Smart Grids Technologies
Module 1, Lab 1 - 24/02/2025
DFT-based Synchrophasor Estimation: DFT, Aliasing and Spectral Leakage

1 Organization of the lab

This laboratory session will cover the fundamentals of DFT-based synchrophasor estimation, including DFT computation, aliasing and spectral leakage. The laboratory will be divided into two parts. **Part A** will expand on Lecture #1 and #2, exploring the following topics:

- The meaning of the Discrete Fourier Transform (DFT);
- How to select the DFT parameters (i.e., the sampling rate F_s and the window length T) to analyse voltage and current waveforms typical of power systems;
- How to avoid/reduce aliasing.

Part B relates to Lecture #3 and focuses on the following topics:

- How to apply special window functions to DFT;
- How special window functions can reduce the effects of spectral leakage;
- How to select the most-suited window function.

At the end of this laboratory session you will develop instruments and practical knowledge that will be used during the next lab sessions.

This report will not be graded; however, its submission is mandatory. The purpose of the questions within this document is to enhance your comprehension of the subject matter. Your acquired knowledge from **all three laboratories of Module 1** will be evaluated in a quiz scheduled for **Monday, March 17th** from **9:15 to 10:00**. The deadline for submission of the reports is **Sunday, March 16th** at **23:55**.

2 Theoretical Background

2.1 Signal sampling and aliasing

As discussed in Lecture #1 and #2, in order to implement signal processing algorithms, the analog signal representing a generic power system quantity has to be converted into its digital equivalent by *sampling*. The sampled signal is ideally represented by an array of equally-spaced samples by the discrete sampling time $T_s = 1/F_s$.

To be able to reconstruct the analog signal based on the acquired samples, the signal has to be sampled correctly. In particular, the main source of error related to sampling is called *aliasing* which may cause the frequency replica of the spectrum image to overlap. Based on *Nyquist-Shannon sampling theorem*, in order to prevent aliasing, the input signal needs to be sampled with a sampling frequency that is at least 2 times higher than the maximum frequency component of the signal (Lecture #2):

$$F_s \geq 2F_m \tag{1}$$

2.2 Discrete Fourier Transform

The Discrete Fourier Transform (DFT) is a method used to determine the frequency content of a discrete signal sequence, obtained by periodic sampling of a continuous signal in time-domain. As covered by Lectures #1 and #2, the DFT implements a Fourier transform at a discrete set of frequencies based on the choice of 2 parameters:

- the sampling rate F_s
- the window length T

In particular the DFT elements are separated by the frequency interval Δf, also called frequency resolution (Lecture #2):

$$\Delta f = \frac{1}{T} = \frac{1}{N \cdot T_s} = \frac{F_s}{N} \tag{2}$$

being N the number of samples per window and T_s the sampling time.

The DFT for frequency bin k is computed as follows (Lecture #2, slide 10):

$$X(k) = \frac{2}{B} \sum_{n=0}^{N-1} w(n) \cdot x(n) \cdot W_N^{kn},\ k \in [0, N-1] \tag{3}$$

where $x(n)$ is the sampled signal under analysis, $w(n)$ is the discrete windowing function, k and n are the indexes of the frequency bins and time-domain samples, respectively, B is the normalization factor:

$$B = \sum_{n=0}^{N-1} w(n) \tag{4}$$

and W_N is the twiddle factor:

$$W_N = e^{-j2\pi/N} = \cos(2\pi/N) - j\sin(2\pi/N) \tag{5}$$

$$W_N^{kn} = e^{-j2\pi kn/N} = \cos(2\pi kn/N) - j\sin(2\pi kn/N) \tag{6}$$

Note that the DFT can be expressed in terms of the real and imaginary components as follows:

$$\mathbb{R}\mathrm{e}\left\{X\left(k\right)\right\} = \frac{2}{B} \sum_{n=0}^{N-1} x\left(n\right) \cdot w\left(n\right) \cdot \cos\left(2\pi kn/N\right), \tag{7}$$

$$\mathbb{I}\mathrm{m}\left\{X\left(k\right)\right\} = -\frac{2}{B} \sum_{n=0}^{N-1} x\left(n\right) \cdot w\left(n\right) \cdot \sin\left(2\pi kn/N\right) \tag{8}$$

