Phasor Measurement Units

Discrete Fourier Transform

$$X = |X| \cdot e^{j\psi} = A \cdot e^{j\psi} = A \cdot [\cos(\psi) + j\sin(\psi)]$$

- $|X| = A = \sqrt{X_{Re}{}^2 + X_{Im}{}^2}$
- $\arg(X) = \psi = \tan^{-1}\left(\frac{X_{Im}}{X_{Re}}\right)$

$$X(k) \triangleq \frac{2}{B}\sum_{n=0}^{N-1} w(n)x(n)W_N^{kn}, 0 \le k \le N-1$$

- Normalization factor $B = \sum_{n=0}^{N-1} w(n)$
- Twiddle factor $W_N = e^{-j2\pi/N}$

Error Metrics

$$FE = |f_{meas} - f_{true}|$$

$$RFE = \left|(\frac{df}{dt})_{meas} - (\frac{df}{dt})_{true}\right|$$

$$TVE = \sqrt{\frac{(\hat{X}_{Re} - X_{Re})^2 + (\hat{X}_{Im} - X_{Im})^2}{X_{Re}^2 + X_{Im}^2}}$$

- Estimated phasor $\hat{X}_{Re} + j\,\hat{X}_{Im}$
- True phasor $X_{Re} + jX_{Im}$

IpDFT - Rectangular Window

$$w_R(n) = 1, \qquad n \in [0, N-1]$$

$$\hat{\delta} = \varepsilon \frac{|X(k_m + \varepsilon)|}{|X(k_m)| + |X(k_m + \varepsilon)|}$$

- $\varepsilon = \pm 1 = sign(|X(k_m + 1)| - |X(k_m - 1)|)$

$$\hat{f}_0 = (k_m + \hat{\delta})\Delta f$$

$$\hat{A}_0 = |X(k_m)| \left|\frac{\pi\hat{\delta}}{\sin(\pi\hat{\delta})}\right|$$

$$\hat{\varphi}_0 = \angle X(k_m) - \pi\hat{\delta}$$

IpDFT - Hann Window

$$w_H(n) = \frac{1 - \cos\left(\frac{2\pi n}{N}\right)}{2}, n \in [0, N-1]$$

$$\hat{\delta} = \varepsilon \frac{2|X(k_m + \varepsilon)| - |X(k_m)|}{|X(k_m)| + |X(k_m + \varepsilon)|}$$

- $\varepsilon = \pm 1 = sign(|X(k_m + 1)| - |X(k_m - 1)|)$

$$\hat{f}_0 = (k_m + \hat{\delta})\Delta f$$

$$\hat{A}_0 = |X(k_m)| \left|\frac{\pi\hat{\delta}}{\sin(\pi\hat{\delta})}\right| |\hat{\delta}^2 - 1|$$

$$\hat{\varphi}_0 = \angle X(k_m) - \pi\hat{\delta}$$

Time Synchronization

Time error: $\Delta t(t) = a + b \cdot t + D_r \cdot \frac{t^2}{2} + \epsilon(t)$

- a – initial time error
- $b = \Delta F_s / F_s$ – normalized frequency error
- D_r – frequency drift constant
- $\epsilon(t)$ – noise

PTP offset (o) and propagation delay (d)

$$o = \frac{(t_{s1} - t_{m1}) - (t_{m2} - t_{s2})}{2}$$

$$d = \frac{(t_{s1} - t_{m1}) + (t_{m2} - t_{s2})}{2}$$

- t_{s1}, t_{m1} – sync message timestamps
- t_{s2}, t_{m2} – delay_req message timestamps

Nodal Admittance Matrix and Per-Unit

$$\bar{\mathbf{Y}} = \mathbf{A}_{\mathfrak{B}}^{T}\bar{\mathbf{Y}}_{\mathcal{L}}\mathbf{A}_{\mathfrak{B}} + \bar{\mathbf{Y}}_{\mathcal{T}}$$

Single-Phase Case

$$\bar{Y}_{nn} = \bar{Y}_{t_n} + \sum_{\ell_k=(n,\cdot)} \bar{Y}_{\ell_k} + \sum_{\ell_k=(\cdot,n)} \bar{Y}_{\ell_k} \ (n \in \mathcal{N})$$

$$\bar{Y}_{mn} = \begin{cases} -\bar{Y}_{\ell_k} \ (\exists \ell_k = (m,n) \text{ or } \exists \ell_k = (n,m)) \\ 0 \qquad\qquad (\text{otherwise}) \end{cases}$$

