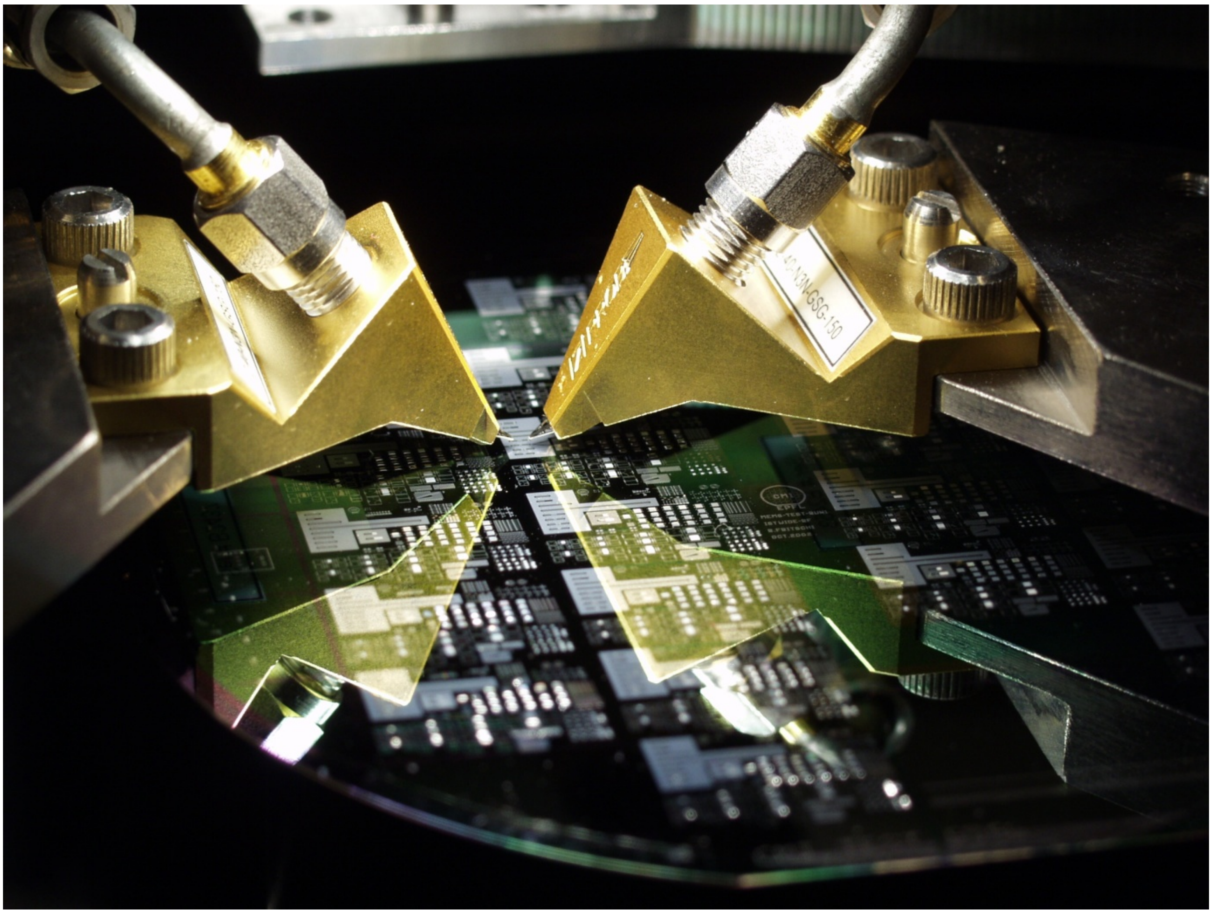
# MICROWAVES





**revision September 2015**

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# 1. Introduction and Applications

## 1.1 Scope

The scope of this course is to familiarize the students with some fundamental aspects related to microwaves. The following topics will be treated:

- Applications of microwaves
- Maxwell's model (brief reminder)
- Transmission line theory, and modal transmission
- Microwave network analysis
- Microwave components
- Introduction to microwave measurements
- Introduction to microwave sources and amplifiers
- Introduction to microwave filters

## 1.2 History

Electromagnetic theory and Microwaves are often considered being mature scientific disciplines, as their fundaments were built by James Clerk Maxwell during the second half of the 19$^{th}$ century. Moreover, they had a tremendous growing during the Second World War due to the research done on radar applications.

Nowadays, work in these fields is very active again, as the boom in telecommunication and data transmisssion has reactivated the interest in microwaves, most of wireless links being done in the microwave frequencies.

1872 : Publication of James Clerk Maxwell's "A Treatise on Electricity and Magnetism". This work unifies all the previous work done on electricity, magnetism and electromagnetism and summarizes the obtained results in four equations.

1887 : Heinrich Hertz performs the first demonstration of a wireless link. To this aim, he built an experimental system using a spark generator coupled to a dipole antenna on the emission side and a coil on the receiver side. These first experiments were done at a frequency of 37 MHz and 1 GHz, and were enthusiastically received by the scientific community, as it was the first experimental validation of Maxwell's theory. The interest however remained confined to academic circles..

1885-1887 : Publication of Oliver Heaviside's comments on Maxwell's work. These comments, introducing the vector notation, made Maxwell's theory more accessible to the scientific community.

1887 : Lord Rayleigh proves theoretically the possibility to transmit through a waveguide.

1901 : Marconi repeats Hertz' experiments, and then decides to work at much lower frequencies. He succeeded in doing the first transatlantic link on the $12^{th}$ of December.

1903: First regular wireless telegraphic link between England and New Scotland..

1920 : First tube amplifiers (triodes) working above 1 MHz.

1921 : The 12th of December (20 years exactly after the first link performed by Marconi), radio amateurs perform the first transatlantic link in medium waves, at 1.5 MHz (nearly short waves). This was possible thanks to a new receiver, the super heterodyne receiver.

1930 : First use of Radar in VHF band (54-88 MHz).

1930 : The crystal detector replaces the needle detector

1936 : Rediscovery of the waveguide by two scientists independently : G.C. Southworth and W.L. Barrow both presented a paper on waveguide propagation at the same conference.

1937 : Invention of the klystron by the Varian brothers. This tube can be used as a generator or an amplifier, and works in the microwave range.

1938 : Motorola develops the first portable phone.

1939-1945 : Comeback of microwaves, thanks to RADAR

1948 : Principles of distributed filter theory by Richards.

1949 : First use of ferrites for the fabrication of non reciprocal components (isolators, circulators)

1950 : First multiple cavity filters, synthesized using Butterworth of Chebyshev characteristics.

1950 : Development of planar microwave transmission lines : first the striplines, then the microstrip lines and the coplanar waveguide

1950 : Apparition of TWT (Traveling Wave Tube) amplifiers and of masers, used as low noise amplifiers.

1960 : First transistors and integrated circuits in microwaves.

1960 : First passive satellite "Echo 1", a metallized balloon of 30m of diameter, which was used as a reflector at an altitude of 1600 km.

1962 : First active satellite, Telstar1, which enabled transatlantic telediffusion. It has an elliptic orbit varying between 950km and 5650 km.

1965 : First geostationary satellite, "Early Bird or Intelsat 1", having 240 phone channels for a satellite weight of38 kg.

1970 : Beginning of MMIC (Monolithic Microwave Integrated Circuits), allowing a better integration of microwave circuits.

1970 : Apparition of first CAD tools for microwaves.

2004 : - Hundreds of geostationary satellites.

-Hundred of LEO and MEO satellites

- 3rd generation of cellular phone

- Wireless data transmission

- ...

- ...

2015 : - Internet of things

- Big Data

- Wearable connectivity

- ...

- ...

## 1.3 Definition of the Microwave band

The electromagnetic spectrum comprises all frequencies between zero and infinity. It is traditionally divided in bands covering a decade, meaning that the upper limit of a band equals ten times its lower limit. These limiting frequencies are chosen in a way that their associated wavelength is a power of 10, when expressed in meters.

### *1.3.1 Electromagnetic spectrum*

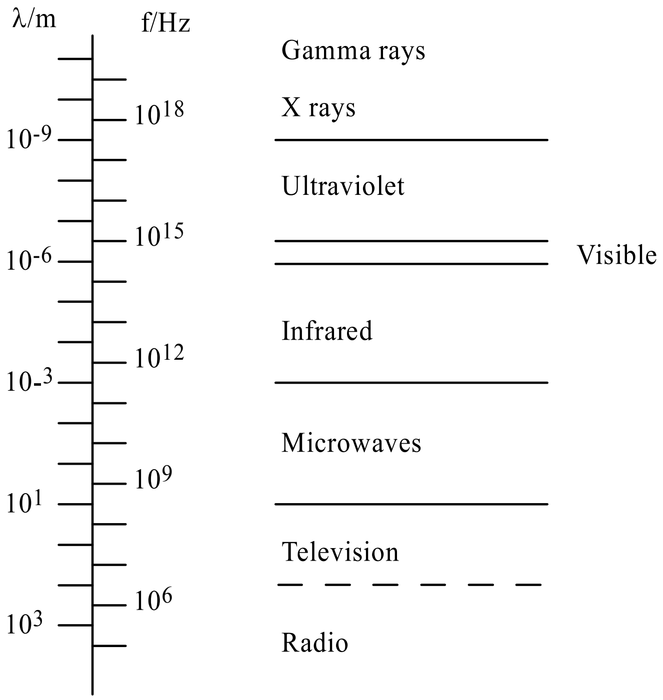



*Fig. 1. 1 : Subdivisions of the electromagnetic spectrum*

| Band | Frequency | Wavelength | Applications |
|---|---|---|---|
| VLF<br>Very Low Frequency | 3 - 30 kHz | 100 -10 km | Navigation, Sonar |
| LF<br>Low Frequency | 30 - 300 kHz | 10 - 1 km | long wave<br>radio |
| MF<br>Medium Frequency | 300 kHz - 3 MHz | 1 km - 100 m | Goniometry,<br>radio AM |
| HF<br>High Frequency | 3 - 30 MHz | 100 - 10 m | CB, short waves,<br>air traffic |

| | | | |
|---|---|---|---|
| VHF<br>Very High Frequency | 30 - 300 MHz | 10 - 1 m | Radio FM, TV, radar,<br>mobile communications |
| UHF<br>Ultra High Frequency | 300 MHz - 3 GHz | 1 m - 10 cm | Cellular phone, Satellite, TV,<br>radar |
| SHF<br>Supra High Frequency | 3 - 30 GHz | 10 - 1 cm | Satellite, hertzian links,<br>radio astronomy |
| EHF<br>Extremely High Frequency | 30 - 300 GHz | 10 - 1 mm | Satellite, radar,<br>radio astronomy,<br>military applications |

*Fig. 1. 2 : Wireless communication frequency bands*

The bands between 300 MHz and 300 GHz are called microwaves. They are characterized by the fact that the size of circuits and components is of the same order of magnitude as the wavelength.

In comparison, at the power network frequency of 50 Hz the wavelength is of 6000km, while at visible optical frequencies the wavelength is in the order of 0.6 micrometers.

The microwave band is subdivided in the following way :

| Band | Frequency |
|---|---|
| L | 1-2 GHz |
| S | 2-4 GHz |
| C | 4-8 GHz |
| X | 8-12 GHz |
| Ku | 12-18 GHz |
| K | 18-26 GHz |

| | |
|---|---|
| Ka | 26-40 GHz |
| Q | 40-60 GHz |
| E | 60-90 GHz |

*Fig. 1. 3 : Microwave sub-bands*

We have seen about that the microwaves band was defined according to frequency. Alternatively, we can also define it according to period, wavelength or energy. We have indeed the following equivalences :

| Frequency (f) | 300 MHz – 300 GHz |
|---|---|
| Period (T) | 3 ns – 3 ps |
| Wavelength ($\lambda$) | 1 m – 1 mm |
| Energy (hf) | $1.2\ 10^{-6}$ eV – $1.2\ 10^{-3}$ eV |

These different points of view in the definition of the microwave band will have different consequences on their properties, as will be seen in the next section.

## 1.4 Properties of microwaves

### *1.4.1 Bandwidth*

The absolute bandwidth of a transmission system is directly linked (proportional) to the carrier frequency. If the latter is in the microwave area, i.e. in the upper part of the electromagnetic spectrum, the bandwidth will be higher than if we work at lower frequencies. The same is of course also true for transmissions using optical fibers.

### *1.4.2 Transparency of the ionosphere*

The ionosphere is formed by several ionized layers, which surround the Earth at an altitude between 50 and 10000 km. The propagation of electromagnetic waves inside the ionosphere is similar to the propagation in a waveguide. Signals at frequencies below some tens of MHz (cut off) are partially or totally reflected. Signals having a higher frequency can cross the ionosphere, and suffer a distortion which becomes smaller as the frequency increases. In the range of microwaves, the distortion is so slight it is negligible.

An effective dielectric permittivity is associated with uniform plasma. It is given by :

$$\varepsilon_e = \varepsilon_0 \left( 1 - \frac{\omega_p^2}{\omega^2} \right) \qquad \omega_p = \sqrt{\frac{Nq^2}{m\varepsilon_0}} \tag{1. 1}$$

With

N : number of ions/volume

q : charge of the electron

m : mass of the electron

$\varepsilon_0 = 8.854\ 10^{-12}$ As/Vm : permittivity of free space.

We can distinguish three regions :

<u>I) $\omega << \omega_p$</u>

The effective permittivity of the medium tends towards $-\infty$, and the characteristic impedance of the medium, given by

$$Z_{milieu} = \sqrt{\frac{\mu}{\varepsilon}}$$

is imaginary and close to 0. An incident wave will thus be totally reflected (Figure 1.4)

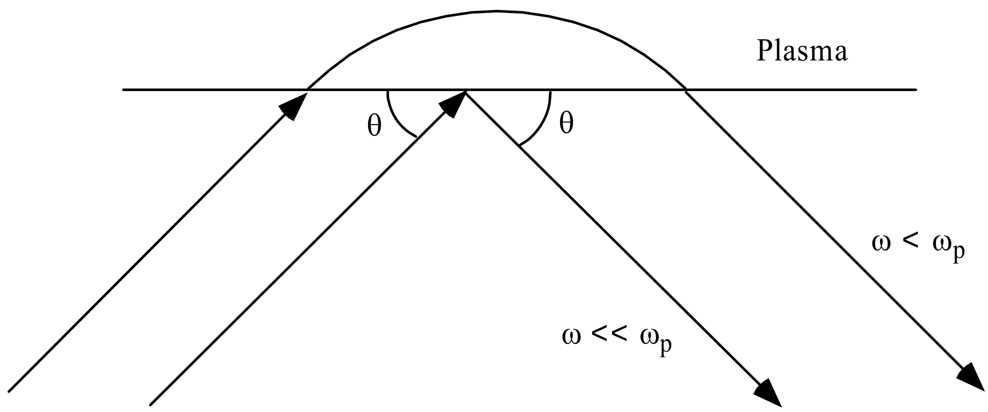



*Fig. 1. 4 : total reflection due to ionosphere at low frequencies*

<u>II) $\omega = \omega_p$</u>

The effective permittivity of the medium is zero, thus the wave number k of the wave is also equal to zero :

$$k = \omega\sqrt{\varepsilon\mu}$$

The wave is absorbed by the plasma (figure 1.5)

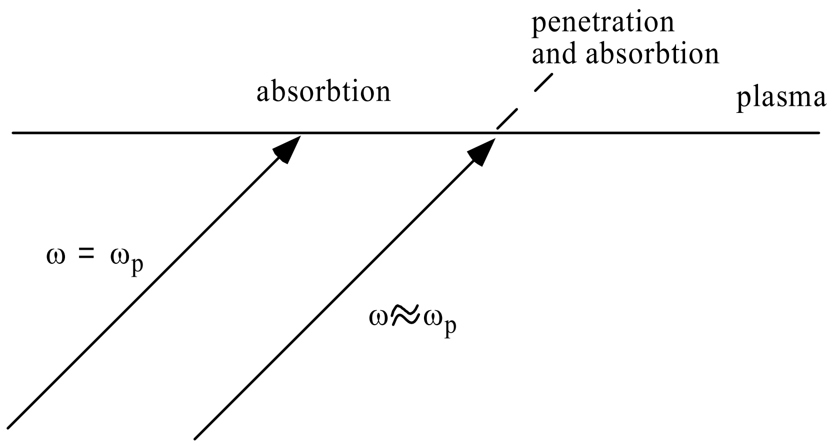



*Fig. 1. 5: absorption of a wave by the plasma*

<u>III) $\omega >> \omega_p$</u>

The effective permittivity of the plasma tends towards unity, and the wave is nit disturbed by the plasma (Fig. 1.6)

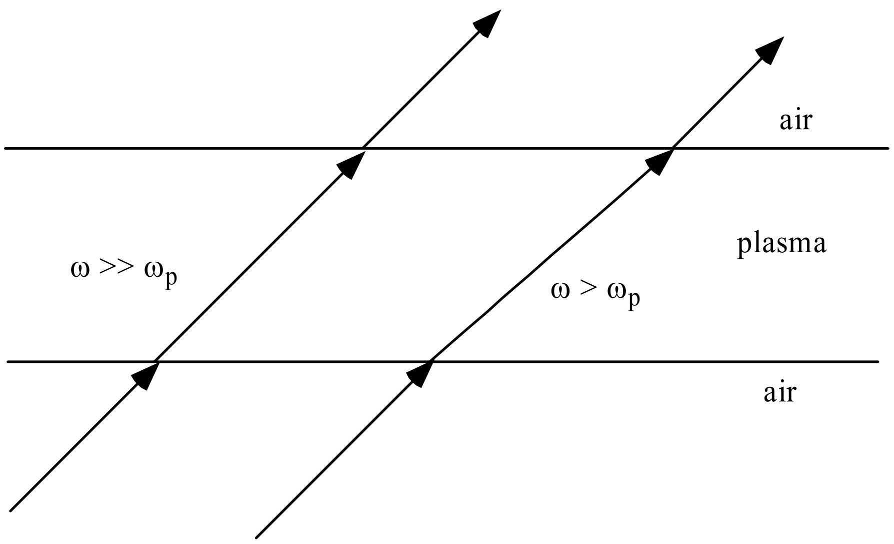



*Fig. 1. 6 : High frequency wave undisturbed by plasma.*

### *1.4.3 Partial transparency of the atmosphere*

The gases composing the atmosphere and the different suspended molecules do not affect signals below 10 GHz. The first absorption ray (24GHz) is corresponding to the presence of water.

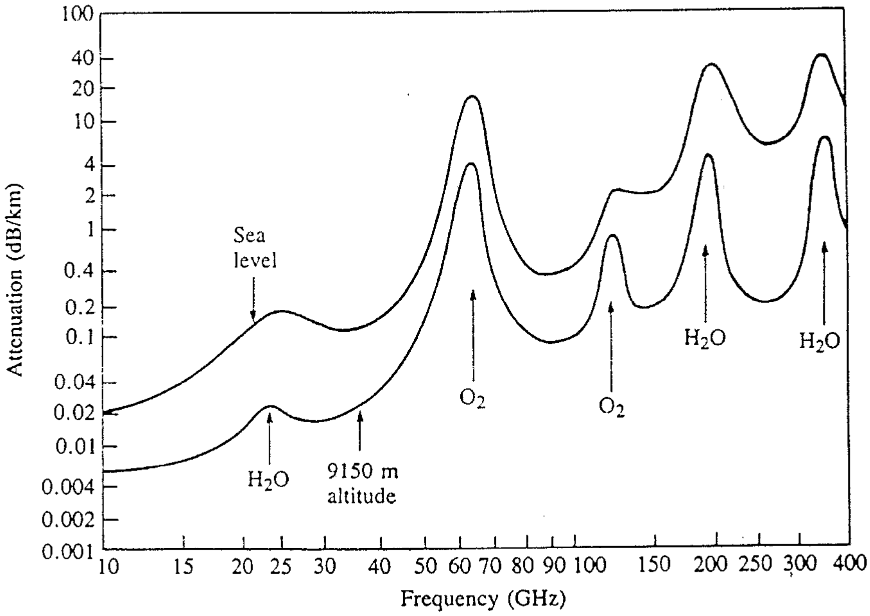



*Fig. 1. 7: absorption due to atmosphere*

To the absorption due to gazes present in the atmosphere, we must add the effect of clouds, rain, snow and hail (Fig. 1.8)

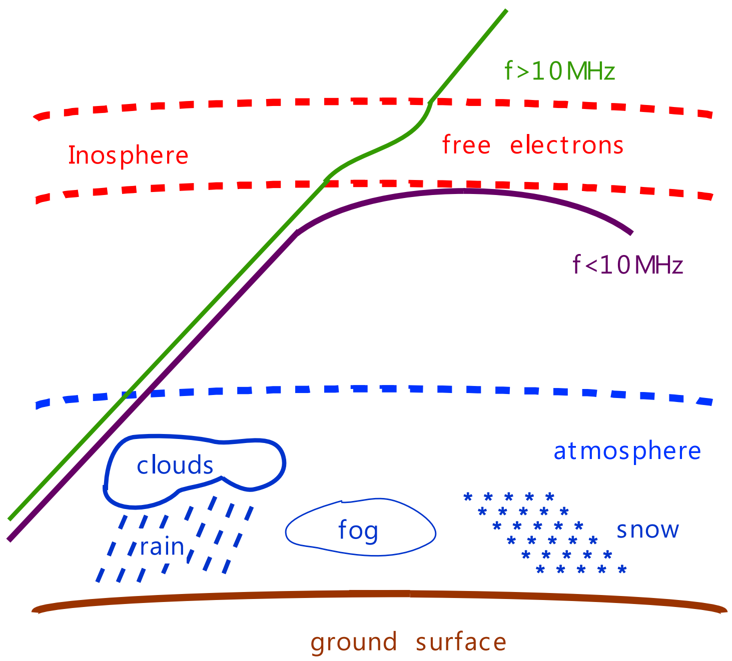



*Fig. 1. 8: effect of atmosphere and ionosphere on microwaves*

### *1.4.4 Inhomogeneity of the atmosphere*

The dielectric constant of the earth atmosphere varies slightly with the altitude, as the air becomes less dense. The refraction index of a typical "average" atmosphere is depicted in figure 1.9. for light and microwaves, and is given in (1.2).

$$\varepsilon_r = n^2 \cong \left[1+\left(\frac{79p}{T}-\frac{11\nu}{T}+\frac{3.8\cdot10^5\nu}{T^2}\right)\cdot10^{-6}\right]^2 \qquad (1.\ 2)$$

where p is the barometric pressure in millibar, T the temperature in Kelvin, ν the water vapor pressure in millibar and n the refraction index.

