Chapter 1 - Generalities on acoustics

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1 Acoustic propagation in an (infinite) waveguide

- $m = \rho_0 S dx$
- Newton's law : $m\frac{\partial v}{\partial t} = F_{tot} = p(x)S - p(x+dx)S$
  Denoting $v = \frac{\partial \xi}{\partial t}$ and $p(x) - p(x+dx) = -\frac{\partial p}{\partial x}dx$, the local law can be formulated as :
  $\frac{\partial p}{\partial x} = -\rho_0 \frac{\partial^2 \xi}{\partial t^2}$
- the equation of compression leads to the differential equation : $\frac{\delta P_{tot}}{P_{tot}} + \Gamma_0 \frac{\delta V}{V}$ where $\frac{\delta a}{a}$ denotes the relative variation of a quantity a.
  Here, the reference pressure is p_s and the reference volume is $V_0 = Sdx$.
  Moreover, the pressure variation is equal to the acoustic pressure variation $\delta P_{tot} = p$, and $\delta V = S(\xi(x+dx) - \xi(x)) = S\frac{\partial \xi}{\partial x}dx$.
  This leads to : $p = -p_s \Gamma_0 \frac{\partial \xi}{\partial x}$
- by derivating the local Newton's law with respect to x and derivating the differential equation of compression twice with respect to t, one gets :
  $\frac{\partial^2 p}{\partial x^2} - \frac{\rho_0}{\Gamma_0 p_s}\frac{\partial^2 p}{\partial t^2} = 0.$
  Leading to $c_0 = \sqrt{\frac{\Gamma_0 p_s}{\rho_0}}$

2 $\lambda/4$ resonator

- The sound wave propagation is given by

$$\frac{\partial^2 p(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 p(x,t)}{\partial t^2} = 0 \quad (1)$$

  General solutions to this equation have the form

$$p(x,t) = A\mathrm{e}^{-j(kx+\omega t)} + B\mathrm{e}^{j(kx-\omega t)} \quad (2)$$

- Newton's law :

$$\frac{\partial p(x,t)}{\partial x} = -\rho_0 \frac{\partial v(x,t)}{\partial t} \quad (3)$$

  If p and v harmonic disturbances,

$$\frac{\partial p(x,t)}{\partial x} = -\rho_0 j\omega v(x,t) \quad (4)$$

- At $x = 0$, $v(0,t) = 0$ since the duct is closed and therefore, $\frac{\partial p(0,t)}{\partial x} = 0$.
- Using separation of variables, only the spatial part of the equations can be kept for the harmonic case. The general solution to the problem therefore has the form

$$p(x) = a\mathrm{e}^{-jkx} + b\mathrm{e}^{jkx} \quad (5)$$

Introducing the boundary condition at $x = L$, one gets b as a function of a :

$$p(L) = a\mathrm{e}^{-jkL} + b\mathrm{e}^{jkL} \Rightarrow b = -a\mathrm{e}^{-2jkL} \tag{6}$$

Therefore

$$p(x) = a\mathrm{e}^{-jkL}(\mathrm{e}^{jk(L-x)} - \mathrm{e}^{-jk(L-x)}) = 2ja\mathrm{e}^{-jkL}\sin(k(L-x)). \tag{7}$$

Taking the partial derivative with respect to x yields

$$\frac{\partial p(x)}{\partial x} = -2jak\mathrm{e}^{-jkL}\cos(k(L-x)). \tag{8}$$

Introducing the boundary condition at $x = 0$:

$$\frac{\partial p(0)}{\partial x} = -2jak\mathrm{e}^{-jkL}\cos(kL) = 0. \tag{9}$$

This equation has non trivial solutions only for $cos(kL) = 0$ and therefore :
$k_nL = (2n+1)\frac{\pi}{2}, n \in \mathbb{N}^*$.
Since $k = \omega/c = 2\pi f/c$, the eigenfrequencies are given by

$$\frac{2\pi f_n L}{c} = \frac{(2n+1)\pi}{2} \Rightarrow f_n = (2n+1)\frac{c}{4L}. \tag{10}$$

This is equivalent to $L = (2n+1)\lambda/4$, hence the name.