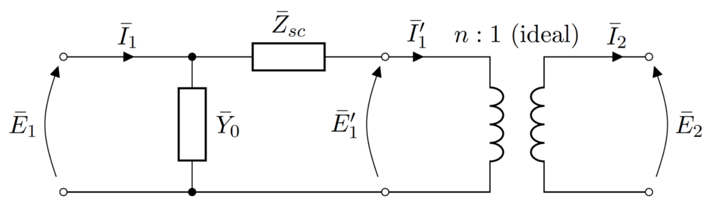
$(m, n \in \mathcal{N},\ m \neq n)$

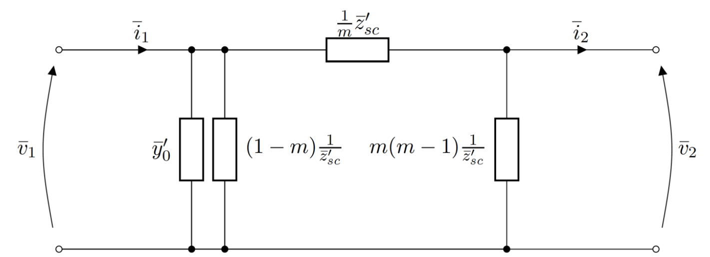
- $\mathbf{A}_{\mathfrak{B}}$ – incidence matrix with elements
  - $a_{kn} = \begin{cases} +1 \ (\text{if } \ell_k = (n,\cdot) \in \mathcal{L}) \\ -1 \ (\text{if } \ell_k = (\cdot,n) \in \mathcal{L}) \\ 0 \qquad (\text{otherwise}) \end{cases}$

- $\bar{\mathbf{Y}}_{\mathcal{L}} = \mathrm{diag}_{\ell_k \in \mathcal{L}}(\bar{Y}_{\ell_k})$ – primitive branch admittance matrix
- $\bar{\mathbf{Y}}_{\mathcal{T}} = \mathrm{diag}_{t_n \in \mathcal{T}}(\bar{Y}_{t_n})$ – primitive shunt admittance matrix
- $\bar{Y}_{\ell_k}$ – branch admittances
- $\bar{Y}_{t_n}$ – shunt admittances
- $\mathcal{N}$ – set of nodes
- $\mathcal{L}$ – set of lines

- A_b – per-phase base power
- V_b – phase-to-ground base voltage

$$I_b = \frac{A_b}{V_b}, \qquad Z_b = \frac{V_b}{I_b} = \frac{V_b^2}{A_b}$$





$$m = n\left(\frac{V_{b1}}{V_{b2}}\right)^{-1}$$

- $\bar{Z}_{sc}$ – short-circuit impedance
- $\bar{Y}_0$ – no-load admittance

Three-Phase Case

- $\mathbf{A}_{\mathfrak{B}}$ – incidence matrix with elements
  - $\mathbf{A}_{\mathfrak{B},\mathrm{kn}} = \begin{cases} +\mathbf{I}_3 \ (\text{if } \ell_k = (n,\cdot) \in \mathcal{L}) \\ -\mathbf{I}_3 \ (\text{if } \ell_k = (\cdot,n) \in \mathcal{L}) \\ 0 \qquad (\text{otherwise}) \end{cases}$

$\bar{\mathbf{Y}}_{\mathcal{L}} = \mathrm{diag}_{\ell_k \in \mathcal{L}}(\bar{\mathbf{Y}}_{\ell_k})$ – primitive shunt admittance (block) matrix

$\bar{\mathbf{Y}}_{\mathcal{T}} = \mathrm{diag}_{t_n \in \mathcal{T}}(\bar{\mathbf{Y}}_{t_n})$ – primitive shunt admittance (block) matrix

- $\bar{\mathbf{Y}}_{\ell_k}$ – 3x3 branch admittance matrix
- $\bar{\mathbf{Y}}_{t_n}$ – 3x3 shunt admittance matrix
- $\mathcal{N}$ – set of three-phase (a,b,c) nodes
- $\mathcal{L}$ – set of three-phase (a,b,c) lines

$$\bar{\mathbf{Y}}_{\ell_k} = \begin{bmatrix} \bar{Y}_{\ell_k,aa} & \bar{Y}_{\ell_k,ab} & \bar{Y}_{\ell_k,ac} \\ \bar{Y}_{\ell_k,ba} & \bar{Y}_{\ell_k,bb} & \bar{Y}_{\ell_k,bc} \\ \bar{Y}_{\ell_k,ca} & \bar{Y}_{\ell_k,cb} & \bar{Y}_{\ell_k,cc} \end{bmatrix}$$

$$\bar{\mathbf{Y}}_{t_n} = \begin{bmatrix} \bar{Y}_{t_n,aa} & \bar{Y}_{t_n,ab} & \bar{Y}_{t_n,ac} \\ \bar{Y}_{t_n,ba} & \bar{Y}_{t_n,bb} & \bar{Y}_{t_n,bc} \\ \bar{Y}_{t_n,ca} & \bar{Y}_{t_n,cb} & \bar{Y}_{t_n,cc} \end{bmatrix}$$

- A_b – three-phase power base power
- V_b – phase-to-phase base voltage

$$I_b = \frac{A_b}{\sqrt{3}V_b}, Z_b = \frac{V_b}{\sqrt{3}I_b} = \frac{V_b^2}{A_b}$$

Numerical Solution of the Load-Flow

Polar

$\bar{V}_i = V_i e^{j\vartheta_i}$ - voltage at the *i*-th node, s - number of nodes