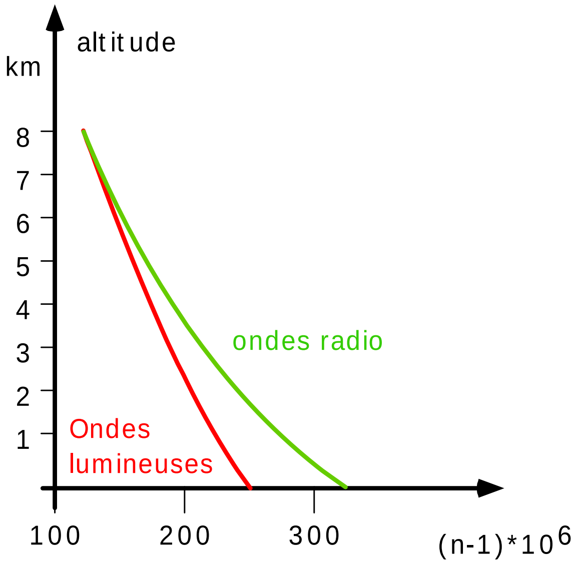


*Fig. 1. 9 : inhomogeneity of the atmosphere*

The effect of this inhomogeneity is that microwaves do not travel along a straight line, but are curved as they gain altitude. Indeed, let us suppose a slab of atmosphere which is made of three layers, having each a different refraction index (Fig. 1.10) :

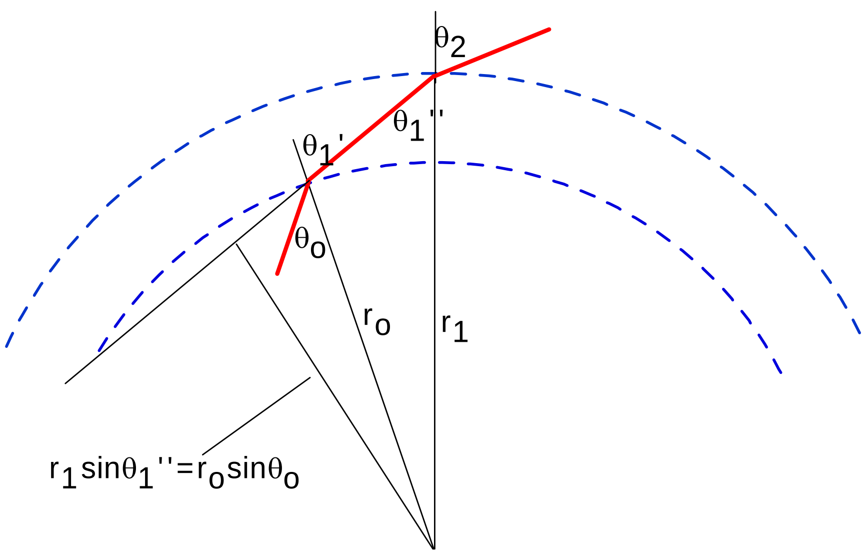


*Fig. 1. 10: inhomogeneous atmosphere*

Let us consider the spherical interfaces given by $r_o$ and $r_1$, which separates the three regions. Snell's law tells us that :

$$n_o \sin\theta_o = n_1 \sin\theta_1' \qquad n_1 \sin\theta_1'' = n_2 \sin\theta_2 \tag{1. 3}$$

Moreover, geometry tells us that :

$$r_1 \sin\theta_1'' = r_o \sin\theta_1' \tag{1. 4}$$

If combine those relations, we get

$$n_o r_o \sin\theta_o = n_1 r_o \sin\theta_1' = n_1 r_1 \sin\theta_1'' = n_2 r_2 \sin\theta_2 \tag{1. 5}$$

which can be generalized for a continuously varying medium into :

$$nr \sin\theta = n_o r_o \sin\theta_o \tag{1. 6}$$

The practical significance of this for engineering purposes is, that in the planning of a Hertzian link (see applications), the curvilinear path of the wave through the air is replaced by a straight line, but where the Earth is considered to have a fictive radius of

$$R_T = \frac{4}{3} R \tag{1. 7}$$

Thus, if we consider the case depicted in figure 1.11, the two path lengths are considered equivalent.

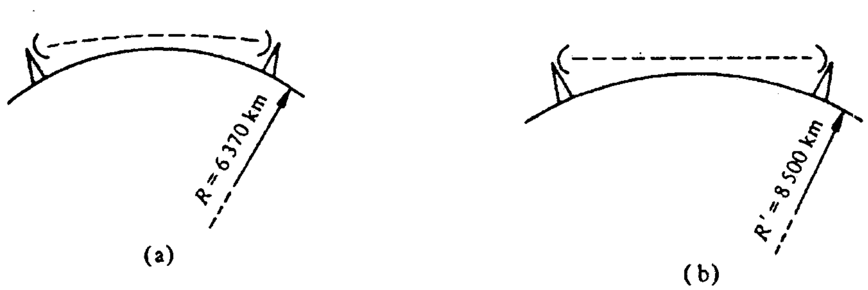



*Fig. 1. 11 : Real versus fictive Earth radius.*

More generally, we can define an equivalence between a rectilinear and a curved path for different atmosphere configurations in the following way:

$$R_T = kR$$

with k given in the following table

| k | zone | weather |
|---|---|---|
| 1.33 | temperate | without fog |
| 1-1.33 | arid mountainous | without fog |
| 0.68-1 | temperate | light fog |

| | | |
|---|---|---|
| 0.5-0.68 | littoral | heavy fog |
| 0.4-0.5 | tropical, water | fog and rain |

### *1.4.5 Electromagnetic noise*

The noise power received by an antenna pointing towards the sky is minimal between 1 and 10 GHz. In this band, the equivalent noise temperature is below 10 K. The corresponding noise power received is obtained by multiplying the noise temperature by Bolzmann's constant ($k_B$=1.3804 $10^{-23}$ J/K) and by the bandwidth of the receiver. This means that it is in the band between 1 and 10 GHz that we will be able to detect the signals with the smallest amplitude, thus having the most sensitive receivers. For instance, signals used for deep space observation use a band close to 3 GHz.

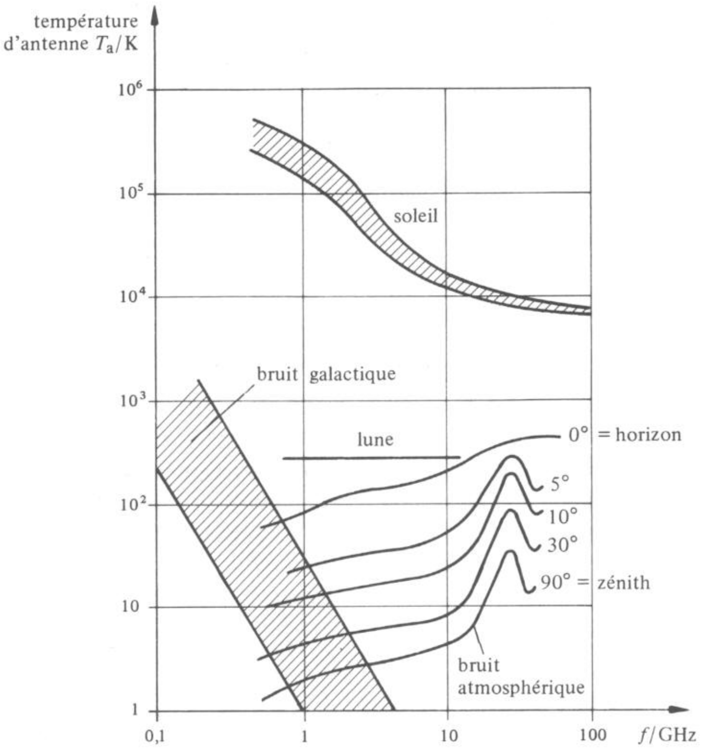



*Fig. 1. 12 : Antenna noise temperature*

### *1.4.6 Antenna directivity*

The antenna beamwidth is proportional to the wavelength divided by the largest dimension of the antenna. For the same physical dimensions, an antenna will be more directive at higher frequencies.

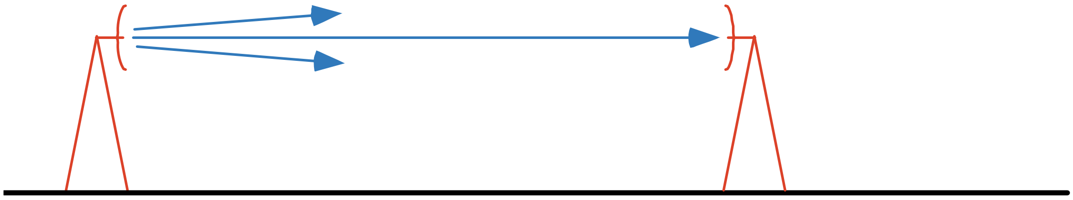


*Fig. 1. 13 : Point to point link (Hertzian link or satellite link) using microwaves*

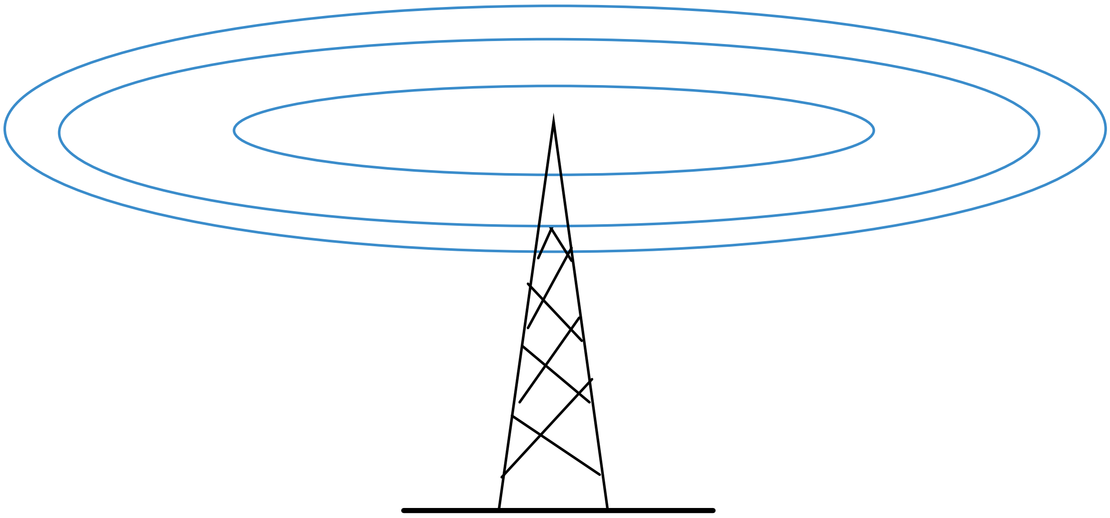


*Fig. 1. 14: omnidrectional Radio diffusion using ultra short waves*

Indeed, in a first approximation the 3dB Beamwidth of antenna is given by

$$\alpha \geq \frac{\lambda}{d} \tag{1. 8}$$

where $\alpha$ is the beamwidth, $\lambda$ the wavelength and d the largest dimension of the antenna.

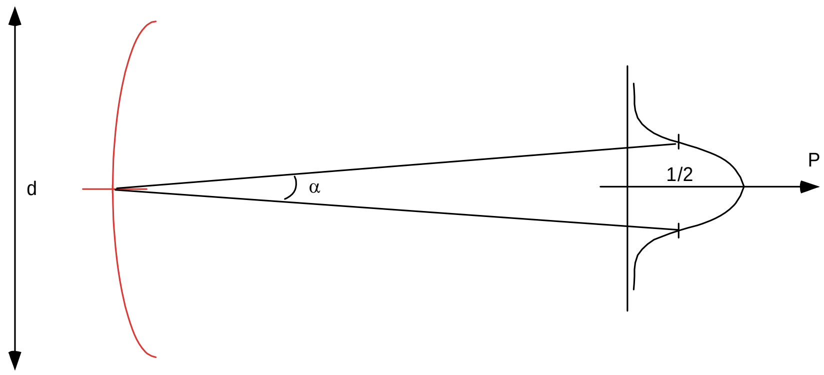



*Fig. 1. 15 : Half power beamwidth*

### *1.4.7 Interaction with matter*

The absorption of electromagnetic waves by matter depends on the frequency. In particular, water absorbs microwaves over the entire band, a property which allows applications like microwave heating or the thermal treatment of certain illnesses.

### 1.4.8 *Non ionizing radiation*

The molecular cohesion energy is much larger than the photonic energy in the microwave band. This means that a photon in the microwave band is not able to modify a chemical link in a molecule, by inducing a photoelectrical effect. Microwave radiation is thus non ionizing ( in comparison, X rays which have a much larger photonic energy can produce ionization of the matter).

## 1.5 Applications

### 1.5.1 *RADAR*

RADAR stands for RA Detection And Ranging. It is based on the use of the echo produced by an obstacle located on the trajectory of an electromagnetic wave. In most cases, the system is as depicted on the schema of figure 1.16.

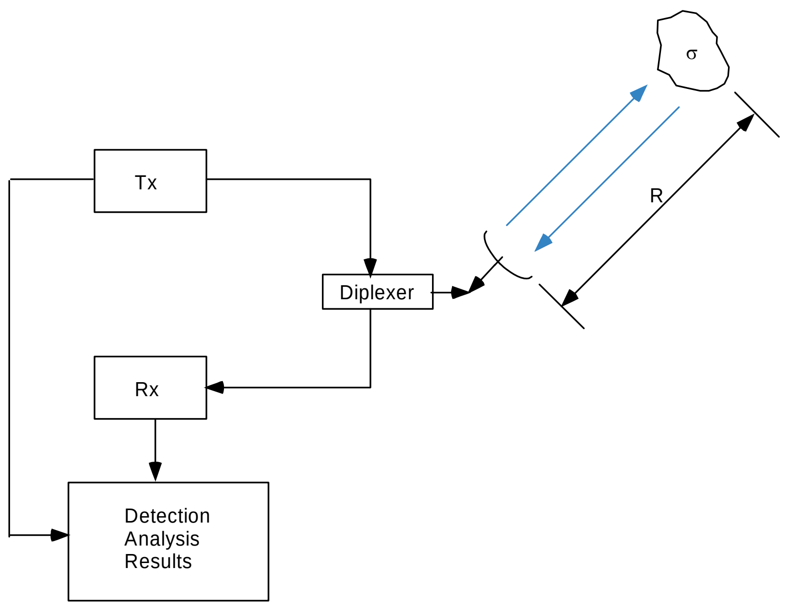



*Fig. 1. 16 : Principle of RADAR*

where the received power is linked to the transmitted power by the RADAR equation :

$$\frac{P_r}{P_f} = \frac{g^2 \lambda^2 \sigma}{(4\pi)^3 R^4} \quad (1.9)$$

where $P_r$ is the power at the receiver, $P_f$ the transmitted power, g the antenna gain, $\lambda$ the wavelength, $\sigma$ the equivalent radar surface and R the distance to the target.

The equivalent radar surface depends on the shape of the object and its material. It is usually also frequency dependent : Fig 1.17 gives the equivalent radar surface of a metallic sphere- We see that when the sphere is much larger than the wavelength, its equivalent radar surface is equal to its surface projected on the plane normal to the direction of propagation. On the other hand, when the object is much smaller than the wavelength, the radar signal hardly sees the object, thus its equivalent radar surface is very low. In the area where the object is of the same size than the wavelength, we can see that the behavior of the object shows resonances.

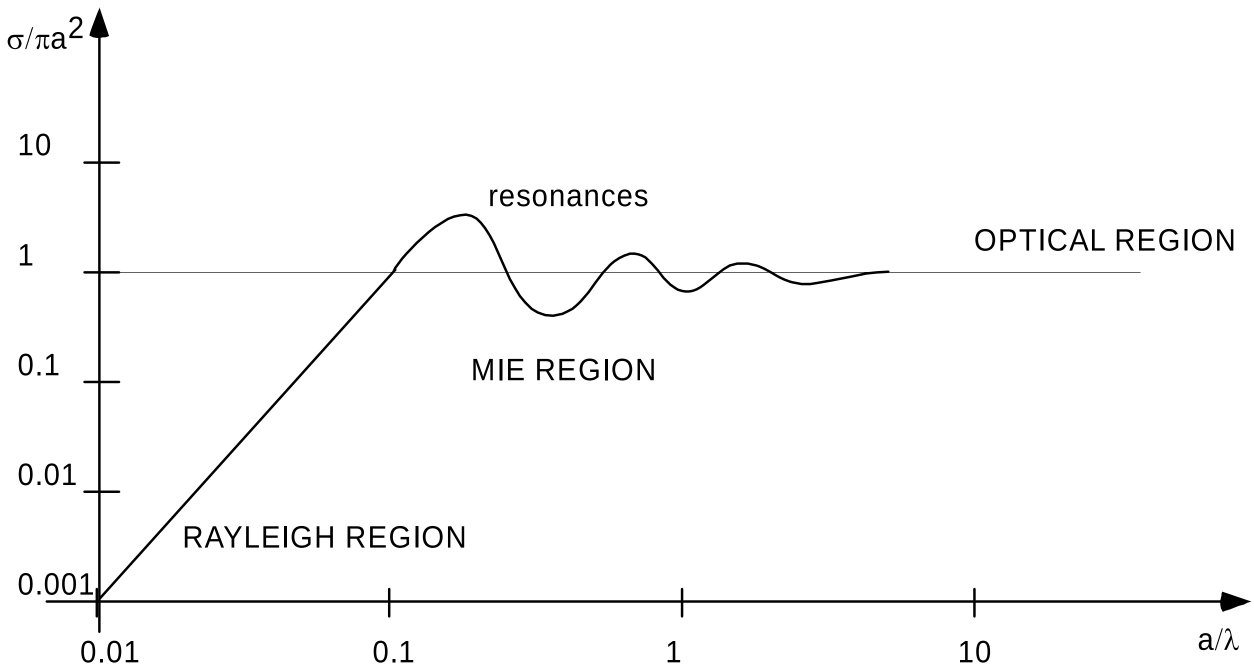



*Fig. 1. 17 : Equivalent RADAR surface of a perfect conductor sphere*

It is not possible to find closed form formulas for most of radar targets. For some canonical structures however, it is possible to find analytical expression for the equivalent radar surface in the optical region. Some examples are shown in figure 1.18.

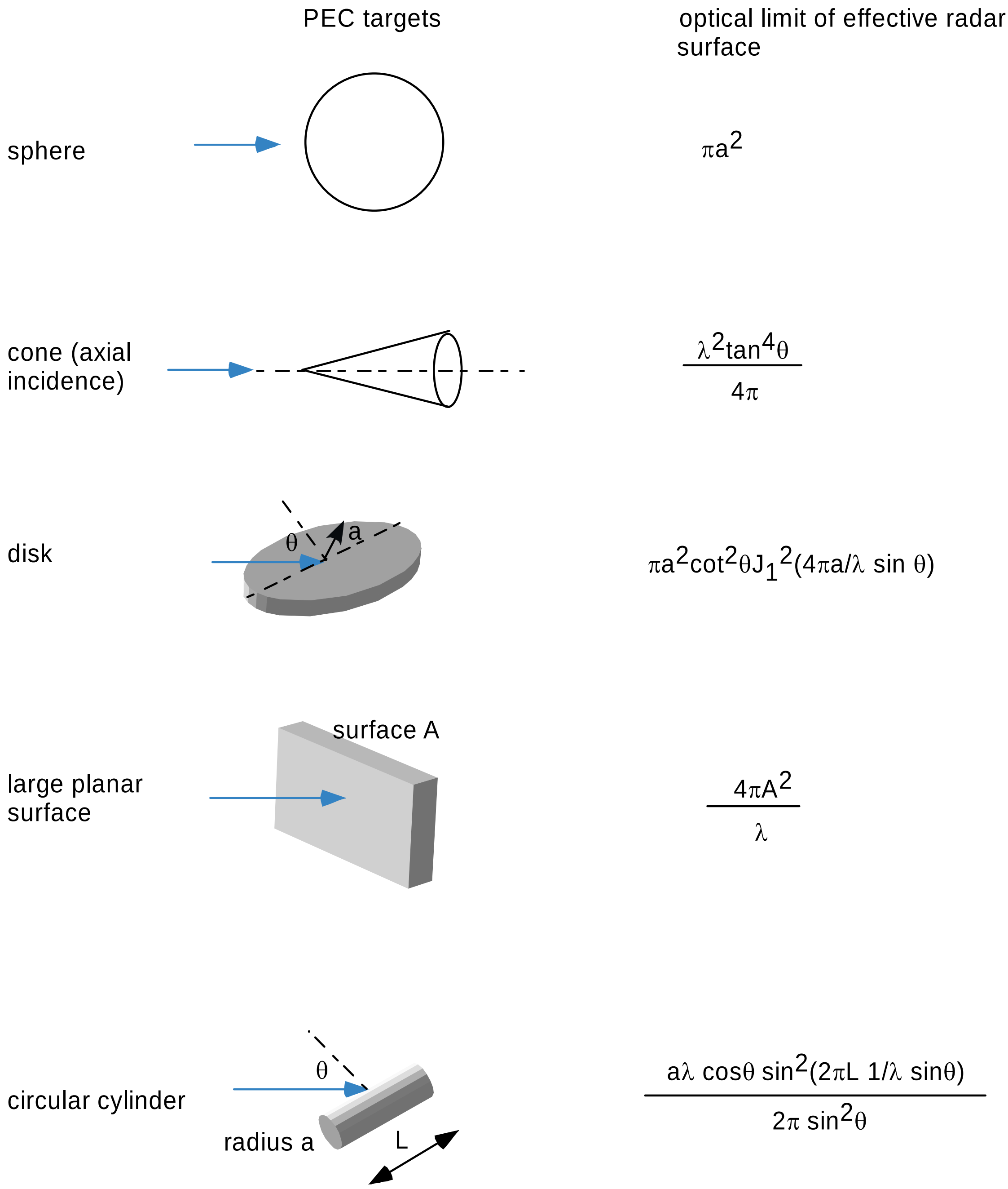



*Fig. 1. 18 : optical limit of equivalent radar surface for some canonical perfect electric conductors*

### *Pulse radar*

For the measurement of the distance to the target, we use a pulse modulation of the radar (fig. 1.19). The distance to the target is simply given by

$$R = \frac{c_o t_{ar}}{2} \qquad (1.\ 10)$$

where R is the distance, $t_{ar}$ the time between the emission of the pulse until the arrival of the reflected pulse and $c_o$ the velocity of light in free space. The advantage of pulsed radars is that the peak power can be much larger than the average power used by the system (typically a factor of 1000). The frequency of repetition of the pulses $f_r$ is set by the longest distance the radar has to detect (see fig. 1.19).