— $f_1 = 340 \cdot 1/(4 \cdot 0.025) = 3400$ Hz

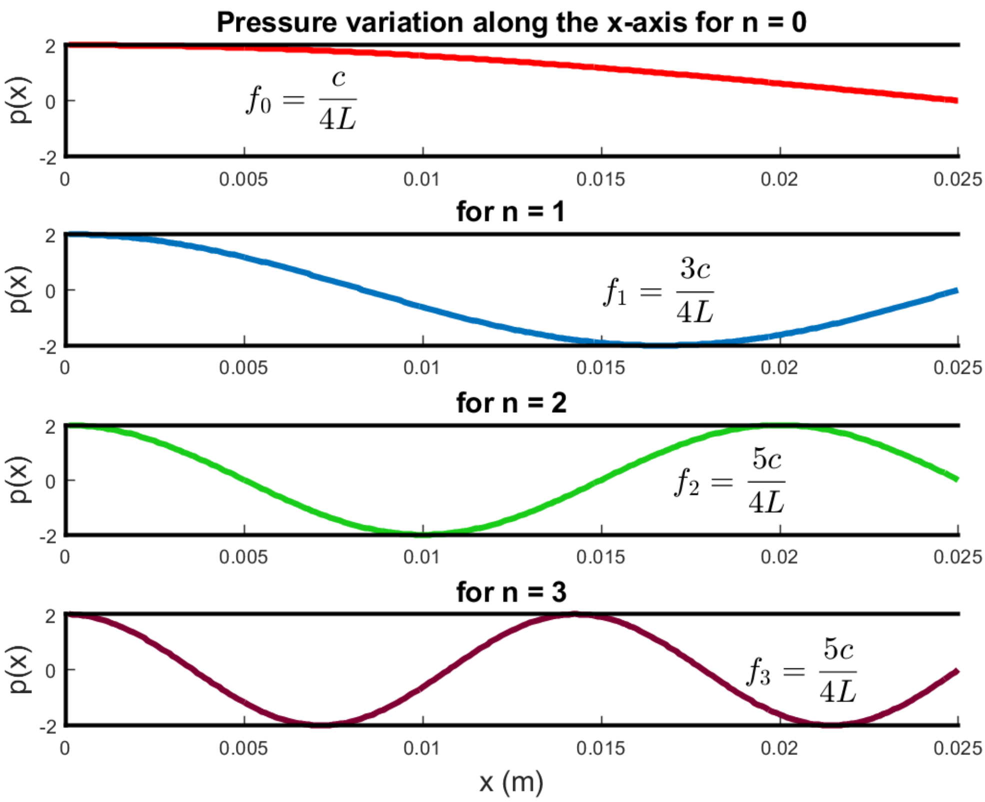

Figure 1 – Pressure field along the duct for the first 4 resonance frequencies.

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Matlab script
% Audio Engineering (MA1) Generalities on acoustics, Chap I, exercise 2
% Quarter wavelength mode shapes for a straight duct filled with air
%

% Duct parameters
L = 0.025;  % duct length
S = 0.0038; % duct cross-section

% Parameters pf the acoustic domain
rho = 1.2;  % density of air
c = 340;    % sound wave celerity