$\bar{Y}_{i\ell} = Y_{i\ell} e^{j\gamma_{i\ell}}$ - element *il* of the admittance matrix $[\mathbf{Y}]$

$$\begin{cases} P_i = \sum_{\ell=1}^{s} V_i V_\ell Y_{i\ell} \cos\left(\vartheta_i - \vartheta_\ell - \gamma_{i\ell}\right) \\ Q_i = \sum_{\ell=1}^{s} V_i V_\ell Y_{i\ell} \sin\left(\vartheta_i - \vartheta_\ell - \gamma_{i\ell}\right) \end{cases}$$

$$\begin{bmatrix} J_{PV} & J_{P\vartheta} \\ J_{QV} & J_{Q\vartheta} \end{bmatrix}^{(\nu)} \times \begin{bmatrix} \Delta V \\ \Delta\vartheta \end{bmatrix}^{(\nu+1)} = \begin{bmatrix} \Delta P \\ \Delta Q \end{bmatrix}^{(\nu)}$$

$$J_{PV}: \begin{cases} \dfrac{\partial P_i}{\partial V_\ell} = Y_{i\ell} V_i \cos\left(\vartheta_i - \vartheta_\ell - \gamma_{i\ell}\right) \\ \dfrac{\partial P_i}{\partial V_i} = 2Y_{ii} V_i \cos\gamma_{ii} + \sum_{\substack{\ell=1 \\ \ell\neq i}}^{s} Y_{i\ell} V_\ell \cos\left(\vartheta_i - \vartheta_\ell - \gamma_{i\ell}\right) \end{cases}$$

$$J_{P\vartheta}: \begin{cases} \dfrac{\partial P_i}{\partial \vartheta_\ell} = Y_{i\ell} V_i V_\ell \sin\left(\vartheta_i - \vartheta_\ell - \gamma_{i\ell}\right) \\ \dfrac{\partial P_i}{\partial \vartheta_i} = -V_i \sum_{\substack{\ell=1 \\ \ell\neq i}}^{s} Y_{i\ell} V_\ell \sin\left(\vartheta_i - \vartheta_\ell - \gamma_{i\ell}\right) \end{cases}$$

$$J_{QV}: \begin{cases} \dfrac{\partial Q_i}{\partial V_\ell} = Y_{i\ell} V_i \sin\left(\vartheta_i - \vartheta_\ell - \gamma_{i\ell}\right) \\ \dfrac{\partial Q_i}{\partial V_i} = -2Y_{ii} V_i \sin\gamma_{ii} + \sum_{\substack{\ell=1 \\ \ell\neq i}}^{s} Y_{i\ell} V_\ell \sin\left(\vartheta_i - \vartheta_\ell - \gamma_{i\ell}\right) \end{cases}$$

$$J_{Q\vartheta}: \begin{cases} \dfrac{\partial Q_i}{\partial \vartheta_\ell} = -Y_{i\ell} V_i V_\ell \cos\left(\vartheta_i - \vartheta_\ell - \gamma_{i\ell}\right) \\ \dfrac{\partial Q_i}{\partial \vartheta_i} = V_i \sum_{\substack{\ell=1 \\ \ell\neq i}}^{s} Y_{i\ell} V_\ell \cos\left(\vartheta_i - \vartheta_\ell - \gamma_{i\ell}\right) \end{cases}$$

Cartesian

$\bar{V}_i = V_i' + jV_i''$ - voltage at the *i*-th node;

$\bar{Y}_{i\ell} = G_{i\ell} + jB_{i\ell}$ - element *il* of the admittance matrix $[\mathbf{Y}]$

$$J_{PR}: \begin{cases} \dfrac{\partial P_i}{\partial V_\ell'} = G_{i\ell} V_i' + B_{i\ell} V_i'' \\ \dfrac{\partial P_i}{\partial V_i'} = 2G_{ii} V_i' + \sum_{\substack{\ell=1 \\ \ell\neq i}}^{s} \left(G_{i\ell} V_\ell' - B_{i\ell} V_\ell''\right) \end{cases}$$

$$J_{PX}: \begin{cases} \dfrac{\partial P_i}{\partial V_\ell''} = -B_{i\ell} V_i' + G_{i\ell} V_i'' \\ \dfrac{\partial P_i}{\partial V_i''} = 2G_{ii} V_i'' + \sum_{\substack{\ell=1 \\ \ell\neq i}}^{s} \left(B_{i\ell} V_\ell' + G_{i\ell} V_\ell''\right) \end{cases}$$

$$\begin{cases} P_i = V_i' \sum_{\ell=1}^{s} \left(G_{i\ell} V_\ell' - B_{i\ell} V_\ell''\right) + V_i'' \sum_{\ell=1}^{s} \left(B_{i\ell} V_\ell' + G_{i\ell} V_\ell''\right) \\ Q_i = -V_i' \sum_{\ell=1}^{s} \left(B_{i\ell} V_\ell' + G_{i\ell} V_\ell''\right) + V_i'' \sum_{\ell=1}^{s} \left(G_{i\ell} V_\ell' - B_{i\ell} V_\ell''\right) \end{cases}$$

$$J_{QR}: \begin{cases} \dfrac{\partial Q_i}{\partial V_\ell'} = -B_{i\ell} V_i' + G_{i\ell} V_i'' \\ \dfrac{\partial Q_i}{\partial V_i'} = -2B_{ii} V_i' - \sum_{\substack{\ell=1 \\ \ell\neq i}}^{s} \left(B_{i\ell} V_\ell' + G_{i\ell} V_\ell''\right) \end{cases}$$