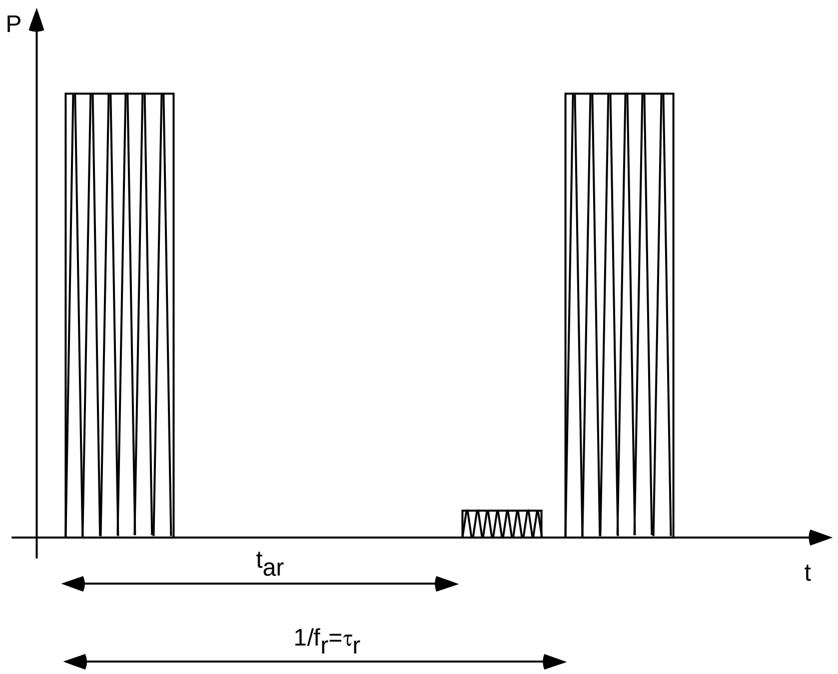
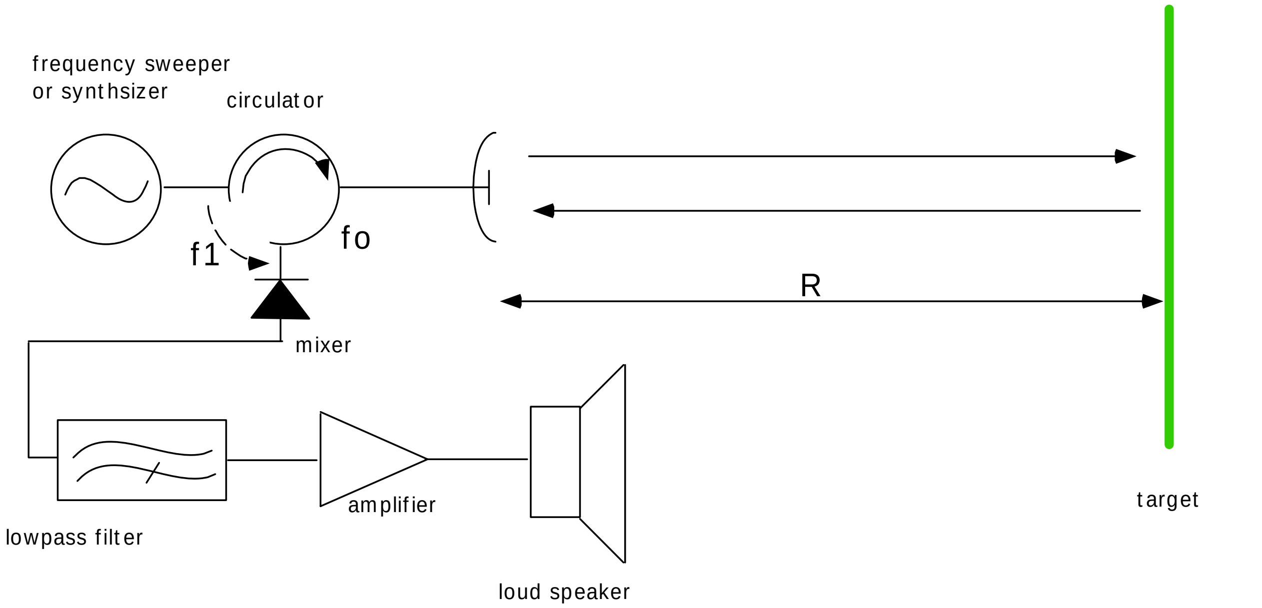



*Fig. 1. 19 : pulse radar*

### *Chirp radar*

For the measurement of short distances (automatic door opening systems, steering aids), another type of radar, the chirp radar is used. The principle of this radar system is depicted in figure 1.20. It consists of a frequency sweeper connected via a circulator to an antenna. The frequency of the signal emitted by the antenna changes thus linearly with the time. The signal reflected back by the target received by the antenna will thus have a different frequency than the signal emitted by the antenna at a specific instant of time (see fig. 1.21). This returning signal is mixed with the transmitted signal, and the difference in frequency is detected, giving a measure of the distance of the target.

# Radar "chirp"




$$R = \frac{c_0 \, t_{ar}}{2} = \frac{c_0 \Delta f}{2m}$$

*Fig. 1. 20 : Principle of the chirp radar*

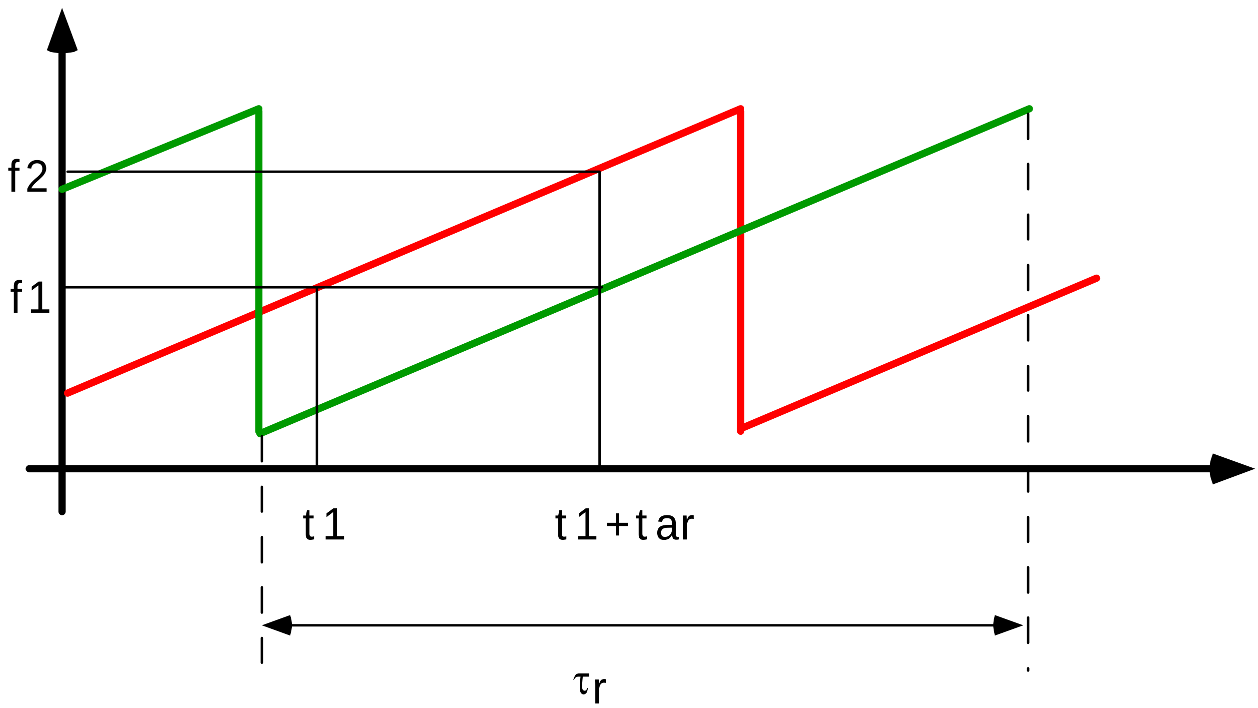



*Fig. 1. 21 : Transmitted and received signals at the antenna of the chirp radar*

*Doppler effect and Doppler radar*

The Doppler ef fect is the shift in wavelength, thus in frequency, due to a relative movement between a source and a receiver. This phenomenon was first described four acoustic waves by Christian Doppler in 1848.

Let us consider the four cases of an acoustic wave due to a point source shown in figure 1.22 :

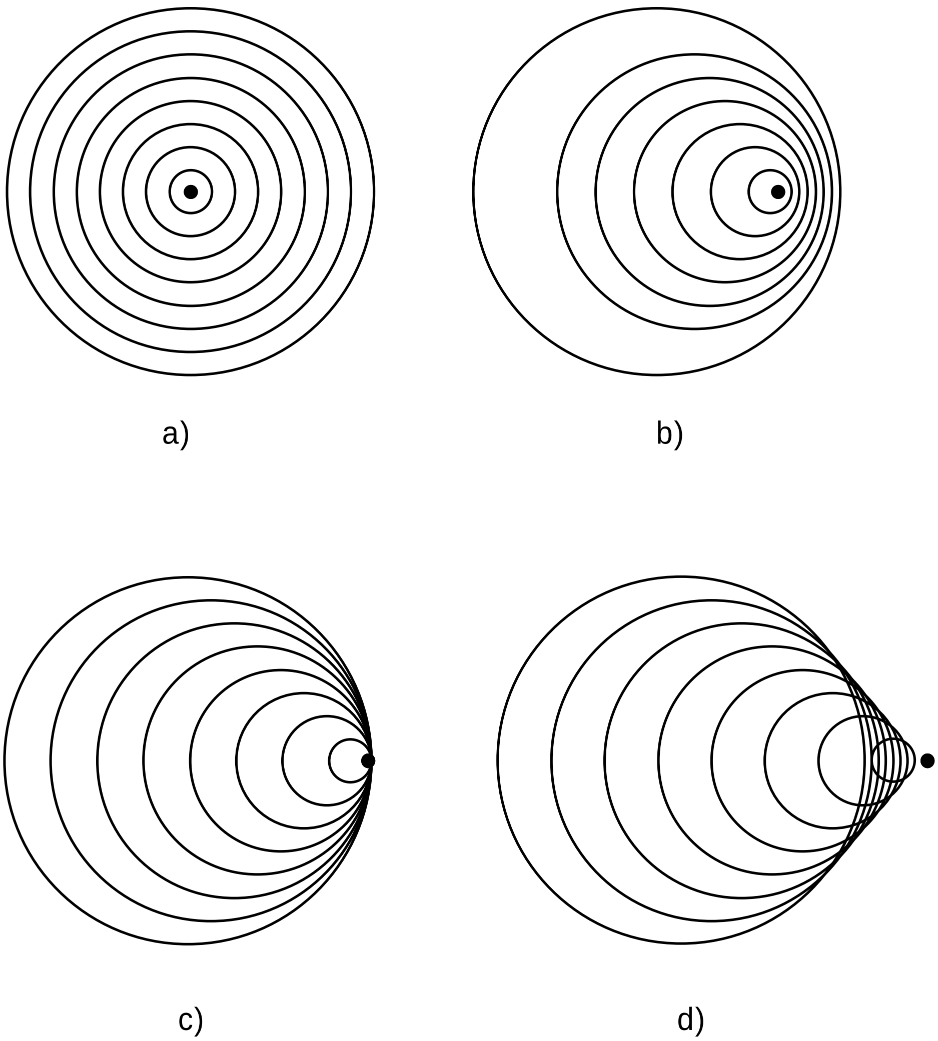



*Fig. 1. 22 : Illustration of the Doppler effect for an acoustic wave*

The first case illustrates an immobile source, on fig 1.22b) the source is moving with a velocity which is lower than the speed of sound, on fig1.22 c) the source is moving with a velocity equal to the speed of sound while on fig1.22d) the velocity of the source is larger than the speed of sound. In the second case, we see clearly that an observer towards which the source is traveling will perceive a signal with a wavelength shorter that the effective wavelength of the emitted signal, while in an observer placed in a position such as the sound source moves away from him will perceive a signal with a larger wavelength. The first observer will thus perceive a higher frequency, while the second a lower frequency. The frequency shift is directly correlated to the relative velocity of the source and the receiver.

This phenomenon exists of course also for electromagnetic waves, but in this case, we will have only the situations describes in figs 1.22 a) and b), as the velocity of the source can not be larger than the speed of light.

The Doppler effect in electromagnetic waves is used for velocity measurement, anti intrusion systems, etc.

*Doppler radar*

Let us consider the situation of figure 1.23, where a static emitter is transmitting a signal of frequency $f_o$ towards a receiver travelling with a certain velocity **v**.

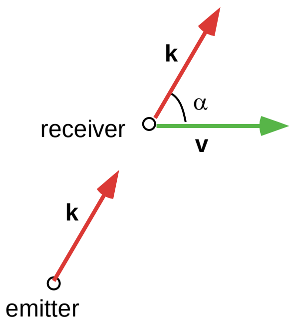


*Fig. 1. 23 : Doppler radar*

The frequency of the signal perceived at the receiver is given by :

$$f_r = f_o\left(1+\frac{v}{c}\cos\alpha\right) \qquad (1.\ 11)$$

Conversely, consider a moving emitter is transmitting towards a static receiver as presented in fig 23.b.

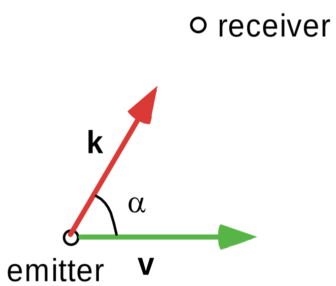


*Fig. 1. 24 : Doppler radar*

The frequency of the signal perceived at the receiver is given by :

$$f_r = \frac{f_o}{\left(1 + \frac{v}{c}\cos\alpha\right)}$$

where c is the speed of light.

### *1.5.2 Telecommunication*

Microwaves are often used for telecommunication links, either between two fixed terrestrial antennas (Hertzian links), for space applications or for mobile communications. Indeed, for the two latter applications, waves are the only alternative.

In a transmission in free space, between two fixed antennas, the received power is given by Friis' formula :

$$P_r = P_f g_1 g_2 \left(\frac{\lambda}{4\pi L}\right)^2 \qquad (1.\ 12)$$

where $P_r$ is the received power, $P_f$ the transmitted power, $g_1$,$g_2$ the antenna gains, $\lambda$ the wavelength and L the distance between the antennas. The term in brackets is called the link path loss, and is caused by the spherical wave nature of electromagnetic waves propagating in free space.

As we have seen in §1.4.4, the atmosphere is slightly inhomogeneous, which leads to a slight change in permittivity with altitude at microwave frequencies. This is taken into account in the planning of Hertzian links (point to point links on Earth) by substituting the radius of the earth by a corrected value when computing the length L of the link.

Another characteristic that has to be considered when planning point to point Hertzian links is the fact the propagation takes place over the Earth. Moreover, other obstacles (mountains, trees, buildings, etc) can perturb the transmission. The reflected wave will arrive with a different phase to the receiver than the direct wave, as the length of these paths is usually different (figure 1.24).

DIFFRACTION ON OBSTACLES

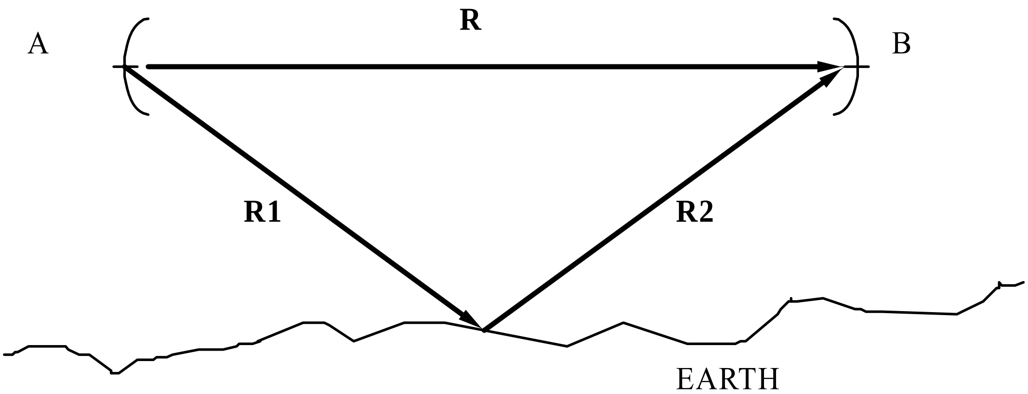


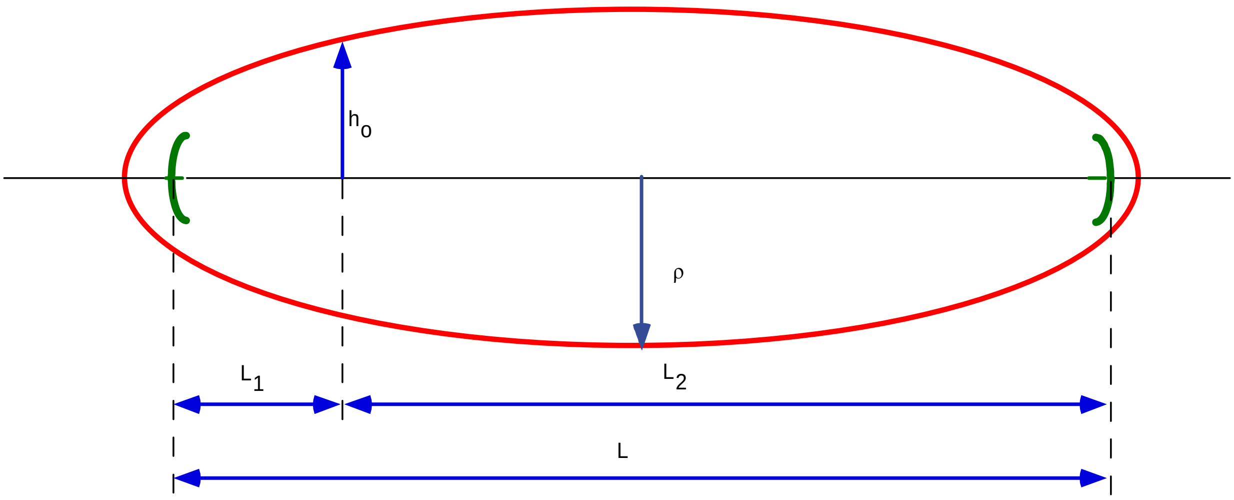

*Fig. 1. 25 : Diffraction on the Earth*

In order to avoid destructive interference between the direct and the reflected waves, Hertzian links are planned such that there is no obstacle in the link that would yield a reflected path which has a path length difference of half a wavelength with the direct path, as a difference of half a wave length would lead to a reflected wave being in phase opposition with direct path (destructive interference). This destructive interference will of course also happened for path length differences of $3\lambda/2$, $5\lambda/2$, etc., but as the path length of the reflected wave becomes much longer than the path length of the direct wave, the former will be sufficiently attenuated with respect to the latter to avoid a complete destructive interference.

We want thus to avoid to have obstacles such that (see fig. 1.24)

$$|\mathbf{R_1}| + |\mathbf{R_2}| - |\mathbf{R}| = \frac{\lambda}{2} \qquad (1.\ 13)$$

which is the equation of an ellipsoid of revolution : The Fresnel ellipsoid. From this equation, the height of the antenna masts such that the ellipsoid is empty of obstacles can be determined (figure 1.25)




1st Fresnel ellipsoid : $h_o = \sqrt{\dfrac{\lambda L_1 L_2}{L}}$ $\rho = \dfrac{1}{2}\sqrt{\lambda L}$

*Fig. 1. 26 : Fresnel ellipsoid*

In the case of mobile communication, it is not possible to avoid the reflection of the obstacles. Multi path communication is the rules, and different models exist to approximate the characteristics of the channel according to the environment : urban, suburban or countryside.

### *1.5.3 Other microwave applications*

- Microwave heating
- Material characterization
- Radiometry
- Remote sensing.

# 2. Maxwell's model: short summary

In this chapter the basics of Maxwell's model necessary for the following of the course will be summarized.

## 2.1 Definition

### *2.1.1 Electric Field **E***

A motionless particle with an electric charge q sustains an electrical force **Fe** called electrostatic force due to all other electrical charges. This force is proportional to q, thus the ratio Fe/q is independent of the considered particle, but indicates a local property of space. This property is called the electric field:

$$\mathbf{Fe}(t, \mathbf{r})/q=\mathbf{E}(t,\mathbf{r}) \ [V/m] \qquad (2.\ 1)$$

The electrostatic force and the electric field are vector quantities, which depend on time and position.

### *2.1.2 Induction field **B***

In addition to the electrostatic force, a moving loaded particle sustains a magnetic force **Fm**. This force is orthogonal to the velocity and to another vector property of space, called the induction field **B**(t,**r**). :

$$\mathbf{Fm}(t,\mathbf{r})/q = \mathbf{v}(t,\mathbf{r}) \times \mathbf{B}(t,\mathbf{r}) \ [V/m] \qquad (2.\ 2)$$

The dimension of the induction field **B**(t,**r**) is the tesla [T = Vs/$m^2$].