a = 1;                          % arbitrary amplitude

x = 0.0001:0.0001:L;            % 1D vector along x axis
N = 4;                          % N first modes order
p = zeros(N,length(x));         % initialization p(x)
for n = 1:N
    for m = 1:length(x)
        p(n,m) = 2*a*cos((2*n-1)*pi/2/L*x(m)); % pressure field along x axis
    end
end
```

3 Sound absorption in a duct

1. General expression for the specific acoustic impedance
The general solution for the 1D wave equation can be written as :

$$p(x) = \underbrace{p_+\, e^{-jkx}}_{\text{forward travelling wave}} + \underbrace{p_-\, e^{jkx}}_{\text{backward travelling wave}}$$

Using the 1D Euler equation : $\dfrac{\partial p}{\partial x} = -\mathrm{j}\omega\rho v$, the particle velocity can be expressed as :

$$v(x) = -\frac{1}{j\omega\rho}\frac{\partial p}{\partial x} = -\frac{1}{j\omega\rho}(-\mathrm{j}k)\left(p_+\, e^{-jkx} - p_-\, e^{jkx}\right) = \frac{1}{\rho c}\left(p_+\, e^{-jkx} - p_-\, e^{jkx}\right)$$

The specific acoustic impedance can therefore be written as :

$$Z_s(x) = \frac{p(x)}{v(x)} = \underbrace{\rho c}_{Z_c}\,\frac{p_+\, e^{-jkx} + p_-\, e^{jkx}}{p_+\, e^{-jkx} - p_-\, e^{jkx}}$$

where Z_c is the characteristic impedance of the fluid medium.

2. General expression for the sound reflection coefficient

$$r(x) = \frac{p_-\, e^{jkx}}{p_+\, e^{-jkx}} = \frac{p_-}{p_+}\, e^{2jkx}$$

3. Express $Z_s(x)$ as a function of $r(x)$

$$Z_s(x) = Z_c\,\frac{p_+\, e^{-jkx} + p_-\, e^{jkx}}{p_+\, e^{-jkx} - p_-\, e^{jkx}} = Z_c\,\frac{1 + \frac{p_-\, e^{jkx}}{p_+\, e^{-jkx}}}{1 - \frac{p_-\, e^{jkx}}{p_+\, e^{-jkx}}} = Z_c\,\frac{1 + r(x)}{1 - r(x)}$$

4. Derive the expression of Z_s at $x = L$ when $r(x = L) = 0$
$Z_s(x = L) = Z_c\,\dfrac{1 + r(x = L)}{1 - r(x = L)} = Z_c$

5. if $r(x = L) = 0$, then $\alpha = 1 - |r(x = L)|^2 = 1$.
A material with a specific acoustic impedance equal to Z_c can be qualified as perfectly absorbent.

6. In the cas where $Z_s(x = L) = Z_c$, then the general expression of the sound pressure inside the duct simplifies to give :

$$p(x,t) = p_+\, e^{-jkx}\, e^{\mathrm{j}\omega t}$$

4 Sound levels

1. The sound power level is given by :
   $L_w = 10\log\frac{P}{P_0} = 10\log\frac{3}{10^{-12}} = 125$ dB
2. $I = P_a/S$ with $S = 4\pi r^2$. The sound intensity for the source $d_1 = 5$ m :
   $\mathrm{I}_1(5\text{ m}) = (4\pi 5^2)^{-1}\cdot 3 = 9.55\cdot 10^{-3}\quad \mathrm{W{\cdot}m^{-2}}$
   For $d_2 = 10$ m :
   $\mathrm{I}_2(10\text{ m}) = (4\pi 10^2)^{-1}\cdot 3 = 2.39\cdot 10^{-3}\quad \mathrm{W{\cdot}m^{-2}}$
3. The sound intensity for $d_1 = 5$ m and $d_2 = 10$ m : $L_{I1}(5\text{ m}) = 10\log\frac{I_1}{I_0} = 100$ dB
   $L_{I2}(10\text{ m}) = 94$ dB

5 Sound levels and acoustic quantities