$$J_{QX}: \begin{cases} \dfrac{\partial Q_i}{\partial V_\ell''} = -G_{i\ell} V_i' - B_{i\ell} V_i'' \\ \dfrac{\partial Q_i}{\partial V_i''} = -2B_{ii} V_i'' + \sum_{\substack{\ell=1 \\ \ell\neq i}}^{s} \left(G_{i\ell} V_\ell' - B_{i\ell} V_\ell''\right) \end{cases}$$

$$\begin{bmatrix} J_{PR} & J_{PX} \\ J_{QR} & J_{QX} \\ J_{VR} & J_{VX} \end{bmatrix}^{(\nu)} \times \begin{bmatrix} \Delta V' \\ \Delta V'' \end{bmatrix}^{(\nu+1)} = \begin{bmatrix} \Delta P \\ \Delta Q \\ \Delta\left(V^2\right) \end{bmatrix}^{(\nu)}$$

$$J_{VR}: \begin{cases} \dfrac{\partial\left(V_i^2\right)}{\partial V_\ell'} = 0 \\ \dfrac{\partial\left(V_i^2\right)}{\partial V_i'} = 2V_i' \end{cases} \qquad J_{VX}: \begin{cases} \dfrac{\partial\left(V_i^2\right)}{\partial V_\ell''} = 0 \\ \dfrac{\partial\left(V_i^2\right)}{\partial V_i''} = 2V_i'' \end{cases}$$

Mixed

$$\begin{cases} P_i = V_i \sum_{l=1}^{s} V_l (G_{il} \cos\vartheta_{il} + B_{il} \sin\vartheta_{il}) \\ Q_i = V_i \sum_{l=1}^{s} V_l (G_{il} \sin\vartheta_{il} - B_{il} \cos\vartheta_{il}) \end{cases} \qquad \vartheta_{il} = \vartheta_i - \vartheta_l$$

$$\begin{bmatrix} J_{PV} & J_{P\vartheta} \\ J_{QV} & J_{Q\vartheta} \end{bmatrix}^{(\nu)} \times \begin{bmatrix} \Delta V \\ \Delta\vartheta \end{bmatrix}^{(\nu+1)} = \begin{bmatrix} \Delta P \\ \Delta Q \end{bmatrix}^{(\nu)}$$

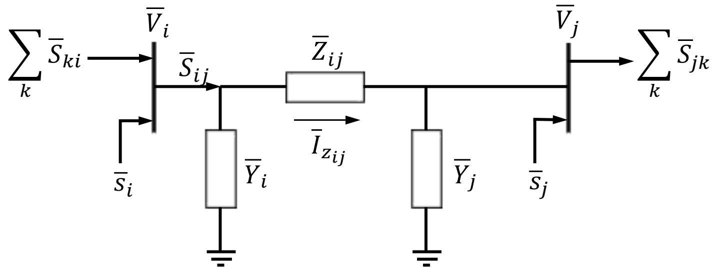
$$J_{PV}: \begin{cases} \dfrac{\partial P_i}{\partial V_\ell} = V_i \left(G_{i\ell} \cos\vartheta_{i\ell} + B_{i\ell} \sin\vartheta_{i\ell}\right) \\ \dfrac{\partial P_i}{\partial V_i} = 2G_{ii} V_i + \sum_{\ell\neq i} V_\ell \left(G_{i\ell} \cos\vartheta_{i\ell} + B_{i\ell} \sin\vartheta_{i\ell}\right) \end{cases}$$

$$J_{P\vartheta}: \begin{cases} \dfrac{\partial P_i}{\partial \vartheta_\ell} = V_i V_\ell \left(G_{i\ell} \sin\vartheta_{i\ell} - B_{i\ell} \cos\vartheta_{i\ell}\right) \\ \dfrac{\partial P_i}{\partial \vartheta_i} = -V_i \sum_{\ell\neq i} V_\ell \left(G_{i\ell} \sin\vartheta_{i\ell} - B_{i\ell} \cos\vartheta_{i\ell}\right) \end{cases}$$

$$J_{QV}: \begin{cases} \dfrac{\partial Q_i}{\partial V_\ell} = V_i \left(G_{i\ell} \sin\vartheta_{i\ell} - B_{i\ell} \cos\vartheta_{i\ell}\right) \\ \dfrac{\partial Q_i}{\partial V_i} = -2B_{ii} V_i + \sum_{\ell\neq i} V_\ell \left(G_{i\ell} \sin\vartheta_{i\ell} - B_{i\ell} \cos\vartheta_{i\ell}\right) \end{cases}$$

$$J_{Q\vartheta}: \begin{cases} \dfrac{\partial Q_i}{\partial \vartheta_\ell} = -V_i V_\ell \left(G_{i\ell} \cos\vartheta_{i\ell} + B_{i\ell} \sin\vartheta_{i\ell}\right) \\ \dfrac{\partial Q_i}{\partial \vartheta_i} = V_i \sum_{\ell\neq i} V_\ell \left(G_{i\ell} \cos\vartheta_{i\ell} + B_{i\ell} \sin\vartheta_{i\ell}\right) \end{cases}$$