The magnetic force produced by the induction field **B** on moving charge is the basis for all electromechanical conversions (motors, generators, etc.)

### *2.1.3 Electric charge density*

The electric charge can be a point or distributed in space. The following four charge densities exist:

q [C=As] point charge

$\rho_l$ [As/m] line charge density

$\rho_s$ [As/m$^2$] surface charge density
$\rho$ [As/m$^3$] volume charge density

### *2.1.4 Electric current density*

In some media, the electric charges may move freely. The application of an electric force leads to a motion of the charges, which creates an electric current, whose density is defined by :

$$\mathbf{J}(t,\mathbf{r}) = \sum_i \rho_i \mathbf{v_i} \quad [As/m^2] \tag{2. 3}$$

The summation ports on all the types of charges particles moving in the considered medium. In many situations, it is the mean velocity $\mathbf{v}_i$ and not the acceleration which is proportional to the electric field. This is due to collisions between the particles. We define the conductivity σ [S/m]

$$\mathbf{J}(t,\mathbf{r}) = \sigma\mathbf{E}(t,\mathbf{r}) \quad [A/m^2] \tag{2. 4}$$

This is Ohm's law. We find free particles in classical conductors (metals), semiconductors, salt solutions, plasmas and electric arcs.

### *2.1.5 Surface current density Js*

A surface charge density $\rho_s$ can also move, producing a surface current density $\mathbf{J}_s(\mathbf{r},t)$. A surface current can flow on the surface between two different media, in particular when one of the media is a perfect electric conductor.

### *2.1.6 Dielectric properties*

In insulating media, the charges are bound to the atoms and molecules. When an electric field is applied, the charges sustain a force but are tied by the atom's cohesion forces. They can thus only moves slightly. Small dipoles form then in the medium, producing a polarisation field **P**(t,**r**), which depends on the applied electrical field. The combined effect of the electric field and the polarisation is called electric flux density.

$$\mathbf{D}(t,\mathbf{r}) = \varepsilon_0 \mathbf{E}(t,\mathbf{r}) + \mathbf{P}(t,\mathbf{r}) \tag{2. 5}$$

where $\varepsilon_0 = 8.854 \cdot 10^{-12}$ is the dielectric constant. In free space, and by extension in air, there is no polarisation.
In a lossless, isotropic and linear medium, the polarisation is a linear function of E(**r**, t). We can thus write:

$$\mathbf{D}(t,\mathbf{r}) = \varepsilon\mathbf{E}(t,\mathbf{r}) = \varepsilon_0 \varepsilon_r \mathbf{E}(t,\mathbf{r}) \tag{2. 6}$$

where $\varepsilon_r$ is the relative permittivity of the medium.

### *2.1.7 Magnetic properties*

The magnetic properties of materials result of a quantum property of the electron, called magnetic spin, which can be positive or negative.

In most elements, the number of positive spins equals the number of negative spins and the considered medium does not have any magnetic properties. In some materials, the so-called ferromagnetic materials (iron, nickel, cobalt, some rare earths, their oxides and their alloys), the numbers of positive and negative magnetic spins are different. The resulting magnetic moment yields a magnetization moment M(t,**r**). The magnetic field is then defined as:

$$\mathbf{B}(t,\mathbf{r}) = \mu_o[\mathbf{H}(t,\mathbf{r}) + \mathbf{M}(t,\mathbf{r})] \tag{2. 7}$$

where $\mu_0=4\pi\cdot10^{-7}$ Vs/Am. In free space and non ferromagnetic materials, there is no magnetization. In a lossless, isotropic and linear medium, the magnetization is a linear function of the magnetic field. Thus:

$$\mathbf{B}(t,\mathbf{r}) = \mu_o\mu_r\mathbf{H}(t,\mathbf{r}) \tag{2. 8}$$

where $\mu_r$ the relative permeability of the medium.

### *2.1.8 Properties of vacuum*

Vacuum, and by approximation air, are linear. There is neither polarization nor magnetization, and their relative permittivity and permeability are equal to one.

The constants $\varepsilon_o$ and $\mu_o$ are linked by :

$$\frac{1}{\sqrt{\varepsilon_o\mu_o}} = c_o \cong 3\cdot10^8 \quad [m/s] \tag{2. 9}$$

which yields the free space light velocity, and

$$\sqrt{\frac{\mu_o}{\varepsilon_o}} = Z_o \cong 120\pi \cong 376.\bar{6} \quad [\Omega] \tag{2. 10}$$

which yields the characteristic impedance of free space.

## 2.2 Maxwell's equations

### *2.2.1 In time domain*

The four vector fields defined above are independent. They are linked by Maxwell's equations in all points which belong not to an interface between two media:

$$\nabla \times \mathbf{E}(t,\mathbf{r}) = -\frac{\partial \mathbf{B}(t,\mathbf{r})}{\partial t} \qquad \nabla \cdot \mathbf{D}(t,\mathbf{r}) = \rho(t,\mathbf{r})$$
$$\nabla \times \mathbf{H}(t,\mathbf{r}) = \frac{\partial \mathbf{D}(t,\mathbf{r})}{\partial t} + \mathbf{J}(t,\mathbf{r}) \qquad \nabla \cdot \mathbf{B}(t,\mathbf{r}) = 0 \tag{2. 11}$$

If we take the divergence of the second equation and use the third, we get the continuity equation:

$$\frac{\partial \rho(t,\mathbf{r})}{\partial t} + \nabla \cdot \mathbf{J}(t,\mathbf{r}) = 0 \tag{2. 12}$$

### *2.2.2 In frequency domain*

In this course, we will consider time harmonic waves. Indeed, even non harmonic phenomena like transients are often studied by decomposing the time domain wave in its frequency spectrum using Fourier's transform. For a sinusoidal wave of pulsation $\omega = 2\pi f$, the time dependence is of the type cos $(\omega t + \phi)$ and we can write

$$A\cos(\omega t + \phi) = \mathrm{Re}\,[Ae^{j\phi}\exp(j\omega t)] = \sqrt{2}\,\mathrm{Re}\,[\frac{A}{\sqrt{2}}e^{j\phi}\exp(j\omega t)] = \sqrt{2}\,\mathrm{Re}\,[A_e e^{j\phi}\exp(j\omega t)] \tag{2. 13}$$

A is the peak value and $A_e = A/\sqrt{2}$ is the effective value.

The true time dependent physical values f(**r**,t) (f=**E, D, H, B, J**,$\rho$) are replaced by complex time independent values, called phasors <u>f(**r**)</u> (f=<u>**E**</u>, <u>**D**</u>, <u>**H**</u>, <u>**B**</u>, <u>**J**</u>, <u>ρ</u>), using the relation :

$$f(t,r) = \sqrt{2}\,\mathrm{Re}\left[\underline{f(r)}e^{j\omega t}\right] \quad (2.\ 14)$$

The factor $\sqrt{2}$ is introduced in the definition in order that the modulus of the phasor corresponds to the effective value of the signal. *In some textbooks, this factor is not introduced in the phasors's definition. The norm of the phasor is then the peak value of the signal, and a factor ½ appears in power and energy related formulas.*

The introduction of phasors allows replacing time domain derivation by a multiplication by jω. Maxwell's equations in phasor notation become:

$$\begin{aligned} \nabla \times \underline{\mathbf{E}}(\mathbf{r}) &= -j\omega \underline{\mathbf{B}}(\mathbf{r}) & \nabla \cdot \underline{\mathbf{D}}(\mathbf{r}) &= \underline{\rho}(\mathbf{r}) \\ \nabla \times \underline{\mathbf{H}}(\mathbf{r}) &= j\omega \underline{\mathbf{D}}(\mathbf{r}) + \underline{\mathbf{J}}(\mathbf{r}) & \nabla \cdot \underline{\mathbf{B}}(\mathbf{r}) &= 0 \end{aligned} \quad (2.\ 15)$$

and the continuity equation :

$$\nabla \cdot \underline{\mathbf{J}}(\mathbf{r}) + j\omega \underline{\rho}(\mathbf{r}) = 0 \quad (2.\ 16)$$

*From this point on, we will always work in the frequency domain, and in order to simplify notation phasors will not be underlined anymore.*

In this course, we will limit ourselves to linear media. We have seen that in this case

$$\mathbf{D} = \varepsilon \mathbf{E} \quad ; \quad \mathbf{B} = \mu \mathbf{H} \quad (2.\ 17)$$

where now ε and μ are two constants defining the medium (permittivity and permeability), which can in general be frequency dependent and complex.

$$\varepsilon = \varepsilon' - j\ \varepsilon'' \quad ; \quad \mu = \mu' - j\ \mu'' \quad (2.\ 18)$$

A complex value for ε implies, when we transform back to time domain, that **D**(**r**,t) et **E**(**r**,t) have the same pulsation ω but are not in phase. Moreover, if Im(ε) < 0, **D** is late with respect to **E**. A negative imaginary part of ε is linked to causality (Kramers-König relations which are similar to Bode's relations in circuit theory) and correspond physically to the existence of losses in the medium. The same consideration can be done for **B**, **H**. As an example water at 1 GHz has $\varepsilon_r = (80 - j10)$ , $\mu_r = 1$.

Finally we must note that the pulsation of an electromagnetic phenomenon is unchanged by a linear medium.

Relations (2.17) imply that only two vectors **E, H,** are necessary to describe an electromagnetic phenomenon in a linear medium. We can thus rewrite Maxell's equations as :

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \qquad \nabla \bullet \mathbf{E} = \rho/\varepsilon$$
$$\nabla \times \mathbf{H} = \mathbf{J} + j\omega\varepsilon\mathbf{E} \qquad \nabla \bullet \mathbf{H} = 0 \qquad (2.19)$$

## 2.3 Boundary conditions

In presence of a boundary separating two different media #1 and #2, Maxwell's equations must be completed by the following boundary conditions (Fig 2.1)

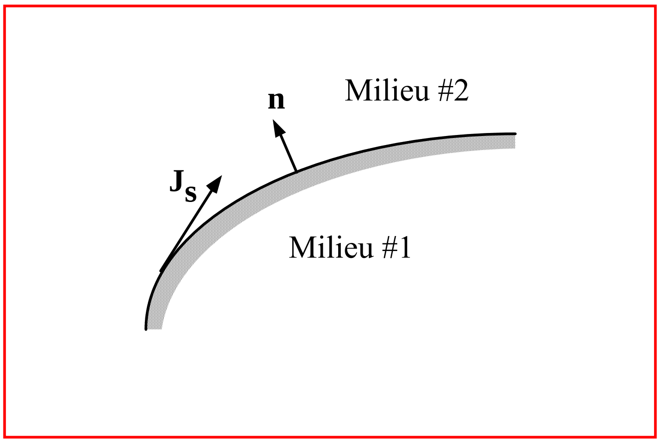



*Fig. 2. 1: Boundary conditions*

$$\mathbf{n} \times [\mathbf{E_1} - \mathbf{E_2}] = \mathbf{0} \qquad \mathbf{n} \times [\mathbf{H_1} - \mathbf{H_2}] = \mathbf{J_s} \qquad (2.20)$$

for the tangential components

$$\mathbf{n}\cdot\left[\varepsilon_{r1}\mathbf{E_1}-\varepsilon_{r2}\mathbf{E_2}\right]=\rho_s \qquad \mathbf{n}\cdot\left[\mu_{r1}\mathbf{H_1}-\mu_{r2}\mathbf{H_2}\right]=0 \tag{2. 21}$$

for the normal components. **n** is the unitary vector normal to the surface pointing from medium #1 to medium #2, $\mathbf{J_s}$ is an eventual surface current [A/m] exiting at the interface and $\rho_s$ is the surface charge density which my exist between the media. .

## 2.4 Electric and magnetic energy

The following definitions are valid for time harmonic fields:

| | | |
|---|---|---|
| $w_e = (1/2)\ \varepsilon\mathbf{E}\bullet\mathbf{E}$ | [J/m$^3$] : | Mean value of the electric energy density at one point |
| $w_m = (1/2)\ \mu\mathbf{H}\bullet\mathbf{H}$ | [J/m$^3$] : | Mean value of the magnetic energy density at one point |
| $\mathbf{S} = \mathbf{E} \times \mathbf{H}^*$ | [W/m$^2$] : | Poynting vector ( mean value of the power flux at one point) |

The integration of Maxwell's equations over a volume *v*, enclosed by a surface *s* yields Poynting's theorem:

$$\int_s ds\ \mathbf{S}\bullet\hat{\mathbf{n}} \ + \ j\omega\int_v dv\ (w_e + w_m) = -\int_v dv\ \ \mathbf{J}\bullet\mathbf{E} \quad , \quad [w] \tag{2. 22}$$

where **n** is the unitary normal vector to s pointing towards outside. The first term to the left is the flux of the Poynting vector, i.e. the power escaping from the volume through the surface *s*. The second term on the left side corresponds to the reactive power in the volume. The sum of those two powers is equal to the power given by the current sources.

## 2.5 Potentials

### *2.5.1 Magnetic vector potential*

Knowing that the divergence of a rotational is always identical to zero and that the divergence of the induction field **B(r)** is zero, we define the magnetic vector potential in the following way :

:

$$\nabla \cdot \nabla \times \mathbf{A} \equiv \mathbf{0} \text{ and } \nabla \cdot \mathbf{B} = \mathbf{0}, \text{ thus } \mathbf{B} = \nabla \times \mathbf{A} \qquad (2.\ 23)$$

This relation defines the magnetic vector potential short to an irrotational factor. We may thus replace **A** by $\mathbf{A} + \nabla\Phi$ where Φ is an arbitrary function, as the rotational of a gradient is always equal to zero. On have thus a certain degree of freedom in choosing the definition of **A**.

### *2.5.2 Electric scalar potential*

Combining (2. 24) with Maxwell's first equation, we get :

$$\nabla \times \left(E + j\omega\mathbf{A}\right) = 0 \qquad (2.\ 25)$$

As the rotational of a gradient is always zero, we can define a scalar function V such that :

$$\mathbf{E} + j\omega\mathbf{A} = -\nabla V \qquad (2.\ 26)$$

This function is called electric scalar potential and is defined short to a constant. We have chosen a negative sign, because the convention states that the field lines go from the positive potential to the negative potential

## 2.6 Wave equation

### *2.6.1Source and induced currents*

In all electromagnetic problems, we admit the existence of source currents $\mathbf{J_{src}}$ which are not modified by the fields they create or by any other field. These sources generate the electromagnetic excitation fields. If any object is placed close to these excitation fields, they will produce induced currents $\mathbf{J_{ind}}$ on the object. In turn, these induced currents in the object will generate diffracted fields. The total fields are the sum of the excitation and the diffracted fields.

There is no physical difference between source currents and induced currents, as in both cases they are merely moving electrons. We will however establish a conceptual difference, subtle but essential :

* $\mathbf{J}_{src}$ is a known imposed current. it is not affected by the existing fields. It is the fundamental source of the problem.
* $\mathbf{J}_{ind}$ is a current depending on the total field, in general unknown.

In Maxwell's equation

$$\nabla \times \mathbf{H} = j\omega\varepsilon\,\mathbf{E} + \mathbf{J}$$

the current $\mathbf{J}$ is total current and depend thus on the fields. We want to put in evidence the field independent part of the current, which will play the mathematical role of the inhomogeneous term in the differential equation. We write thus $\mathbf{J} = \mathbf{J}_{src} + \mathbf{J}_{ind}$ where $\mathbf{J}_{src}$ is the known "source" part independent of the field and $\mathbf{J}_{ind}$ is the induced part. For object made of linear materials, this induced current is linked exclusively to the total electric field via Ohm's law : $\mathbf{J}_{ind}= \sigma\mathbf{E}$ . We can then write :

$$j\omega\varepsilon\,\mathbf{E} + \mathbf{J} = j\omega\varepsilon\,\mathbf{E} + \mathbf{J}_{ind} + \mathbf{J}_{src} = (j\omega\varepsilon + \sigma)\,\mathbf{E} + \mathbf{J}_{src} = j\omega\varepsilon_T\mathbf{E} + \mathbf{J}_{src} \quad (2.\ 27)$$

and we obtain finally Maxwell's equation in the wanted form:

$$\nabla \times \mathbf{H} = = j\omega\varepsilon_T\mathbf{E} + \mathbf{J}_{src} \quad (2.\ 28)$$

where a global permittivity has been introduced.

$$\varepsilon_T = \varepsilon - j\,\sigma/\omega. \quad (2.\ 29)$$

$\nabla.\mathbf{E} = \rho/\varepsilon$ is replaced $\nabla.\mathbf{E} = \rho_{src}/\varepsilon_T$ in an analogue way.

### *2.6.2 Maxwell's equations far away from the sources*

In this course we shall focus on the propagation phenomena that occur when the signal has quit the generator (source) and propagated towards a receiver. Thus, we will consider a medium without sources and write Maxwell's equations in the following way, which we will often use during this course:

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \qquad \nabla \bullet \mathbf{E} = 0$$
$$\nabla \times \mathbf{H} = j\omega\varepsilon_T\mathbf{E} \qquad \nabla \bullet \mathbf{H} = 0 \qquad (2.30)$$

However and in order to simplify the notation, the permittivity will be write as ε instead of $\varepsilon_T$, implicitly meaning that this complex value takes into account not only the dielectric losses but also an imaginary part σ/ω in the presence of ohmic losses.

### *2.6.3 Wave equation*

We take the rotational of the two first Maxwell's equations, and using vector calculus we show that in a source free region the electric and magnetic field satisfy the following wave (or Helmoltz) equations:

$$\nabla^2\mathbf{E} + \omega^2\mu\varepsilon\mathbf{E} = 0 \quad ; \quad \nabla^2\mathbf{H} + \omega^2\mu\varepsilon\mathbf{H} = 0 \qquad (2.31)$$

or in compact notation

$$(\nabla^2 + k^2)\mathbf{E} = 0 \quad ; \quad (\nabla^2 + k^2)\mathbf{H} = 0 \qquad (2.32)$$

For a given medium and frequency, $k = \omega\sqrt{(\mu\varepsilon)}$ is a complex constant called the *wave number*.

### *2.6.4 Plane waves*

The simplest solution to the wave equations in an infinite unbounded medium is the electromagnetic plane wave, whose fields are given by:

$$\mathbf{E}(\mathbf{r}) = \mathbf{E_0} \exp(-jk\hat{\mathbf{n}}\cdot\mathbf{r}), \quad \text{or} \quad \mathbf{E}(\mathbf{r},t) = \sqrt{2}\mathbf{E}_0 \cos(\omega t - k\hat{\mathbf{n}}\cdot\mathbf{r}) \qquad (2.33)$$

**n** being the unitary vector in the propagation direction. The product of k**n**, written **k**, is sometimes called the propagation vector. The propagation velocity of a wave is given by

$$v = \frac{k}{\omega} = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c_0}{\sqrt{\mu_r \varepsilon_r}} \tag{2. 34}$$

where $c_0$ is the free space light velocity. The associated wavelength is then given by

$$\lambda = 2\pi/k = v/f \tag{2. 35}$$

Plane waves are characterised by the fact that the three vectors **E**, **H** and **k** (propagation vector) are mutually orthogonal and form a direct system of reference. **E** × **H** is in the direction of **k**. Moreover, the phase of **E** and **H** are constant on planar surfaces, which are named equiphase planes, and are orthogonal to the direction of propagation of the wave.

**Example:**

Lets define a Cartesian system of reference along **E**, **H** and **k** respectively, so that the only non-zero components of the fields and the wave vector are $E_x$, $H_y$ and $k_z$. The wave equation becomes:

$$\frac{d^2 E_x}{dz^2} + k^2 E_x = 0 \tag{2. 36}$$

The solutions to this equation are linear combination of $e^{jkz}$ et $e^{-jkz}$. Let consider the solution

$$E_x = E_0 e^{-jkz} \tag{2. 37}$$

which is a wave travelling in the positive direction of z. The associated magnetic field is obtained by

$$j\omega\mu\mathbf{H} = -\nabla \times \mathbf{E} = \hat{\mathbf{y}} \;\; jk \;\; E_x$$

$$E_x = \sqrt{\frac{\mu}{\varepsilon}} H_y \tag{2. 38}$$

The proportionality factor between the two fields has the dimension of impedance. It is the characteristic impedance of the medium.

$$Z_c = \sqrt{\frac{\mu}{\varepsilon}} \qquad \text{In free space}: Z_c = 120\pi \cong 377\Omega \tag{2. 39}$$

This quantity has for radiating waves the same role than the characteristic impedance for transmission lines.

The fields in time domain are given by

$$\begin{aligned} \mathrm{E}_x &= E_0 \cos(\omega t - kz) \\ \mathrm{H}_y &= \frac{E_o}{Z_c} \cos(\omega t - kz) \end{aligned} \tag{2. 40}$$

In a lossless medium (free space for instance), the Poynting vector is purely real and directed along **k**. The transmitted power density is equal to

$$P = \frac{E_0^2}{Z_c} = Z_c H_0^2 \tag{2. 41}$$

*2.6.5 Spherical waves*

Another very useful solution to the wave equations is obtained by resolving the latter in spherical coordinates. The obtained solution is then called spherical wave, and its equiphase surfaces are spheres. This means that, if the origin of a spherical coordinate system is placed on the source of a spherical wave, the propagation is radial.

The electric and magnetic fields and the propagation vector are mutually orthogonal and form a direct reference system, so we can choose the reference system in a way that :

$$\begin{aligned} \mathbf{k} &= k\mathbf{e_r} \\ \mathbf{E} &= E_\theta \mathbf{e}_\theta \\ \mathbf{H} &= H_\varphi \mathbf{e}_\varphi \\ E_\theta &= Z_c H_\varphi = \sqrt{\frac{\mu}{\varepsilon}} H_\varphi \end{aligned} \tag{2. 42}$$

A spherical wave has the following form:

$$E_\theta = E_0 \frac{e^{-jkr}}{r} \tag{2. 43}$$

for a wave travelling away from the origin

### *2.6.6 Wave polarisation*

The orientation of the electric field is called the polarisation of the electromagnetic wave. It can be of three types: linear, circular or elliptical.