1. The sound pressure level is given by $L_p = 20\log_{10}(p/p_0)$. The sound pressure level of another source with a given relative sound pressure $p' = C \cdot p$ can be computed as $L_{p'} = 20\log_{10}(p'/p_0) = 20\log_{10}(C \cdot p/p_0) = 20\log_{10}(C) + 20\log_{10}(p/p_0) = 20\log_{10}(C) + L_p$.
   For a source with $p' = 2$ Pa, $L_{p'} = 94 + 6 = 100$ dB.
   For $p' = 0.1$ Pa, $L_{p'} = 74$ dB.
   For $p' = 10$ Pa, $L_{p'} = 114$ dB.
2. $I = \tilde{p}^2/Z_c = \tilde{p}^2/(\rho_0 c) \rightarrow 2.5 \cdot 10^{-3}$ W/m^2.
3. $L_I = 10\log_{10}(I/I_0) \rightarrow 94$ dB. The same calculations can be made as for sound pressure level, so if $I' = 2I$, $L_{I'} = L_I + 3$ dB.

6 Addition and substraction of decibels

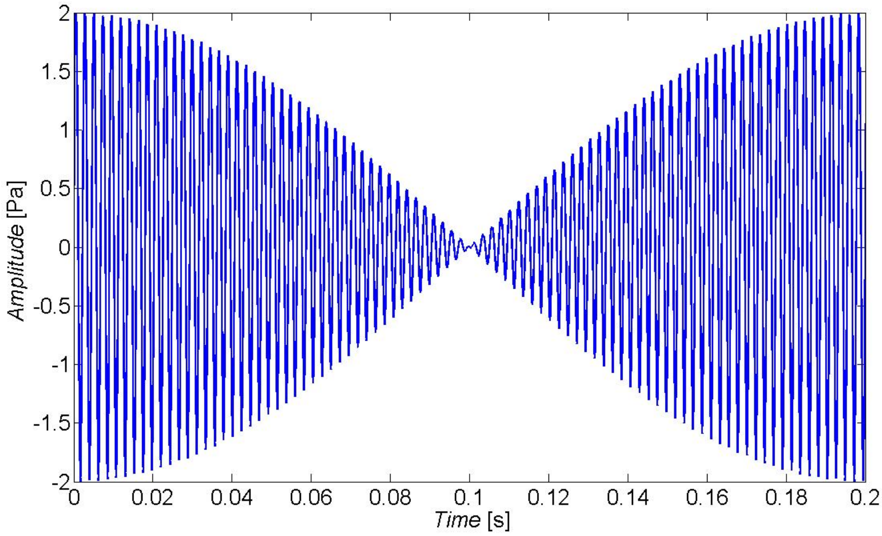
$$L_p(\text{car}_1 + \text{car}_2) = 10\log_{10}\left(\frac{I(\text{car}_1) + I(\text{car}_2)}{I_0}\right) = 10\log_{10}\left(10^{\frac{L_p(\text{car}_1)}{10}} + 10^{\frac{L_p(\text{car}_2)}{10}}\right) \quad (11)$$

The combined sound pressure level is therefore $L_p = 81.8$ dB since the two sources are considered to be uncorrelated and intensities must therefore be summed.
The sound pressure level of the motorcycle can be computed as

$$L_p(\text{motorcycle}) = 10\log_{10}\left(10^{\frac{L_p(\text{car}_1+\text{car}_2+\text{motorcycle})}{10}} - 10^{\frac{L_p(\text{car}_1)}{10}} - 10^{\frac{L_p(\text{car}_2)}{10}}\right) \rightarrow 80.0 \text{ dB} \quad (12)$$

7 Beats

The total sound pressure reads : $p(t) = sin(2\pi f_1 t) + sin(2\pi(f_1 + \Delta f)t)$
Since $sina + sinb = sin\frac{a+b}{2}.cos\frac{a-b}{2}$, then $p(t) = 2sin(2\pi(f + \frac{\Delta f}{2})t).cos\pi\Delta f t$
For $f_1 = 440$ Hz and $f_2 = 445$ Hz, we have “beats” : amplitude modulation of a pure tone at $\frac{f_1+f_2}{2}$, with modulation frequency of $\frac{\Delta f}{2}$:



If the two frequencies are far enough, we have two behaviors :

— if f_2 is close to a multiple of f_1, we still have beats

— in other cases, we do not have beats

8 Pythagore's scale

— The Do one octave higher (f_{12}) corresponds to a doubling of frequency f_0 : $f_{12} = 2f_0$. If all intervals $\frac{f_{i+1}}{f_i}$ are equal, it yields : $\frac{f_{12}}{f_0} = (\frac{f_{i+1}}{f_i})^{12} = 2$, then : $\frac{f_{i+1}}{f_i} = 2^{1/12}$

— $\frac{f_7}{f_0} = 2^{7/12} = 1.4983$

— see table below

— see table below

Note	Equa-temperament scale		Pythagore's scale	
	interval	frequency	interval	frequency
Do	1	262 Hz	1	262 Hz
Do#	1.0595	277.6 Hz	1.0679	279.8 Hz
R	1.1225	294.1 Hz	1.1250	294.7 Hz
R#	1.1892	311.6 Hz	1.2014	314.7 Hz
Mi	1.2599	330.1 Hz	1.2656	331.6 Hz
Fa	1.3348	349.7 Hz	1.3515	354.1 Hz
Fa#	1.4142	370.5 Hz	1.4238	373.0 Hz
Sol	1.4983	392.6 Hz	1.5000	393.0 Hz
Sol#	1.5874	415.9 Hz	1.6018	419.7 Hz
La	1.6818	440.6 Hz	1.6875	442.1 Hz
La#	1.7818	466.8 Hz	1.8020	472.1 Hz
Si	1.8877	494.6 Hz	1.8984	497.4 Hz