Branch Flow Model



$$\overline{S}_{ij} - \overline{Z}_{ij} i_{z_{ij}} - \overline{Y}_i v_i - \overline{Y}_j v_j + \overline{s}_j = \sum_k \overline{S}_{jk}$$

$$v_j = v_i + \left|\overline{Z}_{ij}\right|^2 i_{z_{ij}} - 2\Re\left[\underline{Z}_{ij}\left(\overline{S}_{ij} - \overline{Y}_i v_i\right)\right]$$

$$\sum_k \overline{S}_{ki} + \overline{s}_i = \overline{S}_{ij}$$

$$v_i = \left|\overline{V}_i\right|^2, v_j = \left|\overline{V}_j\right|^2 \text{ and } i_{z_{ij}} = \left|\overline{I}_{z_{ij}}\right|^2$$

$$i_{z_{ij}} = \frac{\left|\overline{S}_{ij} - \overline{Y}_i v_i\right|^2}{v_i}$$

$$arg(\overline{V}_i) + arg(\underline{V}_j) = arg\left(v_i - \underline{Z}_{ij}\left(\overline{S}_{ij} - \overline{Y}_i\left|\overline{V}_i\right|^2\right)\right)$$

Load Flow Approximations

Ward-Hale

Hypothesis:

$$\frac{\partial P_i}{\partial V_l'} = \frac{\partial P_i}{\partial V_l''} = 0 \qquad i \neq l$$

$$\frac{\partial Q_i}{\partial V_l'} = \frac{\partial Q_i}{\partial V_l''} = 0 \qquad i \neq l$$

Carpentier

Hypothesis:

$$\left[\frac{\partial P}{\partial V}\right] \approx 0$$

$$\left[\frac{\partial Q}{\partial \theta}\right] \approx 0$$

Stott

Hypothesis:

$$\left[\frac{\partial P}{\partial V}\right] \approx 0 \qquad \left[\frac{\partial Q}{\partial \theta}\right] \approx 0$$

$$B_{il} \cos\theta_{il} \approx B_{il}$$

$$G_{il} \sin\theta_{il} \ll B_{il}$$

$$Q_i \ll B_{ii} V_i^2$$

DC load flow

- active power through the branch il is

$$P_{il} = \frac{1}{x_{il}} \theta_{il}$$

- the injection of power in a generical node i is:

$$P_i = \sum_{l \neq i} P_{il} = \frac{\theta_{i1}}{x_{i1}} + \cdots \frac{\theta_{is}}{x_{is}}$$

$$P_i = \left(\frac{1}{x_{i1}} + \cdots \frac{1}{x_{is}}\right)\theta_i - \sum_{\substack{l=1 \\ l \neq i}}^{s} \frac{1}{x_{il}} \theta_l = \sum_{l=1}^{s} B_{il}\theta_l$$

$$[P] = [B] \times [\theta]$$

$[B]$ - the "susceptance matrix"

Linearized load flow

- $\mathcal{N}$ – set of PQ buses, $\mathcal{H}$ – set of slack buses, $\{1,2,\ldots,s\} = \mathcal{H} \cup \mathcal{N}$, $\mathcal{H} \cap \mathcal{N} = \emptyset$

$$\mathbf{1}_{\{i=l\}} = \frac{\partial \overline{V}_i}{\partial P_l} \sum_{j \in \mathcal{H} \cup \mathcal{N}} \overline{Y}_{ij} \overline{V}^*{}_j + \underline{V}^*{}_i \sum_{j \in \mathcal{N}} \overline{Y}_{ij} \frac{\partial \overline{V}_j}{\partial P_l}$$

$$-j\mathbf{1}_{\{i=l\}} = \frac{\partial \overline{V}_i}{\partial Q_l} \sum_{j \in \mathcal{H} \cup \mathcal{N}} \overline{Y}_{ij} \overline{V}^*{}_j + \underline{V}^*{}_i \sum_{j \in \mathcal{N}} \overline{Y}_{ij} \frac{\partial \overline{V}_j}{\partial Q_l}$$

$$[\overline{\boldsymbol{I}}_{abc}] = [\overline{\boldsymbol{Y}}_{abc}] \cdot [\overline{\boldsymbol{V}}_{abc}]$$

Nodal voltage module sensitivities:

$$K_{P,V}^{il} = \frac{\partial |\overline{V}_i|}{\partial P_l} = \frac{1}{|\overline{V}_i|} \Re\left(\underline{V}_i \frac{\partial \overline{V}_i}{\partial P_l}\right)$$

$$K_{Q,V}^{il} = \frac{\partial |\overline{V}_i|}{\partial Q_l} = \frac{1}{|\overline{V}_i|} \Re\left(\underline{V}_i \frac{\partial \overline{V}_i}{\partial Q_l}\right)$$