**Linear polarisation**

The orientation of the electric field remains unchanged as a function of time at a specified point of the space. For a wave travelling close to the earth's surface, we often use the terms vertical or horizontal polarisations for a vertical or horizontal electric field.

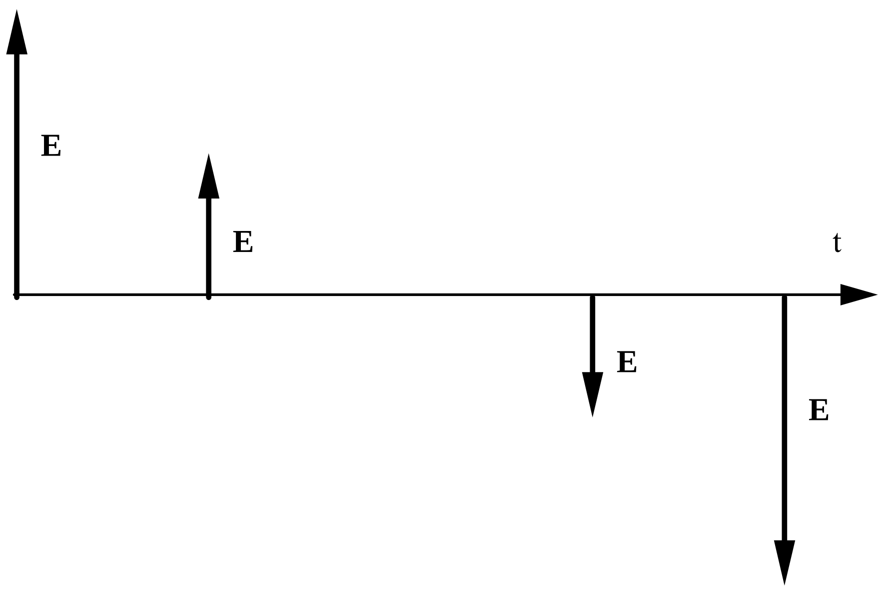



*Fig. 2. 2: linear polarisation*

**Circular polarisation**

The polarisation of a wave is circular when at a fixed point in space the extremity of the phasor representing the electric field describes a circle. If the phasor turns clockwise we talk about right hand circular polarisation (RHCP), id it turns counter clockwise we talk about left hand circular polarisation (LHCP).

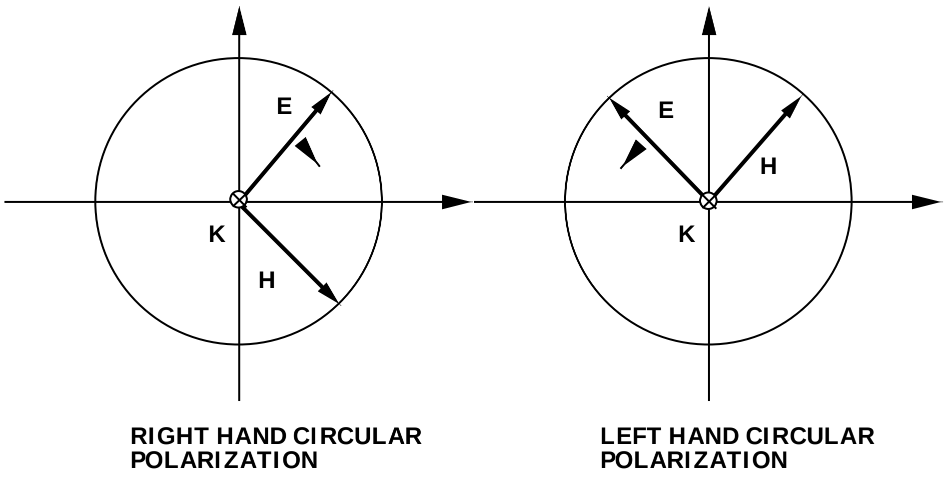



*Fig. 2. 3: Circular polarisation*

**Elliptical polarisation**

The polarisation of a wave is elliptical if the extremity of the phasor representing the electric field describes an ellipse at a fixed point in space. This is the most general case.

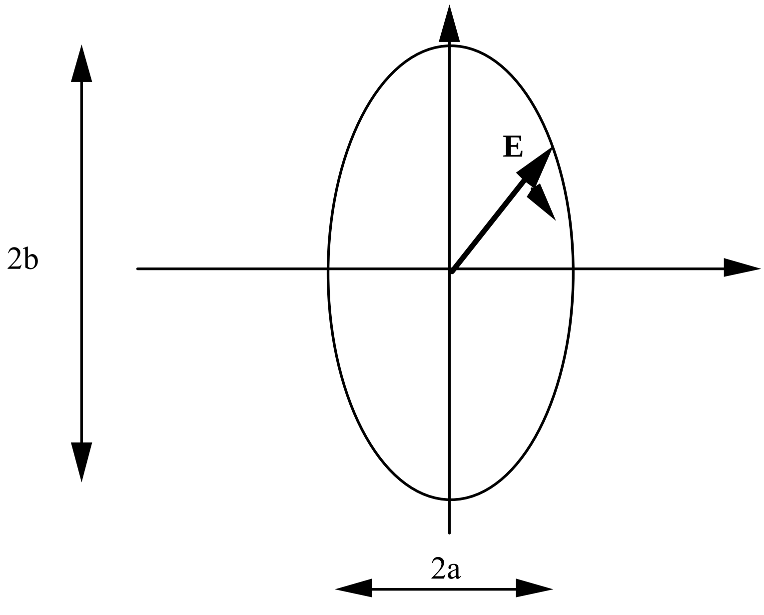



*Fig. 2. 4: Elliptical polarisation*

## *2.6.7 Polarisation characteristics of a field.*

A time harmonic electric field is defined as:

$$\boldsymbol{E}(t) = \sqrt{2}\left[\mathbf{e_x}E_{0x}\cos(\omega t + \varphi_x) + \mathbf{e_y}E_{0y}\cos(\omega t + \varphi_y) + \mathbf{e_z}E_{0z}\cos(\omega t + \varphi_z) + \right] \tag{2. 44}$$

which can be written as

$$\boldsymbol{E}(t) = E(0)\cos(\omega t) + E(T/4)\sin(\omega t) \tag{2. 45}$$

where

$$\begin{aligned} \boldsymbol{E}(0) &= \sqrt{2}\left[\mathbf{e_x}E_{0x}\cos(\varphi_x) + \mathbf{e_y}E_{0y}\cos(\varphi_y) + \mathbf{e_z}E_{0z}\cos(\varphi_z)\right] \\ \boldsymbol{E}(T/4) &= -\sqrt{2}\left[\mathbf{e_x}E_{0x}\sin(\varphi_x) + \mathbf{e_y}E_{0y}\sin(\varphi_y) + \mathbf{e_z}E_{0z}\sin(\varphi_z)\right] \end{aligned} \tag{2. 46}$$

or in term of phasor vector

$$\begin{aligned} \boldsymbol{E}(0) &= \mathrm{Re}\left[\sqrt{2}\mathbf{E}\right] \\ \boldsymbol{E}(T/4) &= -\mathrm{Im}\left[\sqrt{2}\mathbf{E}\right] \end{aligned} \tag{2. 47}$$

Vectors $\boldsymbol{E}(0)$ and $\boldsymbol{E}(T/4)$ are two conjugated axes of the polarisation ellipse. In the case of a linear polarisation they are collinear, which can be written as :

$$\begin{aligned} &\boldsymbol{E}(0) \times \boldsymbol{E}(T/4) = 0 \\ &\left\langle E^2 \right\rangle \neq 0 \end{aligned} \tag{2. 48}$$

In the case of a circular polarisation, the two half axes of the ellipse are orthogonal and have the same length. We write thus:

$$\begin{aligned} &\boldsymbol{E}(0) \cdot \boldsymbol{E}(T/4) = 0 \\ &\left|\boldsymbol{E}(0)\right| = \left|\boldsymbol{E}(T/4)\right| \neq 0 \end{aligned} \tag{2. 49}$$

Which is term of phasors yields:

$$\mathbf{E} \cdot \mathbf{E} = 0 \tag{2. 50}$$

# Chapter 3: Transmission lines

Transmission line theory has a great kinship to standard circuit analysis, with one major difference, being the electrical size : In circuit analysis we assume that the physical dimensions of a circuit are much smaller than the wavelength, whereas a transmission line can have any dimension between a fraction of wavelength (electrically short) to many wavelength (electrically large). Thus, a transmission line is a distributed parameter network.

## 3.1 Incremental model

Let us consider an incremental length of a transmission line, as represented by a section of a two wire line in figure 3.1a. If the segment $\Delta z$ is short, its equivalent circuit can be represented as in figure 3.1b, where R, L, G and C are per unit length quantities :

R = series resistance per unit length, for both conductors [Ω/m]

L = series inductance per unit length, for both conductors [H/m]

G = shunt conductance per unit length [S/m]

C = shunt capacitance per unit length [F/m]

The series inductance represents the total self inductance of the conductors, while the shunt capacitance is due to the proximity of the conductors. The series resistance represents conductive losses and the shunt conductance dielectric losses. A transmission line of finite length can be viewed as a cascade of incremental transmission lines.

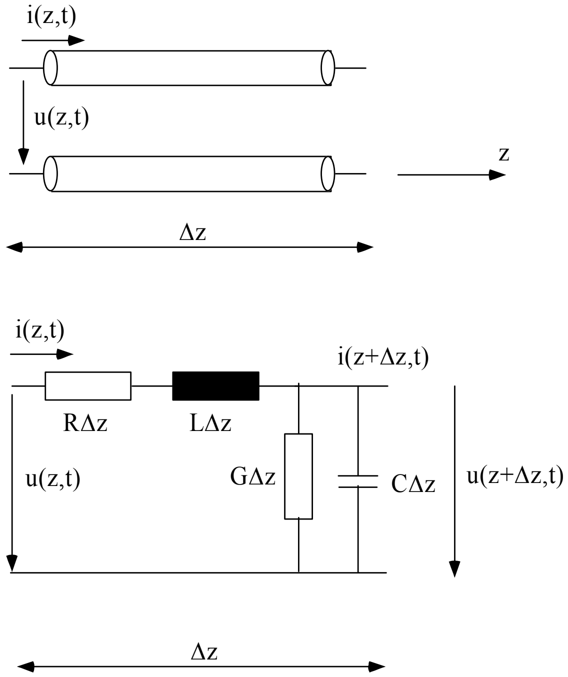


Fig. 3. 1: *Definition of an incremental transmission line*

We can apply Kirchhoff's laws to the circuit of figure 3.1, and obtain :

$$
\begin{aligned}
&v(z,t) - R\Delta z i(z,t) - L\Delta z \frac{\partial i(z,t)}{\partial t} - v(z+\Delta z,t) = 0 \\
&i(z,t) - G\Delta z v(z+\Delta z,t) - C\Delta z \frac{\partial v(z+\Delta z,t)}{\partial t} - i(z+\Delta z,t) = 0
\end{aligned}
\qquad (3.\ 1)
$$

Dividing those relations by Δz and taking the limit as Δz->0 yields the following differential equations for the voltage and the current on the line :

$$
\begin{aligned}
\frac{\partial v(z,t)}{\partial z} &= -Ri(z,t) - L\frac{\partial i(z,t)}{\partial t} \\
\frac{\partial i(z,t)}{\partial z} &= -Gv(z,t) - C\frac{\partial u(z,t)}{\partial t}
\end{aligned}
\qquad (3.\ 2)
$$

These equations are the time domain form of the telegrapher or transmission line equation. For harmonic waves (sinusoidal steady-state condition) and in phasor notation, (3.2) simplifies to :

$$
\begin{aligned}
\frac{dV(z)}{dz} &= -(R + j\omega L) I(z) \\
\frac{dI(z)}{dz} &= -(G + j\omega C) V(z)
\end{aligned}
\qquad (3.\ 3)
$$

These equations can be solved simultaneously to give a wave equation for either V(z) or I(z) :

$$\begin{aligned} \frac{d^2V(z)}{dz^2} - \gamma^2 V(z) = 0 \\ \frac{d^2I(z)}{dz^2} - \gamma^2 I(z) = 0 \end{aligned} \tag{3. 4}$$

where

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)} \tag{3. 5}$$

is the complex propagation constant. The imaginary part, β, is called the phase constant, while the real part, α, is the attenuation constant. Note that the propagation constant is in general a function of frequency.

The solutions to (3.4) are called travelling waves and can be described as :

$$\begin{aligned} V(z) = V_0^+ \mathrm{e}^{-\gamma z} + V_0^- \mathrm{e}^{\gamma z} \\ I(z) = I_0^+ \mathrm{e}^{-\gamma z} + I_0^- \mathrm{e}^{\gamma z} \end{aligned} \tag{3. 6}$$

were the $\mathrm{e}^{-\gamma z}$ term represents a wave travelling in the +z direction, and the $\mathrm{e}^{\gamma z}$ a wave travelling in the –z direction. Applying (3.3) to (3.6) gives the current on the line :

$$I(z) = \frac{\gamma}{R + j\omega L}\left(V_0^+ \mathrm{e}^{-\gamma z} - V_0^- \mathrm{e}^{\gamma z}\right) \tag{3. 7}$$

If we define the characteristic impedance of the line as

$$Z_o = \frac{R + j\omega L}{\gamma} = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \tag{3. 8}$$

we can write :

$$Z_o = \frac{V_0^+}{I_0^+} = \frac{-V_o^-}{I_o^-} \tag{3. 9}$$

Then the second equation of (3.6) can be rewritten as :

$$I(z) = \frac{V_0^+}{Z_o} e^{-\gamma z} - \frac{V_o^-}{Z_o} e^{\gamma z} \tag{3. 10}$$

Converting into time domain, we get :

$$v(z,t) = \sqrt{2}\left|V_o^+\right| \cos\left(\omega t - \beta z + \theta^+\right) e^{-\alpha z} + \sqrt{2}\left|V_o^-\right| \cos\left(\omega t + \beta z + \theta^-\right) e^{\alpha z} \tag{3. 11}$$

where θ is the phase of the complex voltage V.

The wavelength of the travelling wave is defined as the distance between two successive points of equal phase at a fixed instant of time, which is given by :

$$\lambda = \frac{2\pi}{\beta} \tag{3. 12}$$

The phase velocity of the wave is defined as the speed at which constant phase points travel along the line :

$$v_\varphi = \frac{dz}{dt} = \frac{\omega}{\beta} = \lambda f \tag{3. 13}$$

since $\omega=2\pi f$.

## 3.2 Lossless transmission lines

In many practical cases, the loss of the line is very small and can be neglected. Setting R=G=0 in the above results yields

$$\begin{aligned} \gamma &= j\beta = j\omega\sqrt{LC} \\ Z_o &= \sqrt{\frac{L}{C}} \end{aligned} \tag{3. 14}$$

where β and $Z_o$ are real numbers. The general solution for the voltage and the current on the transmission line can be written as

$$V(z) = V_0^+ \mathrm{e}^{-j\beta z} + V_0^- \mathrm{e}^{j\beta z}$$
$$I(z) = I_0^+ \mathrm{e}^{-j\beta z} + I_0^- \mathrm{e}^{j\beta z} = \frac{V_0^+}{Z_o} \mathrm{e}^{-j\beta z} - \frac{V_0^-}{Z_o} \mathrm{e}^{j\beta z} \tag{3. 15}$$

The wavelength on the line is

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{LC}} \tag{3. 16}$$

and the phase velocity of the line is

$$v_\varphi = \frac{dz}{dt} = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} \tag{3. 17}$$

### 3.3 Terminated transmission lines

A lossless transmission line terminated by an arbitrary impedance is depicted in figure 3.2

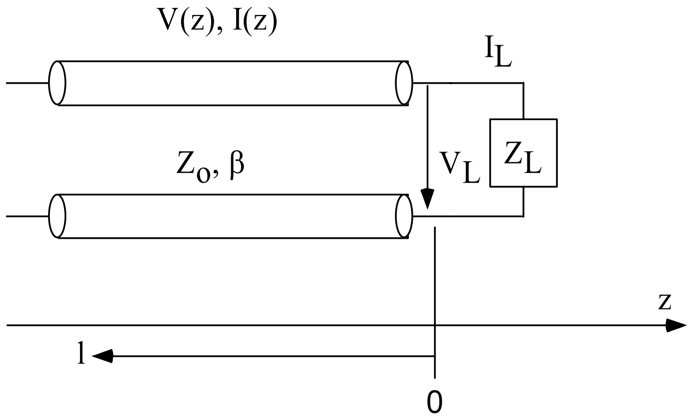


*Fig. 3. 2 : A terminated transmission line*

We suppose that we have an incident wave of the form $V_0^+ \mathrm{e}^{-j\beta z}$ travelling on the line. This wave is generated at a source at z<0. We have seen that the ratio of voltage to current for such a travelling wave is $Z_o$. When the line is terminated in an arbitrary load $Z_L \neq Z_o$, the ratio of voltage to current in the load must equal $Z_L$. Thus, a reflected wave must be generated at the load, with appropriate amplitude to satisfy this condition. The total voltage on the line can be written as in (3.15), as a sum of an incident and a reflected wave. The total voltage and current at the load are related by the load impedance, so at z=0, we must have :

$$Z_L = \frac{V(0)}{I(0)} = \frac{V_0^+ + V_0^-}{V_0^+ - V_0^-} Z_0 \qquad (3.\ 18)$$

Solving for the reflected wave, we get :

$$V_0^- = \frac{Z_L - Z_o}{Z_L + Z_o} V_0^+ \qquad (3.\ 19)$$

We can thus define the voltage reflection coefficient Γ :

$$\Gamma = \frac{V_0^-}{V_0^+} = \frac{Z_L - Z_o}{Z_L + Z_o} \qquad (3.\ 20)$$

A reflection coefficient for the current could also be defined, and will be exactly the negative of the voltage reflection coefficient. We will thus not use it in this course.

The voltage and current on the line can be written as :

$$\begin{aligned} V(z) &= V_0^+ \left( e^{-j\beta z} + \Gamma e^{j\beta z} \right) \\ I(z) &= \frac{V_0^+}{Z_o} \left( e^{-j\beta z} - \Gamma e^{j\beta z} \right) \end{aligned} \qquad (3.\ 21)$$

Thus, the current and voltage on this terminated line are a superposition of an incident and a reflected wave, called standing waves. To avoid reflection, we must have Γ=0, which is obtained when the load impedance is equal to the characteristic impedance of the line.

The time average power flow along the line at a point z is :

$$P_{av} = \mathrm{Re}\left[ V(z) I^*(z) \right] = \frac{\left| V_o^+ \right|^2}{Z_o} \mathrm{Re}\left[ 1 - \Gamma^* e^{-2j\beta z} + \Gamma \ e^{2j\beta z} - |\Gamma|^2 \right] \qquad (3.\ 22)$$

The combination of the two middle terms is purely imaginary, the power flow reduces thus to :

$$P_{av} = \mathrm{Re}\left[V(z)I^{*}(z)\right] = \frac{\left|V_o^{+}\right|^2}{Z_o}\left(1-|\Gamma|^2\right) \tag{3. 23}$$

which shows that the average power flow is constant at any point on the line, and that the total power delivered to the load is equal to the incident power minus the reflected power. If Γ=0, the maximum of power is delivered to the load, while no power is delivered if the modulus of the reflection coefficient equals 1. The preceding results assumed that the source is matched (there are no re reflections at the source).

When the load is mismatched, then not all of the available power is delivered to the load. The "loss" is called return loss, and is defined in dB as :

$$RL = -20\log_{10}|\Gamma| \text{ dB} \tag{3. 24}$$

If the load is matched to the line, the magnitude of the voltage on the line is constant : $|V(z)| = \left|V_0^{+}\right|$.

When the load is mismatched, the reflected wave leads to a standing wave, where the magnitude of the voltage is not constant along the line :

$$|V(z)| = \left|V_0^{+}\right|\left|1+\Gamma\, e^{2j\beta z}\right| = \left|V_0^{+}\right|\left|1+\Gamma\, e^{-2j\beta l}\right| = \left|V_0^{+}\right|\left|1+|\Gamma|\, e^{j(\theta-2\beta)l}\right| \tag{3. 25}$$

where l=-z is the positive distance measured from the load back toward the generator, and θ is the phase of the reflection coefficient. We see that the magnitude of the voltage oscillates with z along the line. The maximum occurs when $e^{j(\theta-2\beta)l} = 1$, and is equal to

$$V_{\max} = \left|V_0^{+}\right|\left(1+|\Gamma|\right) \tag{3. 26}$$

while the minimum occurs at $e^{j(\theta-2\beta)l} = -1$ and is equal to

$$V_{\min} = \left|V_0^+\right|\left(1-|\Gamma|\right) \tag{3. 27}$$

As the magnitude of the reflection coefficient increases, the ratio of $V_{max}$ to $V_{min}$ increases, so a measure of the mismatch of line, called the standing wave ratio, is defined as :

$$SWR = \frac{V_{\max}}{V_{\min}} = \frac{\left(1+|\Gamma|\right)}{\left(1-|\Gamma|\right)} \tag{3. 28}$$

It can be see that the SWR is a real number such that $1 \le SWR \le \infty$ , where SWR=1 implies a matched load.

