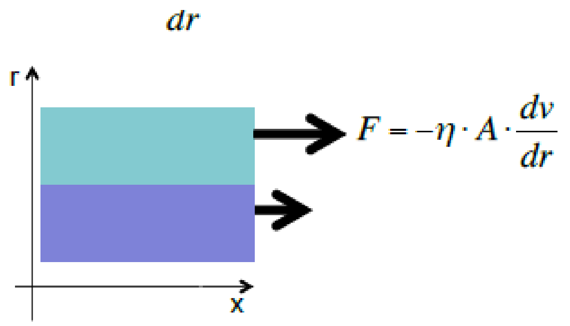
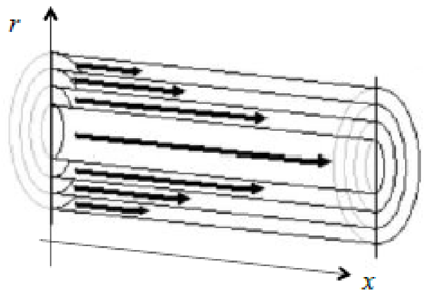
# Annex: Calculation of Hydraulic Resistance using Poiseuille's Law

Let us consider a fluid flowing through a pipe. We assume that the flow is laminar. The force due to viscosity between the two layers of fluid shown in the figure below is given by:

$$F = -\eta A \frac{dv}{dr}$$




Where

$\eta$: viscosity of the fluid

$v(r)$: fluid velocity as a function of pipe-radius $r$

For a layer:

$$F_{pressure} + F_r + F_{r+dr} = 0 \qquad (1)$$

The force due to viscosity can be written as:

$$F_r = -\eta * 2\pi r * \Delta x * \left.\frac{dv}{dr}\right|_r$$

$$F_{r+dr} = +\eta * 2\pi(r + dr) * \Delta x * \left.\frac{dv}{dr}\right|_{r+dr}$$

Putting these expressions in (1), we obtain:

$$-\Delta P * 2\pi r dr - \eta * 2\pi r * \Delta x * \left.\frac{dv}{dr}\right|_r + \eta * 2\pi(r + dr) * \Delta x * \left.\frac{dv}{dr}\right|_{r+dr} = 0$$

For small variations over $dr$, we can write the following using a 1st order Taylor series approximation:

$$\left.\frac{dv}{dr}\right|_{r+dr} = \left.\frac{dv}{dr}\right|_r + \left.\frac{d^2v}{dr^2}\right|_r * dr$$

Using this approximation, and ignoring $(dr)^2$ terms, we obtain:

$$-\Delta P * 2\pi r dr + \eta * 2\pi dr * \Delta x * \frac{dv}{dr} + \eta * 2\pi r dr * \Delta x * \frac{d^2v}{dr^2} = 0$$

$$\frac{1}{\eta} * \frac{\Delta P}{\Delta x} = \frac{d^2 v}{dr^2} + \frac{1}{r} * \frac{dv}{dr}$$

The solution of this differential equation is of the form: $v = A + Br^2$. Putting this form back into the above equation, we obtain:

$$\frac{1}{\eta} * \frac{\Delta P}{\Delta x} = 2B + \frac{1}{r} * 2Br$$

$$B = \frac{1}{4\eta} * \frac{\Delta P}{\Delta x}$$

And, applying the boundary condition at the wall that $v(r = R) = 0$, we obtain:

$$A = -\frac{1}{4\eta} * \frac{\Delta P}{\Delta x} * R^2$$

Thus, the final expression for the velocity profile within the pipe becomes:

$$\boldsymbol{v = -\frac{1}{4\eta} * \frac{\Delta P}{\Delta x} * (R^2 - r^2)}$$

Note that the maximum velocity is obtained at the centre of the pipe, where $r = 0$. Thus, $v_{max} = -\frac{1}{4\eta} * \frac{\Delta P}{\Delta x} * R^2$.

Now, the flow rate is give by:

$$Q(r) = \frac{dV}{dt}$$

$$dQ = v * 2\pi r dr = \frac{1}{4\eta} * \frac{|\Delta P|}{\Delta x} * (R^2 - r^2) * 2\pi r dr = \frac{\pi}{4\eta} * \frac{|\Delta P|}{\Delta x} * (rR^2 - r^3) * dr$$

$$Q = \frac{\pi}{4\eta} * \frac{|\Delta P|}{\Delta x} * \int_0^R (rR^2 - r^3) * dr = \frac{\pi R^4}{8\eta} * \frac{|\Delta P|}{\Delta x}$$

$$\boldsymbol{Q = \frac{\pi R^4}{8\eta l} * \Delta P}$$

We can thereby deduce the resistance of the pipe as being the ratio of the pressure drop across the length of the pipe to the flow rate through the pipe, as follows:

$$\boldsymbol{R = \frac{\Delta P}{Q} = \frac{8\eta l}{\pi R^4}}$$