State Estimation

Weighted Least Squares

The non-linear case

$$\mathbf{z}_t = h(\mathbf{x}_t) + \mathbf{v}_t$$

$$\mathbf{z} = \left[P_{inj}, P_{flow}, Q_{inj}, Q_{flow}, I_{mah}, V_{mag}, , \delta\right]$$

$$\mathbf{G}(\hat{\boldsymbol{x}}_t) = \mathbf{H}^{T}(\hat{\boldsymbol{x}}_t)\mathbf{R}_t^{-1}\mathbf{H}(\hat{\boldsymbol{x}}_t)$$ - Gain matrix

$$\hat{\mathbf{x}}_t^{k+1} = \hat{\mathbf{x}}_t^{k} + \left[\mathbf{G}(\hat{\mathbf{x}}_t^{k})\right]^{-1}\mathbf{H}^{T}\left(\hat{\mathbf{x}}_t^{k}\right)\mathbf{R}_t^{-1}\left[\mathbf{z}_t - h\left(\hat{\mathbf{x}}_t^{k}\right)\right]$$

$$\mathbf{H} = \begin{bmatrix} \frac{\partial P_{inj}}{\partial \delta} & \frac{\partial P_{inj}}{\partial V} \\ \frac{\partial P_{flow}}{\partial \delta} & \frac{\partial P_{flow}}{\partial V} \\ \frac{\partial Q_{inj}}{\partial \delta} & \frac{\partial Q_{inj}}{\partial V} \\ \frac{\partial Q_{flow}}{\partial \delta} & \frac{\partial Q_{flow}}{\partial V} \\ \frac{\partial I_{magn}}{\partial \delta} & \frac{\partial I_{magn}}{\partial V} \\ \frac{\partial V_{magn}}{\partial \delta} & \frac{\partial V_{magn}}{\partial V} \\ \frac{\partial \delta}{\partial \delta} & \frac{\partial \delta}{\partial V} \end{bmatrix}$$

The linear case

$$\mathbf{z} = \mathbf{H}\mathbf{x} + \mathbf{v}$$

$$\hat{\mathbf{x}} = \mathbf{G}^{-1}\mathbf{H}^{T}\mathbf{R}^{-1}\mathbf{z}$$

with

$$\mathbf{G} = \mathbf{H}^{T}\mathbf{R}^{-1}\mathbf{H}$$

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_{\mathbf{V}} \\ \mathbf{H}_{\mathbf{I}} \end{bmatrix}$$

$$p(\mathbf{v}) \sim N(0, \mathbf{R})$$

$$\mathbf{R} = diag\left(\sigma_1^2, \ldots, \sigma_m^2\right)$$

$$\mathbf{x} = \left[V_{1,re}, \ldots, V_{s,re}, , V_{1,im}, \ldots, V_{s,im}\right]^{T}$$

$$\mathbf{z}^{T} = \left[\mathbf{z}_V, \mathbf{z}_I\right]$$

$$\mathbf{z}_V = \left[V_{1,re}, \ldots, V_{d_1,re}, V_{1,im}, \ldots, V_{d_1,im}\right]$$

$$\mathbf{z}_I = \left[I_{1,re}, \ldots, I_{d_2,re}, I_{1,im}, \ldots, I_{d_2,im}\right]$$

$$\mathbf{H}_{\mathbf{V}} = \begin{bmatrix} [\beta] & [\upsilon] \\ [\xi] & [\eta] \end{bmatrix}$$

$$where \begin{cases} \beta = \begin{cases} 1 \ if \ i = \ell \\ 0 \ if \ i \neq \ell \end{cases} \\ \upsilon = \xi = 0 \\ \eta = \begin{cases} 1 \ if \ i = \ell \\ 0 \ if \ i \neq \ell \end{cases} \end{cases}$$

$$I_{i,re} = \sum_{\ell=1}^{s}\left(G_{i\ell}V_{\ell,re} - B_{i\ell}V_{\ell,im}\right)$$

$$I_{i,im} = \sum_{\ell=1}^{s}\left(G_{i\ell}V_{\ell,im} + B_{i\ell}V_{\ell,re}\right)$$

$$\mathbf{H}_{\mathbf{I}} = \begin{bmatrix} [G_{i\ell}] & [-B_{i\ell}] \\ [B_{i\ell}] & [G_{i\ell}] \end{bmatrix}$$

Bad data processing

$$\mathbf{r} = \mathbf{z} - \mathbf{H}\hat{\mathbf{x}}$$ - Residuals

$$\boldsymbol{\Omega} = \mathrm{cov}(\mathbf{r}) = \mathbf{R} - \mathbf{H}\mathbf{G}^{-1}\mathbf{H}^{\mathbf{T}}$$

$$r_{\mathrm{i}}^{\mathrm{N}} = \frac{|\mathrm{r_i}|}{\sqrt{\boldsymbol{\Omega}_{\mathrm{ii}}}}$$ - Normalized residuals

Discreet Kalman Filter

Prediction Equations
Prediction of the state: $\tilde{\mathbf{x}}_t = \mathbf{A}\hat{\mathbf{x}}_{t-1} + \mathbf{B}\mathbf{u}_{t-1}$ $\tilde{\mathbf{P}}_t \equiv \mathbf{A}\hat{\mathbf{P}}_{t-1}\mathbf{A}^{T} + \mathbf{Q}_{t-1}$