The reflection coefficient can be generalized to any point along the line :

$$\Gamma(l) = \frac{V_0^- \mathrm{e}^{-j\beta l}}{V_0^+ \mathrm{e}^{j\beta l}} = \Gamma(0)\mathrm{e}^{-2j\beta l} \tag{3. 29}$$

where Γ(0) is given by (3.20). We have seen that the power flow on the line is constant, while the voltage amplitude on a mismatched line is oscillatory. We can thus deduce that the impedance seen looking into a mismatched line must vary with the position. Indeed, at a distance l=-z from the load, the input impedance looking towards the load is given by

$$Z_{in} = \frac{V(-l)}{I(-l)} = Z_o \frac{V_0^+\left(\mathrm{e}^{j\beta l} + \Gamma\,\mathrm{e}^{-j\beta l}\right)}{V_0^+\left(\mathrm{e}^{j\beta l} - \Gamma\,\mathrm{e}^{-j\beta l}\right)} = Z_o \frac{1+\Gamma\,\mathrm{e}^{-j2\beta l}}{1-\Gamma\,\mathrm{e}^{-j2\beta l}} \tag{3. 30}$$

This result can be transformed using (3.20), and we obtain :

$$\begin{aligned} Z_{in} &= Z_o \frac{\left(Z_L + Z_o\right)\mathrm{e}^{j\beta l} + \left(Z_L - Z_o\right)\mathrm{e}^{-j\beta l}}{\left(Z_L + Z_o\right)\mathrm{e}^{j\beta l} - \left(Z_L - Z_o\right)\mathrm{e}^{-j\beta l}} \\ &= Z_o \frac{Z_L \cos\beta l + jZ_o \sin\beta l}{Z_0 \cos\beta l + jZ_L \sin\beta l} \\ &= Z_o \frac{Z_L + jZ_o \tan\beta l}{Z_o + jZ_L \tan\beta l} \end{aligned} \tag{3. 31}$$

This result is the transmission line impedance equation.

## 3.4 Special cases of terminated transmission lines

Consider first the case where the line is terminated by a short circuit, as depicted in figure 3.3

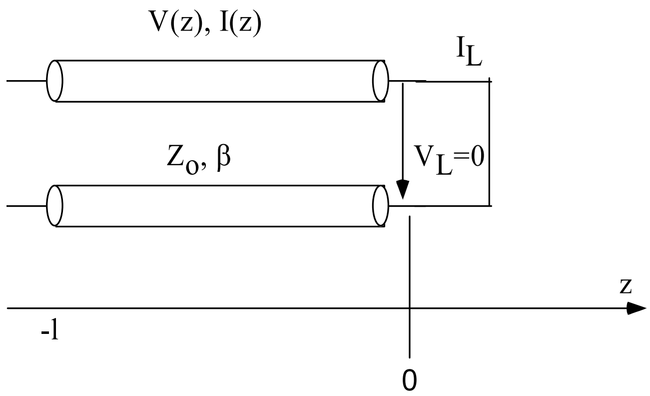



*Fig. 3. 3*

In this case, $Z_L$ =0, and we see immediately that Γ=-1. The voltage and the current along a short circuited line can be written as

$$V(z) = V_0^+ \left( e^{-j\beta z} - e^{j\beta z} \right) = -2jV_0^+ \sin \beta z$$
$$I(z) = \frac{V_0^+}{Z_o} \left( e^{-j\beta z} + e^{j\beta z} \right) = \frac{2V_0^+}{Z_o} \cos \beta z \tag{3. 32}$$

which shows that, as expected for a short circuit, the voltage is 0 at the load while the current is maximum. The input impedance can be found from (3.31) :

$$Z_{in} = jZ_o \tan \beta l \tag{3. 33}$$

which is purely imaginary for any length l, and takes all values between $-j\infty \le Z_{in} \le j\infty$ . For instance, we have $Z_{in}$=0 when l=0, but for l=λ/4, $Z_{in} = j\infty$ . We see also that the impedance is periodic in l, with a periodicity of λ/2.

Consider next an open circuited line, as shown in figure 3.4

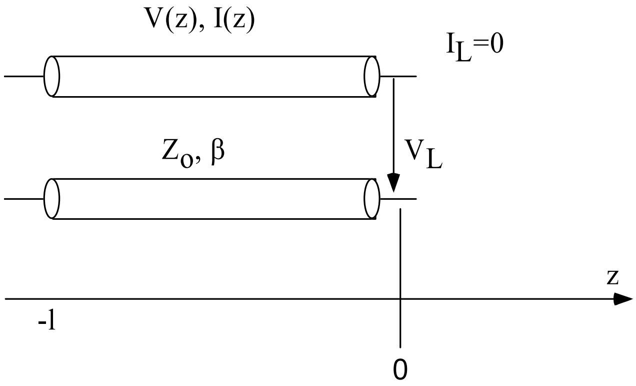



*Fig. 3. 4*

In this case, $Z_L = \infty$, and we obtain Γ=1. The voltage and current along the line are given by

$$
\begin{aligned}
V(z) &= V_0^+\left(e^{-j\beta z} + e^{j\beta z}\right) = 2V_0^+ \cos \beta z \\
I(z) &= \frac{V_0^+}{Z_o}\left(e^{-j\beta z} - e^{j\beta z}\right) = \frac{-2jV_0^+}{Z_o} \sin \beta z
\end{aligned}
\qquad (3.\ 34)
$$

which shows that the current is zero at the load and that the voltage is maximum at the load. The input impedance can be found as

$$
Z_{in} = -jZ_o \cot \beta l \qquad (3.\ 35)
$$

which is also purely imaginary.

## 3.4 Generator and load mismatches

We have assumed above that the generator was matched to the line. We will study now what happens when both the generator and the load are mismatched. Let us consider the circuit depicted in figure 3.5

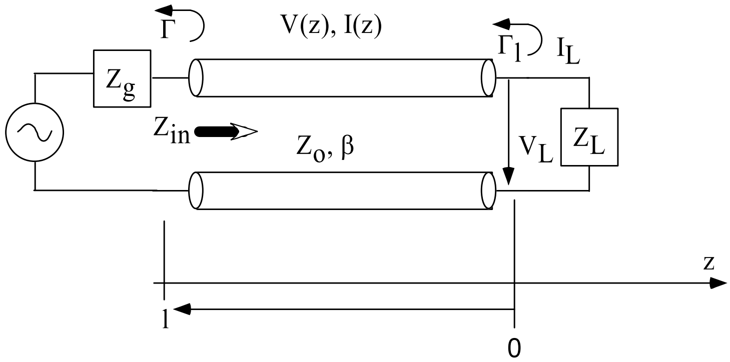



*Fig. 3. 5*

It consists of a transmission line circuit with arbitrary generator and load impedances $Z_g$ and $Z_l$, which may be complex. We assume that the transmission line is lossless, with length l and characteristic impedance $Z_o$. Multiple reflections can occur on the line, as waves reflected from the load can be re reflected by the generator and form an infinite sequence of reflections. In the steady state, the result is a single wave travelling towards the load and a single reflected wave travelling towards the generator. We analyze circuit 3.5 by finding first the impedance looking into the terminated transmission line from the generator end. We get :

$$Z_{in} = Z_o \frac{1+\Gamma_l \, e^{-j2\beta l}}{1-\Gamma_l \, e^{-j2\beta l}} = Z_o \frac{Z_l + jZ_o \tan \beta l}{Z_o + jZ_l \tan \beta l} \tag{3. 36}$$

where

$$\Gamma_l = \frac{Z_l - Z_o}{Z_l + Z_o} \tag{3. 37}$$

is the reflection coefficient of the load. The voltage on the line is given by (3.21), and we can find $V_0^+$, the amplitude of the incident wave from the generator end of the line, where z=-l :

$$V(-l) = V_g \frac{Z_{in}}{Z_{in}+Z_g} = V_0^+ \left( e^{j\beta l} + \Gamma_l \, e^{-j\beta l} \right) \tag{3. 38}$$

so that

$$V_0^+ = V_g \frac{Z_{in}}{Z_{in}+Z_g} \frac{1}{\left( e^{j\beta l} + \Gamma_l \, e^{-j\beta l} \right)} \tag{3. 39}$$

This can be rewritten using (3.36) as

$$V_0^+ = V_g \frac{Z_o}{Z_o + Z_g} \frac{e^{-j\beta l}}{\left(1 - \Gamma_l \Gamma_g e^{-2j\beta l}\right)} \tag{3. 40}$$

where $\Gamma_g$ is the reflection coefficient seen looking into the generator :

$$\Gamma_g = \frac{Z_g - Z_o}{Z_g + Z_o} \tag{3. 41}$$

The power delivered into the load is then obtained as :

$$P_l = \mathrm{Re}\left[V_{in} I_{in}^*\right] = \left|V_{in}\right|^2 \mathrm{Re}\left[\frac{1}{Z_{in}}\right] = \left|V_g\right|^2 \left|\frac{Z_{in}}{Z_{in} + Z_g}\right|^2 \mathrm{Re}\left[\frac{1}{Z_{in}}\right] \tag{3. 42}$$

If we write $Z_{in} = R_{in} + jX_{in}$ and $Z_g = R_g + jX_g$ , we obtain

$$P_l = \left|V_g\right|^2 \frac{R_{in}}{\left(R_{in} + R_g\right)^2 + \left(X_{in} + X_g\right)^2} \tag{3. 43}$$

Let us consider several case of load impedance. First, let us assume the case where the load is matched to the line, so that $Z_l$=$Z_o$. In this case $\Gamma_l = 0$ and SWR=1 on the line. The input impedance is $Z_{in} = Z_o$, and the power delivered to the line is

$$P_l = \left|V_g\right|^2 \frac{Z_o}{\left(Z_o + R_g\right)^2 + {X_g}^2} \tag{3. 44}$$

Next, consider the case when the generator in matched to the input impedance of a mismatched transmission line. In this case we have $Z_{in}$=$Z_g$ and the overall reflection coefficient Γ seen at the input of the line is zero :

$$\Gamma = \frac{Z_{in} - Z_g}{Z_{in} + Z_g} = 0 \qquad (3.45)$$

However, in this case, $\Gamma_g \neq 0$ and $\Gamma_l \neq 0$ in general, and there may be a standing wave on the line. The power delivered to the load is

$$P_l = \left|V_g\right|^2 \frac{R_g}{4\left(R_g{}^2 + X_g{}^2\right)} \qquad (3.46)$$

We see that even if the terminated line is matched to the generator, the power delivered to the load may be less than the power delivered to the load from (3.44), where the line was matched to the load, but not the generator. This leads to the question of what is the optimum load impedance, or equivalently, what is the optimum input impedance to achieve maximum power transfer to the load for a given generator impedance.

Let us assume that $Z_g$ is fixed, and that we may vary the input impedance $Z_{in}$ until we achieve the maximum power delivered to the load. Knowing $Z_{in}$, it is easy to find the corresponding load impedance $Z_l$, via an impedance transformation along the line. To maximize $P_l$, we differentiate with respect to the real and imaginary parts of Zin. Using (3.43), we get :

$$\frac{\partial P_l}{\partial R_{in}} = 0 \rightarrow \frac{2}{\left(R_{in} + R_g\right)^2 + \left(X_{in} + X_g\right)^2} - \frac{4R_{in}\left(R_{in} + R_g\right)}{\left[\left(R_{in} + R_g\right)^2 + \left(X_{in} + X_g\right)^2\right]^2} = 0 \qquad (3.47)$$

or in other terms

$$R_g^2 - R_{in}^2 + \left(X_{in} + X_g\right)^2 = 0 \qquad (3.48)$$

$$\frac{\partial P_l}{\partial X_{in}} = 0 \rightarrow \frac{-4X_{in}\left(X_{in} + X_g\right)}{\left[\left(R_{in} + R_g\right)^2 + \left(X_{in} + X_g\right)^2\right]^2} = 0 \qquad (3.49)$$

or

$$X_{in}\left(X_{in}+X_g\right)=0 \tag{3. 50}$$

Solving (3.48) and (3.50) simultaneously for $R_{in}$ and $X_{in}$ gives $R_{in}=R_g$ and $X_{in}=-X_g$, or

$$Z_{in}=Z_g^* \tag{3. 51}$$

This condition is known as conjugate matching, and results in maximum power transfer to the load, for a fixed generator impedance. Under these conditions, the power delivered to the load is :

$$P_l=\left|V_g\right|^2\frac{1}{4R_g} \tag{3. 52}$$

which is equal to or greater than the powers of (3.44) or (3.46). The reflection coefficients may be nonzero.

Physically, this means that in some situation the multiple voltage reflections on a mismatched line may add in phase to deliver more power to the load than would be delivered if the line were matched (no reflections). If the generator impedance is real ($X_g=0$), then the last two cases produce the same result, which is that the maximum power is delivered to the load when the loaded line is matched to the generator.

Finally, note that neither matching for zero reflection ($Z_l=Z_0$) nor conjugate matching ($Z_l=Z_g$*) necessarily yields the best efficiency for a system. For instance, if $Z_g=Z_l=Z_o$, then both the load and the generator are matched (no reflections), but only half the power produced by the generator is delivered to the load (half is lost in $Z_g$), yielding a transmission efficiency of 50%. This efficiency can only be improved by making $Z_g$ as small as possible.

## 3.5 Impedance matching

The basic idea of impedance matching is to place an impedance matching network between a load impedance and a transmission line (figure 3.6)

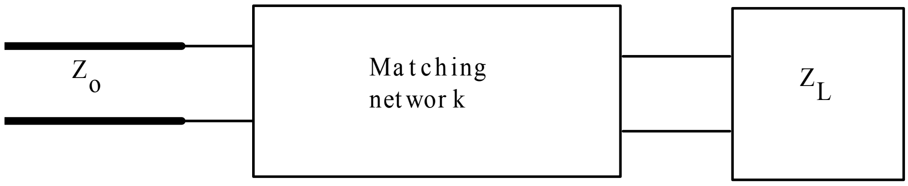



*Fig. 3. 6*

The matching network is ideally lossless, to avoid loss of power, and is designed so that the impedance seen looking into the matching network is equal to the characteristic impedance of the line. Reflections are then eliminated on the transmission line to the left of the matching network, although there will be multiple reflections between the matching network and the load. Impedance matching is important in wireless systems in order to ensure maximum power transfer to the load, to improve the signal to noise ratio and to minimize the RF power required by a system.

As long as the load impedance is passive (having a positive real part), a matching network can always be found, at least for a small frequency band. Some basic matching networks will be described below.

### *3.5.1. The quarter wave transformer*

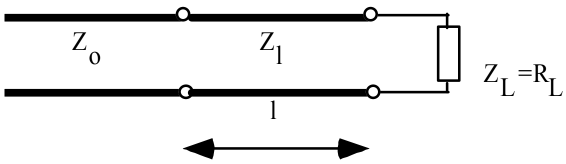



*Fig. 3. 7*

The circuit is shown in figure 3.7, where the impedance of the matching section is given by :

$$Z_l = \sqrt{Z_o Z_L} \qquad (3.\ 53)$$

where $Z_L$ is a real load impedance. At the design frequency $f_0$, the electrical length of the matching section is $\lambda_o/4$, but at other frequencies the electrical length is of course different, and a perfect match is no longer obtained.

The input impedance seen looking into the matching section is given by :

$$Z_{in} = Z_l \frac{Z_L + jZ_l \tan \beta l}{Z_l + jZ_L \tan \beta l} \qquad (3.\ 54)$$

where β corresponds to the design frequency $f_0$, and $\beta l = \pi/2$ at this design frequency. The reflection coefficient seen at the input of the transformer is then :

$$\Gamma = \frac{Z_{in} - Z_o}{Z_{in} + Z_o} = \frac{Z_l\left(Z_L - Z_o\right) + j\tan\beta l\left(Z_l^2 - Z_o Z_L\right)}{Z_l\left(Z_L + Z_o\right) + j\tan\beta l\left(Z_l^2 + Z_o Z_L\right)} \tag{3. 55}$$

Using (3.53), we get :

$$\Gamma = \frac{Z_L - Z_o}{Z_L + Z_o + j2\tan\beta l\sqrt{Z_o Z_L}} \tag{3. 56}$$

The magnitude of the reflection coefficient is

$$\begin{aligned}
\Gamma &= \frac{\left|Z_L - Z_o\right|}{\sqrt{\left(Z_L + Z_o\right)^2 + 4\tan^2\beta l Z_o Z_L}} \\
&= \frac{1}{\sqrt{\frac{\left(Z_L + Z_o\right)^2}{\left(Z_L - Z_o\right)^2} + \frac{4\tan^2\beta l Z_o Z_L}{\left(Z_L - Z_o\right)^2}}} \\
&= \frac{1}{\sqrt{1 + \frac{4Z_o Z_L}{\left(Z_L - Z_o\right)^2} + \frac{4\tan^2\beta l Z_o Z_L}{\left(Z_L - Z_o\right)^2}}} \\
&= \frac{1}{\sqrt{1 + \frac{\left[4Z_o Z_L\right]\sec^2\beta l}{\left(Z_L - Z_o\right)^2}}}
\end{aligned} \tag{3. 57}$$

since $1 + \tan^2\beta l = \sec^2\beta l$

If we assume that we are considering a narrow frequency band around the design frequency $f_0$, then $l \approx \frac{\lambda_0}{4}$ and $\beta l \approx \frac{\pi}{2}$. Then, $\sec^2\beta l >> 1$, and we can write :

$$\Gamma \cong \frac{\left|Z_L - Z_o\right|}{2\sqrt{Z_o Z_L}}\left|\cos\beta l\right| \tag{3. 58}$$

Which gives the approximate mismatch of the quarter-wave transformer near the design frequency (figure 3.8 )

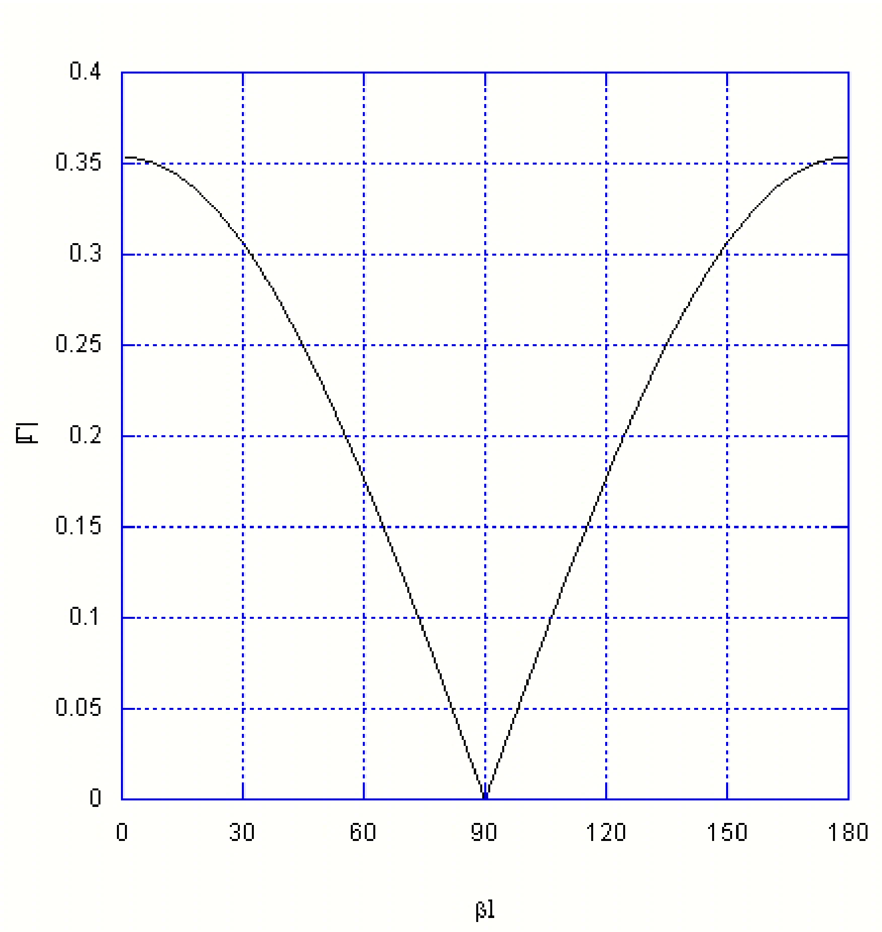
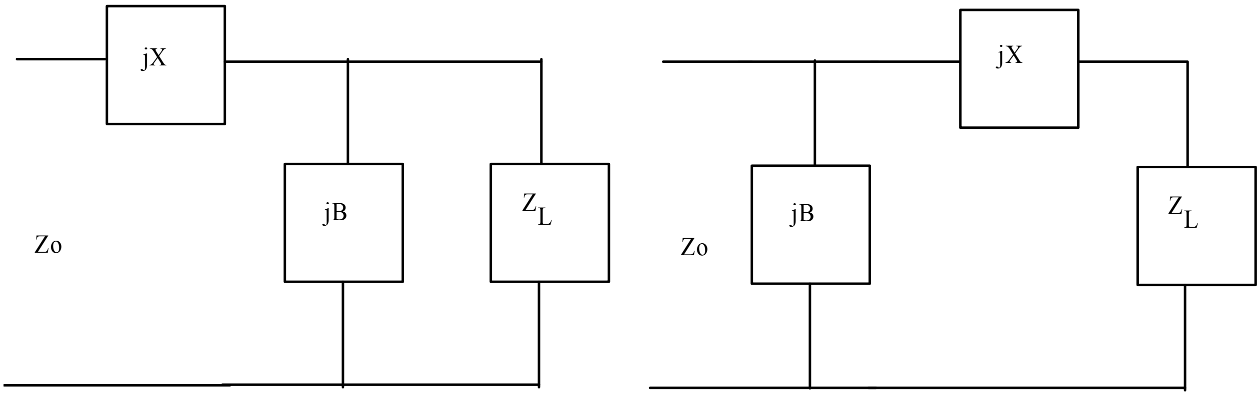



*Fig. 3. 8*

### *3.5.2 Matching using L sections*

Another matching network is the L-section, which uses two reactive elements to match an arbitrary load at a given frequency. In contrary to the quarter wave transformer, the load does not need to be real. This technique is extensively used at lower frequencies, where lumped reactive elements having a good quality factor can readily be found. There are two possible configuration for a L-section network, depicted in figure 3.9.




a) b)

*Fig. 3. 9*

If the normalized impedance $z_L = \frac{Z_L}{Z_o}$ is inside the 1+jx circle of the Smith' chart, then the circuit of figure 3.9a should be used. In either of the configurations, the reactive elements may be either inductive or capacitive, depending on the load impedance.

While analytic solutions for the required values of series reactance jX and shunt susceptance jB are available, it is often more convenient in practice to use the Smith chart to find these values for a given load impedance.