2.3 Signal windowing

When you use the DFT to measure the frequency content of your data, you will have to base the analysis on a finite set of data that can be properly processed by existing computers. *Windowing* is a technique used to section the measurement data into finite-length portions, and you will learn more about it in Lecture #3.

The DFT assumes both the time domain and the frequency domain representations as circular topologies, meaning that the two endpoints of the time waveform are interpreted as though they were connected together. However, the finiteness of the sampling record may result in a truncated waveform with different spectral characteristics from the original continuous-time signal, and the finiteness can introduce discontinuities into the measured data. To minimise their effect, we can apply a *special windowing* *function* to the measured signal in the time domain. This will make the endpoints of the waveform meet and therefore result in a continuous waveform without sharp transitions.

There are different types of window functions available, each with their own advantages and preferred application. Most windows are bell-shaped, beginning and ending at zero and rising to unity in the middle. Generally, the narrowest windows in the time domain have the widest main lobes in the frequency domain, and vice-versa. During this laboratory we will use the *rectangular* window, defined as:

$$w(n) = 1, \quad n \in [0, N-1] \tag{9}$$

and the *Hanning (Hann)* window defined as:

$$w(n) = 0.5 * (1 - \cos(2\pi n/N)), \quad n \in [0, N-1] \tag{10}$$

Based on the convolution theorem, the DFT of the windowed signal exhibits a pair of scaled, shifted and rotated versions of the DFT of the window function: the so-called *positive image* shifted up to the tone frequency f_0, the so-called *negative image* shifted down to $-f_0$ (Lecture #3):

$$X(k) = X_0^+(k) + X_0^-(k) \tag{11}$$

2.4 Spectral leakage

The DFT assumes that the signal is **coherently sampled**, that is to say that the finite data set under analysis contains an integer number of periods of a periodic signal. When this condition is not met (this happens almost all the time) *spectral leakage* arises. You will learn more about it in Lecture #3.

In the case of **incoherent sampling**, the sampling process is not synchronised with the fundamental tone under analysis ($f_0/\Delta f \notin \mathbb{N}$) and, therefore, the DFT bins are not aligned with the signal frequency. This impacts the DFT values of the main lobe (i.e., scalloping loss) and results in nonzero DFT components for all frequency values (i.e., long range spectral leakage). The tails of the negative image of the spectrum tones also leak into the positive frequency range and bias the DFT bins used to perform any signal processing technique (for instance, interpolation). In case of multi-tone signals, this spectral interference is even more severe as it is *replicated for each tone* in the signal. Therefore, the negative and the positive images of all tones may overlap.

To minimize the effects of spectral leakage, we can apply a special windowing function to the acquired signal in the time domain in order to minimize the window's edge effects. The advantage is that in the frequency-domain the tails of every tone image will be smaller, therefore *long range spectral leakage* effect will be minimized. The drawback is that the main lobe of every tone image will become wider which might make it challenging to distinguish two adjacent tones.

2.5 References (in Additional Material)

- Chapter 3 "*DFT-based synchrophasor estimation processes for Phasor Measurement Units applications: algorithms definition and performance analysis*", in the book *"Advanced Techniques for Power System Modelling, Control and Stability Analysis"* edited by F. Milano, IET 2015.

3 LabVIEW Coding

From Moodle, download and **extract** the folder *"SGT-PMUs - Lab1"* and open the LabVIEW project called *"SGT-PMUs.lvproj"*. At `MyComputer` level, this project includes a VI called *"Lab1.vi"*, whereas at `MyComputer/Dependencies` level it includes VIs called *"DFT_bin.vi"* and *"Windowing.vi"*.