Estimation Equations
(1) Computation of the *Kalman Gain:* $\mathbf{K}_t = \tilde{\mathbf{P}}_t\mathbf{H}^{T}(\mathbf{H}\tilde{\mathbf{P}}_t\mathbf{H}^{T} + \mathbf{R})^{-1}$
(2) Estimation of the state $\hat{\mathbf{x}}_t = \tilde{\mathbf{x}}_t + \mathbf{K}_t(\mathbf{z}_t - \mathbf{H}\tilde{\mathbf{x}}_t)$ $\hat{\mathbf{P}}_t \equiv (\mathbf{I} - \mathbf{K}_t\mathbf{H})\tilde{\mathbf{P}}_t$

$$\hat{\mathbf{P}}_t \equiv \mathbb{E}[\hat{\mathbf{e}}_t\hat{\mathbf{e}}_t^{T}]$$

$$\tilde{\mathbf{P}}_t \equiv \mathbb{E}[\tilde{\mathbf{e}}_t\tilde{\mathbf{e}}_t^{T}]$$

$$\tilde{\mathbf{e}}_t \equiv \mathbf{x}_t - \tilde{\mathbf{x}}_t$$

$$\hat{\mathbf{e}}_t \equiv \mathbf{x}_t - \hat{\mathbf{x}}_t$$

- t : time-step
- $\tilde{\mathbf{P}}_t$: prediction error covariance matrix ;
- $\mathbf{K}$: "Kalman gain" ;
- $\hat{\mathbf{P}}_t$: estimation error covariance matrix ;

Linear regression

$$Y_t = \sum_{j=1}^{p} \beta_j f_j(t) + \epsilon_t$$

Where $f_j(t)$ are known functions of time, ϵ_t is iid* white noise $\mathcal{N}(0, \sigma^2)$ being **iid (i.e., independent and identical distributed)** and β_j are the regression coefficients, namely the **unknowns of the problem**.

Theorem #1

1. The $p{\times}p$ matrix $X^T X$ is invertible.
2. The maximum likelihood estimator of $\boldsymbol{\beta}$ is $\widehat{\boldsymbol{\beta}} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\overline{y}$
3. Let us define the ith residual $e_i = (\overline{y}_i - \mathbf{X}_i\widehat{\boldsymbol{\beta}})$ (where $\mathbf{X}_i$ is the ith row of $\mathbf{X}$). Residuals are zero-mean Gaussian and are correlated with a covariance $\sigma^2(\mathbf{Id}_n - \mathbf{H})$ where $\mathbf{H} = \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T$, $\mathbf{Id}_n$ is the identity matrix of order n and σ^2 can be estimated as $\sigma^2 \cong s^2 = \frac{1}{n-p}\sum_i e_i^2$ (i.e., the rescaled sum of residuals).
4. The distribution of $\widehat{\boldsymbol{\beta}}$ is Gaussian with mean $\boldsymbol{\beta}$ and covariace $\sigma^2(\mathbf{X}^T\mathbf{X})^{-1}$

Auto Correlation and Partial Auto Correlation Function

Covariance:

Given a time series $\mathrm{Y}_1, \dots \mathrm{Y}_n$, the covariance of a time series is defined as:

$$\Omega_{\mathrm{Y_i Y_j}} = \mathrm{cov}(\mathrm{Y_i}, \mathrm{Y_j}) = \mathbb{E}\left[\frac{\mathrm{Y_i} - \mathbb{E}(\mathrm{Y_i})}{\mathrm{Y_j} - \mathbb{E}(\mathrm{Y_j})}\right]$$

Autocorrelation function:

The autocorrelation between two observations distanced by a lag h is:

$$\hat{\rho}_h = \frac{\hat{\gamma}_h}{\hat{\gamma}_0},$$

where:

$$\hat{\gamma}_h = \frac{1}{n}\sum_{j=1}^{n-h}(\mathrm{Y}_{j+h} - \overline{\mathrm{Y}})(\mathrm{Y}_j - \overline{\mathrm{Y}})$$

Partial autocorrelation function:

The partial autocorrelation of $\mathrm{Y_i}$ with $\mathrm{Y_j}$, where $\mathrm{j} > \mathrm{i}$ given that $\mathrm{Y}_{i+1}, \dots, \mathrm{Y}_{j-1}$ are known:

$$\mathrm{r_{i,j}} = \frac{-\tau_{i,j}}{\sqrt{\tau_{i,i}\tau_{j,j}}},$$

where $\tau_{i,j}$ is the generic element of Ω^{-1}

Digital filters

Impulse response of a filter:

The **impulse response** of a filter satisfies the following equation:

$$\begin{pmatrix} h_0 \\ h_1 \\ \cdots \\ h_{n-1} \end{pmatrix} = F\delta_n$$

Where δ_n is the Dirac sequence of length n

Backshift operator:

$$B\begin{pmatrix} X_1 \\ X_2 \\ \cdots \\ X_n \end{pmatrix} = \begin{pmatrix} 0 \\ X_1 \\ \cdots \\ X_{n-1} \end{pmatrix}$$

Differencing filter:

$X_t = (\Delta_1 Y)_t = Y_t - Y_{t-1}, t = 1, \dots, n,$

where Δ_1 is the differencing filter at lag 1 and , $Y_j = 0 \; for \; j \le 0$.

Deseasonalizing filter:

The deseasonalizing filter R_T maps the time series Y into a time series $X = R_T Y$, such that

$X_t = \sum_{k=0}^{T-1} Y_{t-k}, t = 1, \dots, n, Y_j = 0 \; for \; j \le 0$

Commutative property of filters:

$$X = R_i R_j Y = R_j R_i Y$$

AR and MA processes

Auto-regressive process

An auto-regressive process AR(p) is described by:

$$X_t = \sum_{i=1}^{p} A_i X_{t-i} + \epsilon_t,$$

where $A_1, \dots, A_p$ are parameters and ϵ_t is iid white.