Let us consider the following example, where we want to design an L-section matching network to match a series RC load having an impedance $Z_L$ =200-j100 Ω, to a 100 Ω line, at a frequency of 500 MHz.

The normalized impedance is $z_L$=2+j, which is plotted on the Smith chart of figure 3.10. This point is inside the 1+jx circle, so we will use the matching circuit of figure 3.9a. Since the first element from the load is a shunt susceptance, it is helpful to convert to a load admittance $y_L$, by drawing the circle representing the amplitude of the load reflection coefficient , and a straight line from the load through the centre of the Smith chart. The load admittance is a the intersection of the circle and the line (figure 3.10). Now, we want to be on the circle 1+jx on the impedance chart after having added a shunt susceptance jB, which means that this susceptance must allow us to reach the 1+jx circle on the admittance chart, which we construct as shown in figure 3.10. (It is the axial symmetry of the 1+jx impedance circle, with respect to a vertical axis going through the centre of the Smith chart). We see, then that the normalized susceptance required is jb=j0.3, and we reach the point y=0.4+j0.5. Converting back to impedance leaves us at z=1-j1.2, indicating that the addition of a series reactance x=j1.2 will bring us to the centre of the Smith chart.

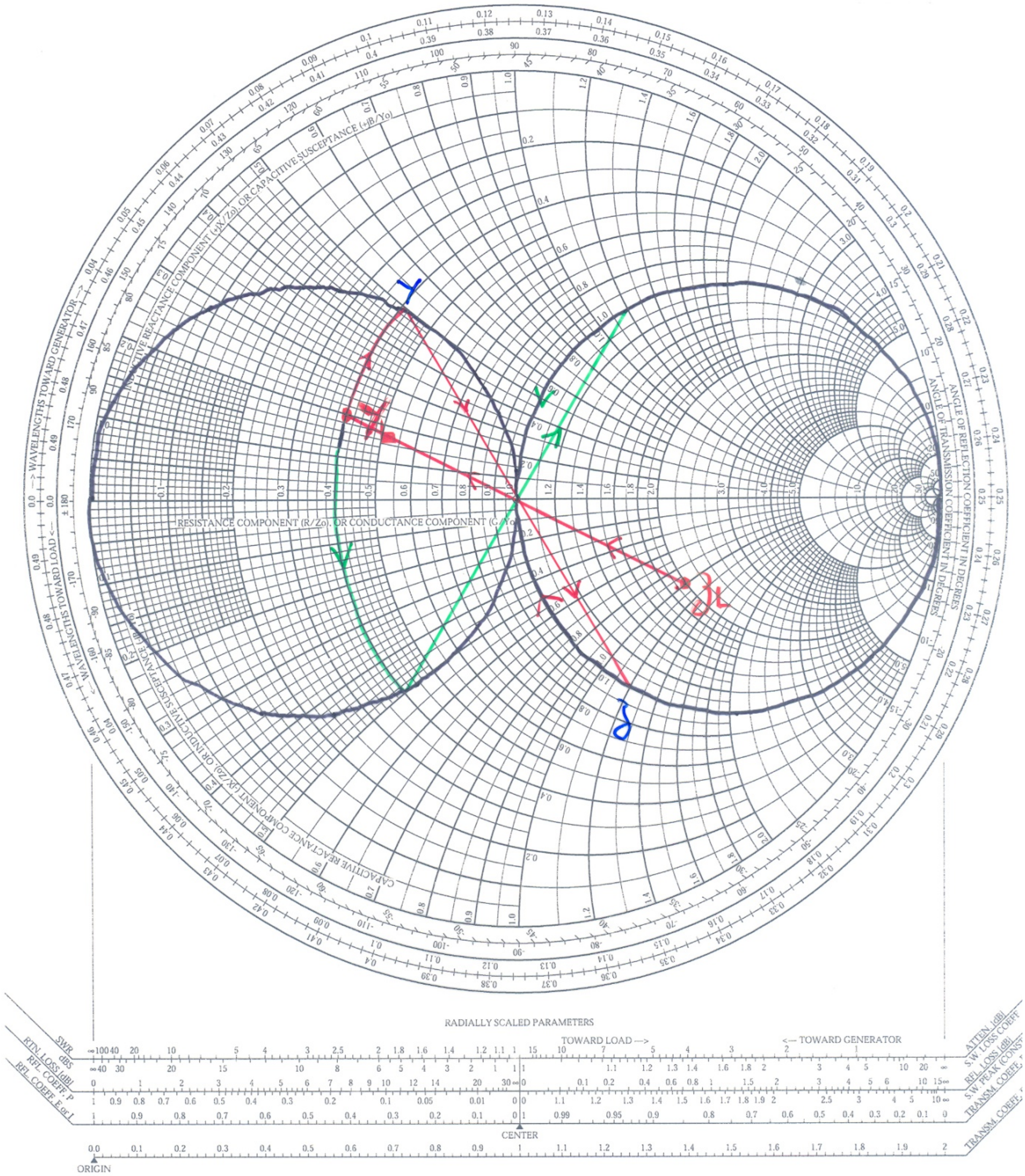
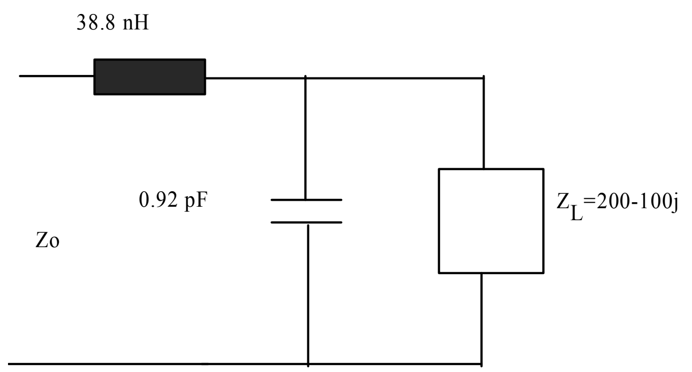


*Fig. 3. 10*

The matching circuit consists of a shunt capacitor and a series inductor, as shown in figure 3.11. At a frequency of 500 MHz, the values are given by

$$C = \frac{b}{2\pi f Z_o} = 0.92\, pF$$

$$L = \frac{x Z_o}{2\pi f} = 38.8 nH$$



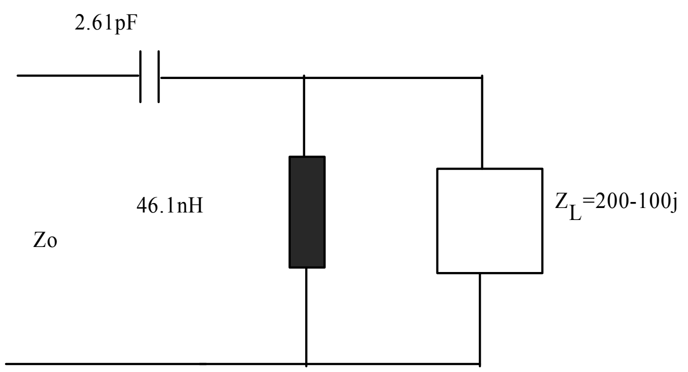




*Fig. 3. 11*

There is a second possible solution for this problem. If instead of adding a shunt suscpetance b=0.3, we use a shunt susceptance of b=-0.7, we will move to a point on the lower half of the rotated 1+jx circle, to y=0.4-j0.5. Converting to impedance yields x=-1.2, which leads to a match as well. This matching circuit is also shown in figure 3.11. The values are given by :

$$C = \frac{-1}{2\pi f x Z_o} = 2.61 pF$$

$$L = \frac{-Z_o}{2\pi f b} = 46.1 nH$$

*3.5.3 Single-Stub tuning*

Finally, we consider a matching technique which uses a single open-circuits or short-circuited length of transmission line (a stub), connected either in parallel or in series with the transmission line at a certain distance from the load (figure 3.12)

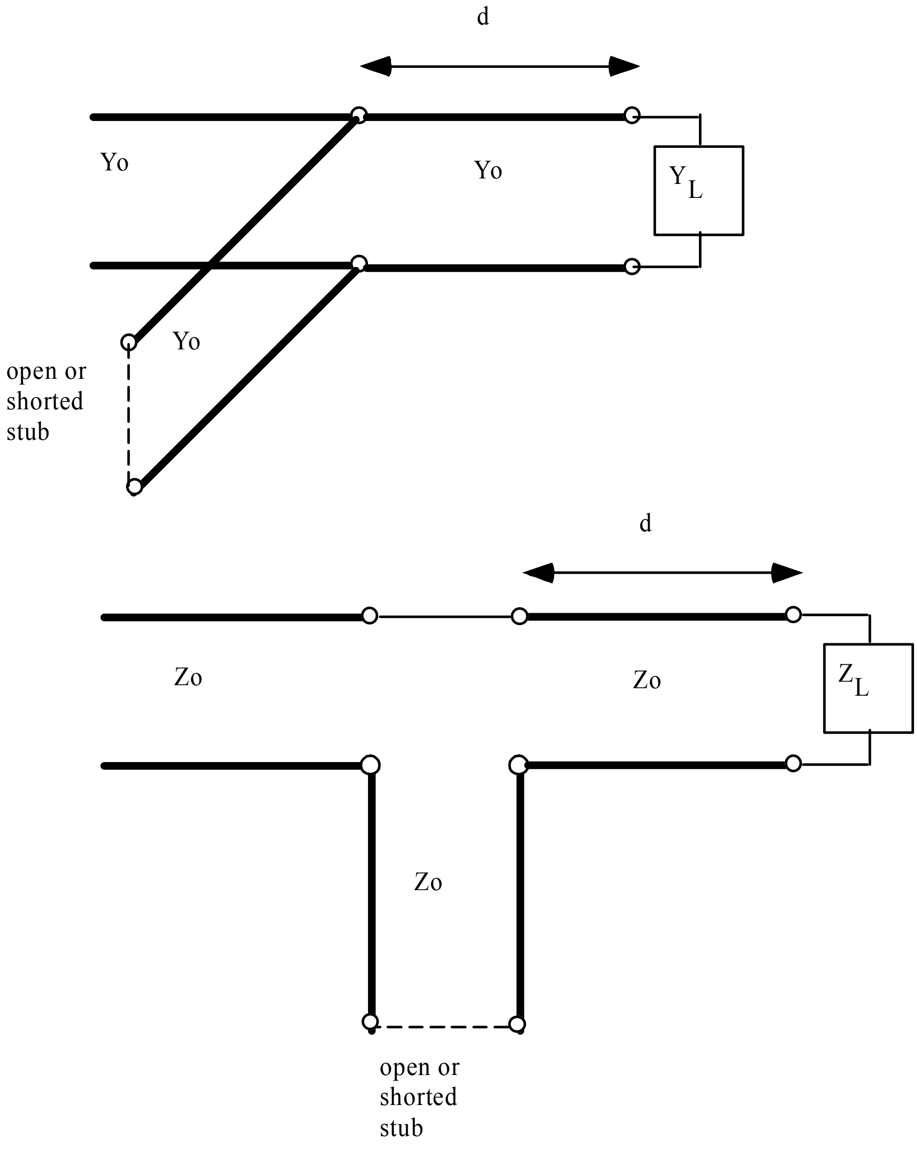



*Fig. 3. 12*

Such a circuit is convenient at microwave frequencies from a fabrication point of view, since no lumped elements are required. In single stub tuning the distance d, from the load to the stub position, and the value of the shunt susceptance (or series reactance) provided by the stub, are adjustable parameters. These two degrees of freedom can be used to match an arbitrary passive load impedance to any feed line. For the shunt stub case, we select d in order to achieve an admittance of $Y=Y_o+jB$ ($Y_o=1/Z_o$)looking towards the load from the end of this section of transmission line of length d. Then, the stub is chosen as $-jB$.

For the series case, the length d is chosen so that the impedance towards the load from that point is $Z=Z_o+jX$. Then, the stub reactance is chosen as $-jX$.

# 4. Guided electromagnetic propagation

In chapter 2, we made no assumptions regarding the geometry of the medium supporting the considered electromagnetic phenomena. The obtained equations were thus very general. In this chapter, we will specialize the results of chapter 2 to the case of guided waves.

**References :**

F.E: Gardiol, Traité d'Electricité de l'EPFL, vol.III : "Electromagnetisme", Presses Polytechniques Romandes

S. Ramo, J.R. Whinnery, T van Duzer : "Fields and Waves in Communication Electronics", Wiley, New-York, 1984

## 4.1 Generalities

### *4.1.1 Reference, coordinates and components*

We will consider that all the geometries we will study have translation symmetry along the z axis. Thus, the z (or longitudinal) axis will play a specific role, which is very different from the role played by the transverse components x,y (or ρ,φ, or any system defined in the transverse plane). We introduce the generic notation $t = (t_1, t_2)$ for these transverse coordinates.

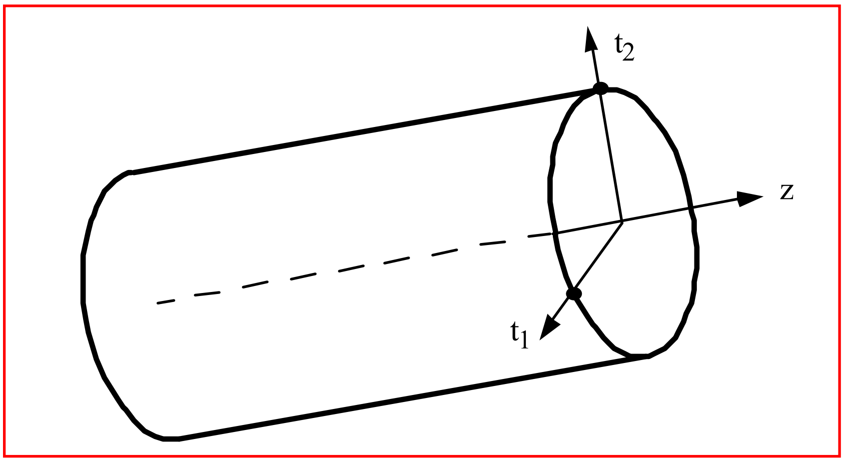



Fig. 4. 1 *: wave guiding structure*

We can write any vector **v** as: $\mathbf{v} = v_z\hat{\mathbf{z}} + \mathbf{v_t}$ , emphasizing the z component and grouping the transverse components in a vector. The vector operator ∇ becomes:

$$\nabla = \left(\frac{\partial}{\partial z}\right)\hat{\mathbf{z}} + \nabla_t$$
$$\nabla^2 = \frac{\partial^2}{\partial z^2} + \nabla_t^2 \qquad (4.\ 1)$$

We use classical variable separation to write

$$f(t_1, t_2, z) = T(t_1, t_2)\ Z(z) \qquad (4.\ 2)$$

All the six scalar components of the electromagnetic fields **E** and **H** have to satisfy a wave equation :

$$(\nabla^2 + k^2) f = 0 \quad ; \quad f = E_{t1}, E_{t2}, E_z, H_{t1}, H_{t2}, H_z \qquad (4.\ 3)$$

We get:

$$\frac{\nabla_t^2 T}{T} + k^2 = -\frac{1}{Z}\frac{d^2 Z}{dz^2} = -\gamma^2 \qquad (4.\ 4)$$

where $\gamma$ is the complex constant associated to the separation process.

### *4.1.2 Longitudinal dependency : propagation exponent*

The equation for Z(z) has a simple analytical solution. We find that the z dependency of guided waves is always of the kind :

$$Z(z) = A \exp(-\gamma z) + B \exp(+\gamma z) \qquad (4.\ 5)$$

exactly as for voltages and currents in transmission lines.

We have incident $\exp(-\gamma z)$ and reflected $\exp(+\gamma z)$ waves, and the physical meaning of the separation constant $\gamma$ becomes clear : Its real part $\alpha$ is the linear attenuation [Np/m] or [dB/m], while its imaginary part is the linear phase constant [rad/m].

To simplify the expressions above, we will consider only incident waves propagating along the positive z axes. In this case, B=0, and we can integrate the amplitude of the incident wave into the transverse function T, and write in compact vector form :

$$\mathbf{E}(t_1,t_2,z) = \mathbf{e}(t_1,t_2)\exp(-\gamma z)$$
$$\mathbf{H}(t_1,t_2,z) = \mathbf{h}(t_1,t_2)\exp(-\gamma z) \qquad (4.\,6)$$

where the transverse vectors **e, h** depend on the transverse section of the guiding structure.

*To find the relations for the reflected wave, we only have to replace $\gamma$ by $-\gamma$.*

### *4.1.3 Transverse dependency: eigenvalues and eigenvectors*

The transverse vectors **e, h** are solution to eigenvalue equations:

$$(\nabla_t^2 + k^2 + \gamma^2)\,\mathbf{e} = (\nabla_t^2 + k_c^2)\,\mathbf{e} = 0$$
$$(\nabla_t^2 + k^2 + \gamma^2)\,\mathbf{h} = (\nabla_t^2 + k_c^2)\,\mathbf{h} = 0 \qquad (4.\,7)$$

The admissible eigenvalues $k_c$ $(k_c^2 = k^2 + \gamma^2)$ are determined by the boundary conditions associated to the transverse geometry of the waveguide. For each eigenvalue, the corresponding eigenvectors **e, h** can be found. The latter are often called the modes of the guiding structure.

Thus, to each eigenvalue we can associate a longitudinal propagation exponent:

$$\gamma = \sqrt{k_c^2 - k^2} = \sqrt{k_c^2 - \omega^2\mu\varepsilon} \qquad (4.\,8)$$

The relation between the eigenvalues and the frequency will determine the nature of $\gamma$ (real, imaginary or complex), and thus the propagation characteristics.

### *4.1.4 Transverse and longitudinal components*

The transverse vectors **e** and **h** contain in general six scalar components. In principle, we have thus to solve 6 scalar eigenvalue problems, with each its particular boundary conditions. It is however evident that these six components are not independent from each other, as they are linked by Maxwell's equations. It should thus be possible to solve the eigenvalue problem for two of the components, and obtain the others from these two.

The most logical choice is to consider the longitudinal components, $e_z$, $h_z$ as basis functions from which we will try to derive the other components. The sought for relations are found easily by introducing the following field expressions in Maxwell's equations:

$$\mathbf{E} = (e_z\hat{\mathbf{z}} + \mathbf{e_t})\exp(-\gamma z) \quad ; \quad \mathbf{H} = (h_z\hat{\mathbf{z}} + \mathbf{h_t})\exp(-\gamma z) \quad ; \quad \nabla = (\partial/\partial z)\hat{\mathbf{z}} + \nabla_t \qquad (4.9)$$

where the transverse-longitudinal decomposition has been used. We find:

$$\nabla_t \times \mathbf{e_t} = -j\omega\mu\ (h_z\hat{\mathbf{z}}) \quad ; \quad \nabla_t \times (e_z\hat{\mathbf{z}}) - \gamma\,\hat{\mathbf{z}} \times \mathbf{e_t} = -j\omega\mu\ \mathbf{h_t}$$
$$\nabla_t \times \mathbf{h_t} = +j\omega\varepsilon\ (e_z\hat{\mathbf{z}}) \quad ; \quad \nabla_t \times (h_z\hat{\mathbf{z}}) - \gamma\,\hat{\mathbf{z}} \times \mathbf{h_t} = +j\omega e\ \mathbf{e_t} \qquad (4.10)$$

and finally:

$$k_c^2\,\mathbf{e_t} = -\gamma\,\nabla_t e_z + j\omega\mu\,\hat{\mathbf{z}} \times \nabla_t h_z$$
$$k_c^2\,\mathbf{h_t} = -\gamma\,\nabla_t h_z - j\omega\varepsilon\,\hat{\mathbf{z}} \times \nabla_t e_z \qquad (4.11)$$

As mentioned, we only have to replace $\exp(-\gamma z)$ by $\exp(+\gamma z)$ and $+\gamma$ by $-\gamma$ to obtain the expressions for a reflected wave.

### *4.1.5 Summary: computing procedure*

The study of a guiding structure will in general include the following steps :

a) The resolution of the eigenvalue problem in the transverse section of the structure

$$(\nabla_t^2 + k_c^2)\,e_z = 0 \quad ; \quad (\nabla_t^2 + k_c^2)\,h_z = 0 \qquad (4.12)$$

with the pertinent boundary conditions. In particular, the eigenvalues $k_c$ and the associated modes $e_z$, $h_z$ have to be found.

b) Compute the longitudinal propagation constant $\gamma = \alpha + j\beta = \sqrt{k_c^2 - k^2}$ . For an incident wave, we choose the sign of the square root so that $\mathrm{Im}(\gamma)>0$.

c) Compute the transverse components

$$k_c^2\, \mathbf{e_t} = -\gamma\, \nabla_t e_z + j\omega\mu\, \hat{\mathbf{z}} \times \nabla_t h_z$$
$$k_c^2\, \mathbf{h_t} = -\gamma\, \nabla_t h_z - j\omega\varepsilon\, \hat{\mathbf{z}} \times \nabla_t e_z \qquad (4.\,13)$$

d) Construct the incident fields:

$$\mathbf{E}(t_1,t_2,z) = (\mathbf{e_t} + e_z\hat{\mathbf{z}}) \exp(-\gamma z)$$
$$\mathbf{H}(t_1,t_2,z) = (\mathbf{h_t} + h_z\hat{\mathbf{z}}) \exp(-\gamma z) \qquad (4.\,14)$$

e) Redo the procedure for the reflected fields. We only have to choose the other branch in the square root defining $\gamma$, which formally is equivalent to substitute $+\gamma$ by $-\gamma$ in the formulas.

f) Find the amplitude if the incident and reflected waves using transmission line theory. To this aim, the load conditions at the end of the line must be known.

g) If time domain expressions are wanted, they are readily obtained using the definition of phasors. For instance, the incident field component $E_Z$ of a complex amplitude $A = |A| \exp(j\varphi_A)$ is given by:

$$E_Z(t_1,t_2,z,t) = \sqrt{2}\, |A|\, e_z(t_1,t_2) \exp(-\alpha z) \cos(\omega t-\beta z+\varphi_A) \qquad (4.\,15)$$

## 4.2 Propagation modes

The possible solutions to the transverse Helmoltz equation are called the modes of the guiding structure. Each mode represents a specific configuration of the electromagnetic fields in which a signal can propagate. To evaluate the characteristics of a transmission channel, it is thus very important to know the modes that may exist for a given combination of geometry, frequency and medium.