TASK 1 - DFT bin computation: Open the subVI called *"DFT_bin.vi"*. Use the preconfigured input/output layout to implement a subVI that computes the k-th DFT bin of the spectrum of a signal using a set of samples representing a portion of an acquired waveform.

TASK 2 - Windowing: Open the subVI called *"Windowing.vi"*. Use the preconfigured input/output configuration to implement a subVI that produces the rectangular and the Hanning window profiles.

4 Exercises

4.1 Part A

Open the VI called *"Lab1.vi"*. For **Part A**, let us only consider the *Analyze signals* tab and the rectangular windowing function. For part A, focus on the **rectangular window (blue)**. **Fix the sampling rate at 500 Hz.**

Q1/ Generate 100 ms of a single-tone signal at 50 Hz. Describe the spectrum and explain its relevant properties (Δf, phase, etc.)

[A1]

Q2/ Generate 100 ms of a multi-tone signal with a fundamental tone at 50 Hz and a 10% harmonic tone at 100 Hz. Describe the spectrum and its relevant properties.

[A2]

Q3/ Generate 100 ms of a multi-tone signal with a fundamental tone at 50 Hz and a 10% harmonic tone at 400 Hz. a) Describe the spectrum and explain why there is a tone at 100 Hz. What phenomena is present? b) How can this analysis be improved?

[A3]

Q4/ Generate a 50 Hz single-tone signal and change the window length in the range (20:dt:200) ms where dt=20 ms. How does the window length affect the spectrum?

[A4]

4.2 Part B

In **Part B**, we will compare the performance of the **rectangular (blue)** and of the **Hanning (red)** windowing functions. **Set the sampling frequency to 5 kHz and the window length to 100 ms**. Run the vi and compute the DFT spectra of the two window functions.

Now open the *Window function* tab. As you can see in the block diagram of *"Lab1.vi"*, the window length N is increased by appending to the window function $10 \cdot N$ zeros. This operation is called "zero-padding" and enables us to increase the frequency resolution of the DFT spectrum for visual appeal (the spectrum is not actually improved as no new information is added).

Q5/ Describe the differences between the two windows. How do you suspect these differences will affect the DFT analysis?

[A5]

Now switch back to the *Analyze signals* tab. **Set the window length to 60 ms.** Note that the magnitude of the three highest amplitude bins is provided in the bottom-left table. When answering the following questions, focus on the magnitude rather than on the phase.

Q6/ Generate a single-tone signal at 50 Hz. Describe the differences in the spectra and report the values of the highest bin for each window. In this situation, is there an advantage to use one window over the other? Justify your answer.

[A6]

Q7/ Generate a single-tone signal at 51.8 Hz. Describe the differences in the spectra and report the values of the highest bin for each window. In this situation, is there an advantage to use one window over the other? Justify your answer.

[A7]

Q8/ Generate a single-tone at 47.9 Hz. Change the window length to 20, 60, and 100 ms. Describe the differences in the spectra and discuss the impact of the window length and the windowing function.

[A8]

Q9/ Generate 60 ms of a multi-tone signal with a fundamental tone at 47.9Hz and a 10% inter-harmonic tone at 80 Hz. Describe the spectra. Which window is better for this analysis? Justify your answer.

[A9]

Q10/ Generate 60 ms of a multi-tone signal with a fundamental tone at 47.9 Hz and a 10% inter-harmonic tone at 25 Hz. Which window is better for this analysis? How does the spectrum change compared to Q9? Give an explanation.

[A10]

What are the take-home messages you have learned during this laboratory session regarding the practical implementation of DFT-based signal processing tools, with respect to the following phenomena?

Q11/ Aliasing.

[A11]

Q12/ Incoherent sampling.

[A12]

Q13/ Windowing.

[A13]

Q14/ Spectral leakage.

[A14]