Moving average process:

A moving average process MA(q) is described by:

$$X_t = \mu + \epsilon_t + \sum_{i=1}^{q} C_i \epsilon_{t-i},$$

where $C_1, \dots, C_q$ are parameters, μ is the expectation of X_t and $\epsilon_t, \dots \epsilon_{t-q}$ are iid white.

ARMA process:

A zero-mean $ARMA(p,q)$ process X_t is a process that satisfies for $t = 1,2, \dots$ a difference equation such as;

$$X_t + A_1 X_{t-1} + \cdots + A_p X_{t-p} = \epsilon_t + C_1\epsilon_{t-1} + \cdots + C_q\epsilon_{t-q} \; where \; \epsilon_t \; is \; iid \sim\mathcal{N}(0, \sigma^2)$$

Unless otherwise specified, $X_{-p+1} = \cdots = X_0 = 0$.

ARIMA process

$ARIMA(p,d,q)$ can be also written as

$$(1 - B)^d(1 + A_1 B + \cdots + A_p B^p)Y = (1 + C_1 B + \cdots + C_q B^q)\epsilon$$

where B is the backshift operator.

Optimal Power Flow

$$\min_{P_{g_1}(t),\dots,P_{g_g}(t),Q_{g_1}(t),\dots,Q_{g_g}(t)} \sum_{t=1}^{T}\sum_{i=1}^{g} C_i\left(P_{g_i}(t)\right)$$

$$s.t.$$

$$\overline{S}_i(t) = \overline{V}_i(t)\sum_{j=1}^{s}\underline{V}_j(t)\underline{Y}_{ij}\,, i = 1, \dots, s$$

$$\overline{S}_i(t) = \left(P_{g_i}(t) + jQ_{g_i}(t)\right) + \left(P_{l_i}(t) + jQ_{l_i}(t)\right),$$

$$i = 1, \dots, s$$

$$P_{g_i}^{min} \le P_{g_i}(t) \le P_{g_i}^{max}, i = 1, \dots, g$$

$$Q_{g_i}^{min} \le Q_{g_i}(t) \le Q_{g_i}^{max}, i = 1, \dots, g$$

$$\left|\overline{V}_1\right| = 1pu,\ \arg\left(\overline{V}_1\right) = 0;$$

$$V_{min} \le \left|\overline{V}_i(t)\right| \le V_{max}, i = 2, \dots, s$$

$$\left|\overline{Y}_{ij}\left(\overline{V}_i(t) - \overline{V}_j(t)\right)\right| \le I_{i,j}^{max}, i \ne j =, 1 \dots, s$$

Stochastic Optimization

$$\min_{u\in U} \mathbb{E}_d(c(u, F(u,d)))$$
$$s.t.\, \mathbb{P}(F(u,d) \in X) \ge 1 - \varepsilon$$

Robust Optimization

$$\min_{u\in U}\ \max_{d\in D}\ c(u, F(u,d))$$
$$s.t.\, (F(u,d) \in X)$$

Quantifiers in Constraints

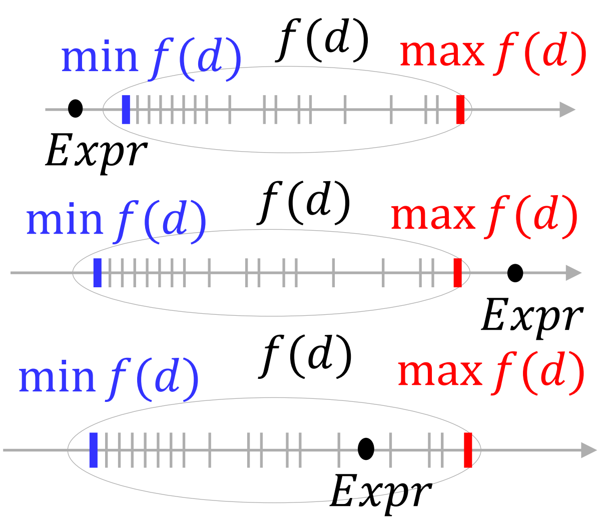
Removal of ∀ and ∃: whenever Expr **does not depend on** d:

$$[\forall d \in D, \text{Expr} \le f(d)] \iff \left[\text{Expr} \le \min_{d\in D} f(d)\right]$$

$$[\forall d \in D, \text{Expr} \ge f(d)] \iff \left[\text{Expr} \ge \max_{d\in D} f(d)\right]$$

$$[\exists d \in D, \text{Expr} \le f(d)] \iff \left[\text{Expr} \le \max_{d\in D} f(d)\right]$$

$$[\exists d \in D, \text{Expr} \ge f(d)] \iff \left[\text{Expr} \ge \min_{d\in D} f(d)\right]$$



Removal of ∃ with supplementary variables

$$\min_{u} \varphi(u)\ \ s.t. \begin{cases} \exists d \in D, C(u,d) \\ C'(u) \end{cases}$$

can be addressed by solving

$$\min_{u,d} \varphi(u)\ \ s.t. \begin{cases} d \in D \\ C(u,d) \\ C'(u) \end{cases}$$