{Node } a\text{:}\quad \frac{V_a - V_s}{R_s} + \frac{1}{L_1}\int (V_a - V_b)\,dt = 0 \tag{7}$$

$$\text{Node } b\text{:}\quad \frac{1}{L_1}\int (V_b - V_a)\,dt + \frac{1}{L_2}\int (V_b - V_c)\,dt + C\frac{dV_b}{dt} = 0 \tag{8}$$

$$\text{Node } c: \quad \frac{1}{L_2}\int (V_c - V_b)\,dt + \frac{V_c}{R_L} = 0 \tag{9}$$

Solution of these by appropriate means yields the three node voltages in terms of the given voltage V_s. Note that no equation for the reference node need be written as it is contained in the above.

In the above we seem to be treating voltage as a potential difference when we take voltage of a node with respect to the chosen reference, but note that this is only after the circuit is defined and we are only breaking up $\int \mathbf{E} \cdot \mathbf{dl}$ into its contributions over the various branches. As illustrated in the preceding section, we do have to define the path carefully whenever there are inductances or other elements with contributions to voltage from Faraday's law.

Finally, a word about sources. The voltage generator most often met in lumped-element circuit theory is a highly localized one. For example, the electrons and holes of a semiconductor diode or transistor may induce electric fields between the conducting electrodes fabricated on the device. The entire device is typically small compared with wavelength so that the electric field, although time-varying, may be written as the gradient of a time-varying scalar potential. The integral of electric field at any instant thus yields an instantaneous potential difference V_s between the electrodes, which is the source voltage (or $V_s - IZ_s$ if current flows). The induced effects from a modulated electron stream passing across a klystron gap are similar, as are those from many other practical devices. There are interesting field problems in the analysis of induced effects from such devices, but from the point of view of the circuit designer, they are simply point sources representable by the V_s used in the circuits.

A quite different limiting case is that in which the fields driving the circuit are not localized but are distributed. An important example is that of a receiving antenna with the fields set down by a distant transmitting antenna. If voltage is taken as the line integral of electric field along the antenna, applied voltage clearly depends upon the circuit configuration and orientation with respect to the applied field. Although quite different from the case with a localized source, it is found that circuit theory is useful here also. A formulation in terms of the retarded potentials will be applied to this case in Sec. 4.11.

Current generators are natural to use as sources in place of voltage generators if emphasis is on the current induced between electrodes of the point source or small-gap device. Similarly for the distributed source, if applied magnetic field at the circuit conductor is given, induced current can be calculated and a current representation is natural. One, however, has a choice in any case since the Thévenin and Norton theorems[1] show that the two representations of Figs. 4.3*d* and 4.3*e* are equivalent with the relations

$$Y_s = Z_s^{-1}, \qquad I_s = V_s Y_s \tag{10}$$

Thus an equivalent to Fig. 4.3*c* is that of Fig. 4.3*f*, utilizing a current generator.

[1] *S. E. Schwarz and W. G. Oldham,* Electrical Engineering: An Introduction, *2nd ed., Saunders, Fort Worth, TX, 1993.*

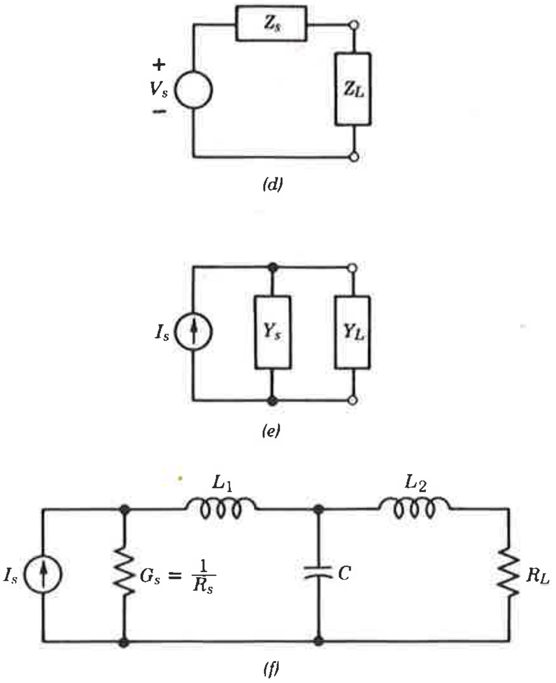


Fig. 4.3 (*d*) Thévenin circuit configuration. (*e*) Norton circuit form. (*f*) Equivalent of circuit in (*c*) using Norton source.

Skin Effect in Practical Conductors

4.4 Distribution of Time-Varying Currents in Conductors of Circular Cross Section

To study the resistive term at frequencies high enough so that current distribution is not uniform, we need to first find the current distribution. This was done in Sec. 3.16 for plane conductors. We now wish to do this for the useful case of round conductors. Recall that a good conductor is defined as one for which displacement current is negligible in comparison with conduction current so that

$$\nabla \times \mathbf{H} = \mathbf{J} = \sigma \mathbf{E} \tag{1}$$

Faraday's law equation is (in phasor form)

$$\nabla \times \mathbf{E} = -j\omega\mu \mathbf{H} \tag{2}$$