**References:**

S. Ramo, J.R. Whinnery, T van Duzer : “Fields and Waves in Communication Electronics”, Wiley, New-York, 1984

### *4.2.1 Classification*

The solutions to Helmoltz' equations

$$(\nabla_t^2 + k_c^2)\, e_z = 0 \quad ; \quad (\nabla_t^2 + k_c^2)\, h_z = 0 \qquad (4.16)$$

can be ordered in the following way :

| | TEM modes | TM or E modes | TE or H modes | Hybrid modes |
|---|---|---|---|---|
| $e_z$ | 0 | $\neq 0$ | 0 | $\neq 0$ |
| $h_z$ | 0 | 0 | $\neq 0$ | $\neq 0$ |

All these mode types exist in nature. In general, a guiding structure with a given geometry can support several mode families. We will for instance see that a coaxial cable supports TEM, TE and TM modes, while Hybrid modes propagate along optic fibres.

The characteristics of each mode family will be briefly described hereafter.

### *4.2.2 TEM Modes*

A TEM (Transverse Electro Magnetic) mode is characterized by the absence of longitudinal components ($e_z = h_z = 0$), and we have purely transverse fields as for a plane wave in an unbounded medium. In any guided structure, the transverse fields are given by (§4.1.4)

$$k_c^2\, \mathbf{e_t} = -\gamma\, \nabla_t e_z + j\omega\mu\, \hat{\mathbf{z}} \times \nabla_t h_z$$
$$k_c^2\, \mathbf{h_t} = -\gamma\, \nabla_t h_z - j\omega\varepsilon\, \hat{\mathbf{z}} \times \nabla_t e_z \qquad (4.17)$$

Thus, we see that when $e_z$ and $h_z$ are zero. The transverse fields are also zero which would be a trivial solution. The only solution allowing non-zero transverse fields is to force the eigenvalue $k_c$=0, which implies $\gamma = jk = j\omega\sqrt{\varepsilon\mu}$ , which is identical to the case of a plane wave in an unbounded medium.

The transverse fields are then computed using directly Maxwell's equations:

$$\nabla_t \times \mathbf{e_t} = 0 \quad ; \quad -\gamma\, \hat{\mathbf{z}} \times \mathbf{e_t} = -j\omega\mu\ \mathbf{h_t}$$
$$\nabla_t \times \mathbf{h_t} = 0 \quad ; \quad -\gamma\, \hat{\mathbf{z}} \times \mathbf{h_t} = +j\omega\varepsilon\ \mathbf{e_t} \tag{4. 18}$$

The rotational of the transverse fields is thus equal to zero, which means that they derive from a potential :

$$\mathbf{e_t} = -\nabla_t V \quad \text{et} \quad \mathbf{E_t} = (-\nabla_t V)\exp(-\gamma z) \quad \text{avec} \quad \nabla_t^2 V = 0 \tag{4. 19}$$

It is thus enough to solve Laplace's equation in the transverse section of the structure using the appropriate boundary conditions, just as in Electrostatics.

In particular, we know that the electrostatic field is equal to zero inside a hollow conductor. It is thus not possible to obtain a TEM propagation mode for metallic waveguide like structures, indeed we need in general two or more distinct conductors to ensure the presence of a TEM wave.

Maxwell's equations show also that for a TEM mode, the magnetic field is linked to the electric field by :

$$\mathbf{h_t} = (\gamma / j\omega\mu)\, \hat{\mathbf{z}} \times \mathbf{e_t} = (j\omega\varepsilon / \gamma)\, \hat{\mathbf{z}} \times \mathbf{e_t} \tag{4. 20}$$

The factor $|\mathbf{e_t}|/|\mathbf{h_t}|$ has the dimension of an impedance, and is called the wave impedance or the mode impedance, $Z_{mod}$. For TEM modes, we have:

$$Z_{mod(TEM)} = j\omega\mu/\gamma = \gamma/j\omega\varepsilon = \sqrt{\mu/\varepsilon}\ . \tag{4. 21}$$

Thus, the wave impedance of a TEM mode is equal to the impedance of the medium supporting the propagation, as for a plane wave.

*Current, voltage and characteristic impedance*

We have seen that the fields of a TEM mode have static behaviour. This implies that a current and a voltage can be univocally defined along the guiding structure. Let's for instance consider the following arbitrary bifilar transmission line :

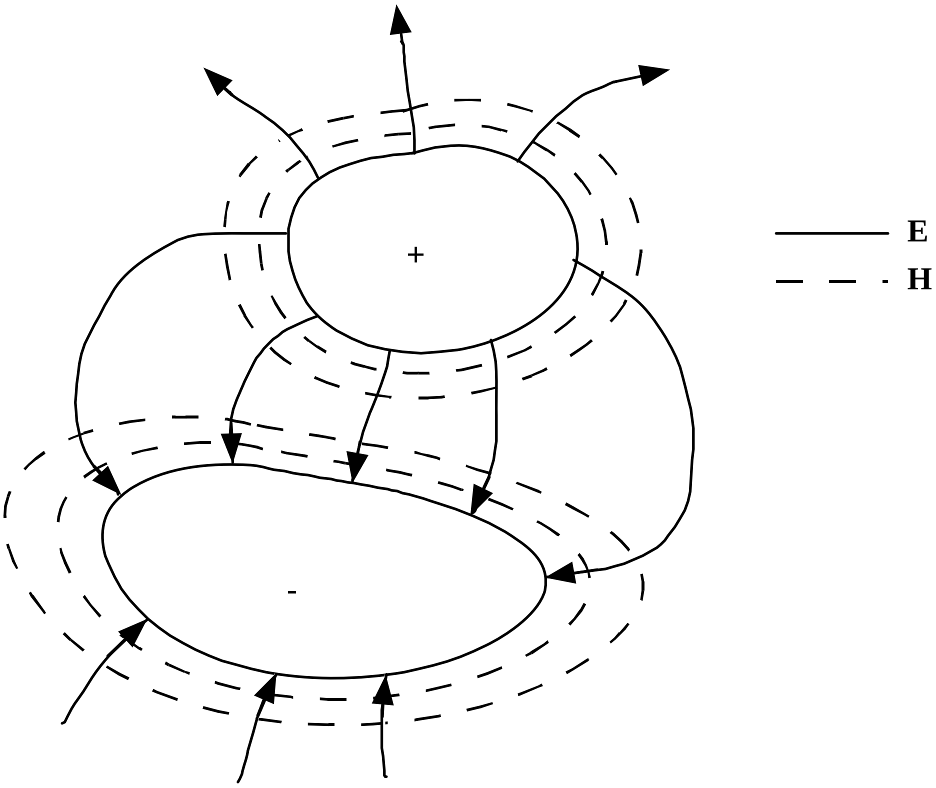



Fig. 4. 2*: Bifilar TEM line*

The potential difference between the positive and the negative conductors is given by the integral of the electric field between the conductors. Because of the static behaviour of the field, this integral will be independent of the chosen integration path, and an unique voltage is defined in the transverse plane as:

$$V = \int_{+}^{-} \mathbf{E} \cdot \mathbf{dl} \tag{4. 22}$$

A voltage wave can thus be defined in the same way as the field waves :

$$V(z) = V^{+}\, \mathrm{e}^{-j\gamma z} + V^{-}\, \mathrm{e}^{j\gamma z} \tag{4. 23}$$

In the same way, the total current circulating on the positive conductor can be obtained using Ampere's law, which for a TEM mode is written :

$$I = \oint_{C+} \mathbf{H} \cdot \mathbf{dl} \tag{4. 24}$$

where C+ in any closed path enclosing the positive conductor but not enclosing the negative conductor. The current wave is then written as :

$$I(z) = I^{+}\,\mathrm{e}^{-j\gamma z} - I^{-}\,\mathrm{e}^{j\gamma z} \tag{4. 25}$$

The factors $\dfrac{V^{+}}{I^{+}}$ and $\dfrac{V^{-}}{I^{-}}$ are constants along the line and have the dimension of an impedance, and represent the characteristic impedance of the line, which depends essentially on the geometry of the conductors.

### *4.2.3 TM Modes*

A TM mode (transverse Magnetic) is characterized by a zero longitudinal component for the magnetic field. The magnetic field is thus transverse, while the electric field is not. The component $e_z$ is solution of :

$$(\nabla_t^2 + k_c^2)\, e_z = 0 \tag{4. 26}$$

with the appropriate boundary conditions. The transverse fields are obtained as :

$$k_c^2\, \mathbf{e_t} = -\,\gamma\, \nabla_t e_z \quad ; \quad \mathbf{h_t} = \frac{j\omega\varepsilon}{\gamma}\, \hat{\mathbf{z}} \times \mathbf{e_t} \tag{4. 27}$$

We note that for TM modes, the propagation constant is equal to $\gamma = \sqrt{k_c^2 - \omega^2\mu\varepsilon}$ and the modal impedance $Z_{mod}$ is given by :

$$Z_{mod} = \gamma \,/\, j\omega\varepsilon = (\sqrt{\omega^2\mu\varepsilon - k_c^2}) \,/\, \omega\varepsilon \tag{4. 28}$$

These parameters depend on the eigenvalue $k_c$.

### *4.2.4 TE Modes*

A TE mode (Transverse electric) is characterized by a zero longitudinal component for the electric field. Thus, the electric field is transverse, but the magnetic field is not. The characteristic equation of these modes is thus:

$$(\nabla_t^2 + k_c^2)\, h_z = 0 \qquad (4.\,29)$$

with the appropriate boundary conditions. The transverse fields are then obtained using:

$$k_c^2\, \mathbf{h_t} = -\gamma\, \nabla_t h_z \quad ; \quad \mathbf{e_t} = \frac{j\omega\mu}{\gamma}\, \mathbf{h_t} \times \hat{\mathbf{z}} \qquad (4.\,30)$$

We note that for TE modes, the propagation constant is equal to $\gamma = \sqrt{k_c^2 - \omega^2\mu\varepsilon}$ and the wave impedance $Z_{mod}$ is given by :

$$Z_{mod} = j\omega\mu / \gamma = \omega\mu / \sqrt{\omega^2\mu\varepsilon - k_c^2} \qquad (4.\,31)$$

These parameters depend on the eigenvalue $k_c$.

### *4.2.5 Recapitulation*

Here are in a compact form the fields and parameters characterizing TEM, TE and TM modes. The square root in the value of the propagation exponent $\gamma$ is always taken in a way to have $\arg(\gamma) \in [0\ ;\ \pi/2]$. This corresponds to a wave travelling in the positive direction of the z axis. The following table is thus valid only for incident waves. In order to obtain the corresponding values for a reflected wave, we only have to replace $\gamma$ by $-\gamma$.

| | **Modes TEM** | **Modes TM** | **Modes TE** |
|---|---|---|---|
| Characteristic equation | $\nabla_t^2 V = 0$ | $(\nabla_t^2 + k_c^2)\, e_z = 0$ | $(\nabla_t^2 + k_c^2)\, h_z = 0$ |
| $\gamma$ | $\sqrt{-\omega^2\mu\varepsilon} = j\omega\sqrt{\mu\varepsilon}$ | $\sqrt{k_c^2 - \omega^2\mu\varepsilon}$ | $\sqrt{k_c^2 - \omega^2\mu\varepsilon}$ |
| Ez | 0 | $e_z \exp(-\gamma z)$ | 0 |
| Hz | 0 | 0 | $h_z \exp(-\gamma z)$ |

| | | | |
|---|---|---|---|
| **Et** | $-\nabla_t V \exp(-\gamma z)$ | $-(\gamma / k_c^2)\, \nabla_t E_z$ | $(j\omega\mu / \gamma)\, (\mathbf{H_t} \times \hat{\mathbf{z}})$ |
| **Ht** | $(\gamma / j\omega\mu)\, (\hat{\mathbf{z}} \times \mathbf{E_t})$ | $(j\omega\varepsilon / \gamma)\, (\hat{\mathbf{z}} \times \mathbf{E_t})$ | $-(\gamma / k_c^2)\, \nabla_t H_z$ |
| Zmod = \|**et**\| / \|**ht**\| | $\sqrt{\mu/\varepsilon}$ | $\sqrt{\omega^2\mu\varepsilon - k_c^2}\, / \,\omega\varepsilon$ | $\omega\mu \,/ \sqrt{\omega^2\mu\varepsilon - k_c^2}$ |

## 4.3 Dispersion and distortion

When the linear phase constant $\beta$ is a non linear function of the frequency, the propagation is said to be dispersive. Waveguides and optic fibres are dispersive transmission lines, and the signal travelling on them will be distorted.

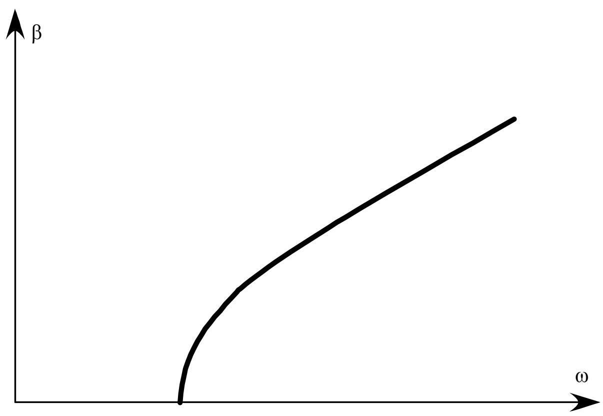



*Fig. 4.3 : Dispersion diagram of a guiding structure*

This effect is illustrated on a Gaussian pulse travelling along a dispersive line, as this gives a simple mathematical development.

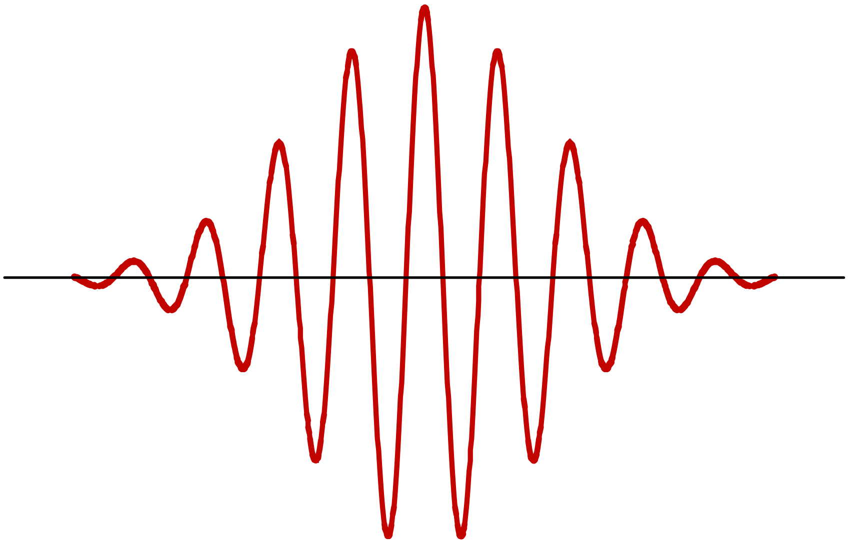


*Fig. 4.4 : modulated Gaussian pulse*

A modulated Gaussian pulse is described by:

$$f(t, z = 0) = \cos(\omega_0 t) e^{-\frac{1}{2}\left(t/\tau\right)^2} \qquad (4.\ 32)$$

where 2τ is the width of the pulse at level $1/\sqrt{e} = 0.606$ of the maximum. The Fourier transform (spectrum) of the Gaussian pulse has also a Gaussian dependenc :

$$F(\omega, z = 0) = \tau\sqrt{2\pi}\, e^{-\frac{1}{2}\left[\tau(\omega-\omega_0)\right]^2} \qquad (4.\ 33)$$

The signal propagates along the dispersive lines and sustains a phase shift β*z*

$$F(\omega, z) = \tau\sqrt{2\pi}\, e^{-\frac{1}{2}\left[\tau(\omega-\omega_0)\right]^2} e^{-j\beta z} \qquad (4.\ 34)$$

To find the corresponding function in time domain, we take the inverse Fourier transform:

$$f(t,z)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}F(\omega,z)e^{j\omega t}d\omega=\frac{\tau\sqrt{2\pi}}{2\pi}\int_{-\infty}^{+\infty}e^{-\frac{1}{2}[\tau(\omega-\omega_0)]^2}e^{-j\beta z}e^{j\omega t}d\omega \quad (4.35)$$

But β is not a simple linear function of ω, so this integral cannot be simply evaluated. The spectrum of the pulse is usually narrow, so we can develop it in a Taylor series in the vicinity of $\omega_0$

$$\beta=\beta_0+\beta_1(\omega-\omega_0)+\frac{\beta_2}{2}(\omega-\omega_0)^2+\ldots \quad (4.36)$$

$$\text{with } \beta_0=\beta(\omega_0) \text{ et } \beta_n=\left.\frac{\partial^n\beta}{\partial\omega^n}\right|_{\omega=\omega_0} \quad (4.37)$$

The integrand takes the following form:

$$e^{-\frac{1}{2}[\tau(\omega-\omega_0)]^2}e^{-jz\left[\beta_o+\beta_1(\omega-\omega_0)+\frac{\beta_2}{2}(\omega-\omega_0)^2\right]}e^{j\omega t} \quad (4.38)$$

Grouping the terms in ω, we obtain a term $e^{j\omega t}e^{-j\beta_1\omega z}=e^{j\omega t'}$, with $t'=t-\beta_1 z$, which corresponds to a translation with velocity $1/\beta_1=v_g$ (group velocity). Grouping the terms in $(\omega-\omega_0)^2$, we obtain :

$$e^{-\frac{1}{2}[\tau(\omega-\omega_0)]^2-j\left[\frac{\beta_2}{2}(\omega-\omega_0)^2\right]z}=e^{-\frac{(\omega-\omega_0)^2}{2}\left[\tau^2+j\beta_2 z\right]}=e^{-\frac{(\omega-\omega_0)^2\tau_e^2}{2}} \quad (4.39)$$

Taking now the inverse Fourier transform (4.35) we find, after some approximations, that the width of the Gaussian pulse becomes

$$\tau'\cong\frac{\left|\tau_e^2\right|}{\tau}=\sqrt{\tau^2+\left(\frac{\beta_2 z}{\tau}\right)^2} \quad (4.40)$$

At a large distance z, if the dispersion $\beta_2$ is important, we tend towards

$$\tau' = \frac{\beta_2 z}{\tau} \tag{4. 41}$$

Thus a very narrow pulse in z=0 widens faster than a large pulse in z=0

## 4.4 Parallel plate waveguide

One of the simplest electromagnetic transmission systems is the parallel plate waveguide, made of two parallel conductive plates separated by a distance a (the height of the guide, figure 4.4)

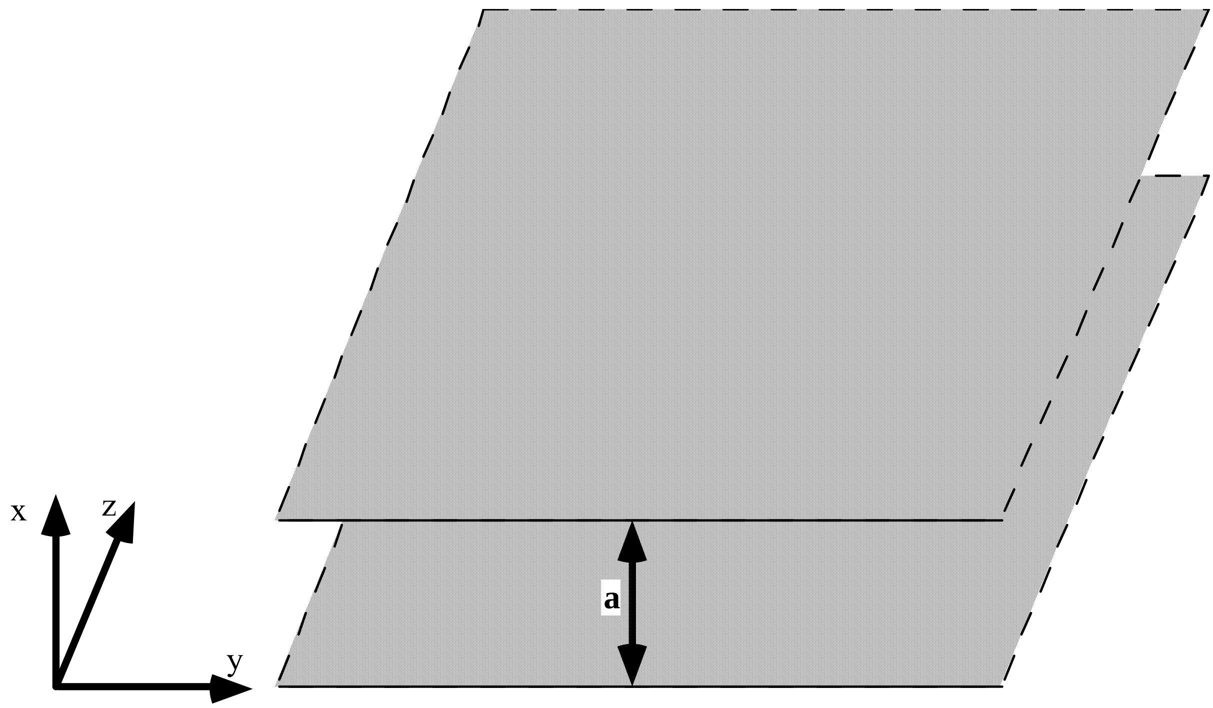



*Fig. 4.5 : parallel plate waveguide*

For the analysis, the plates are considered to be infinite and placed at x=0 and x=a respectively. The propagation occurs as usual along z, and the medium between the plates is defined by $\varepsilon,\mu$.

Because the structure is supposed to be infinite in the y direction, the fields will be independent of this coordinate. The solution of the relations given in §4.2.5 is then simple, and the following results are obtained:

| | TEM Mode | TM Modes | TE Modes |
|---|---|---|---|
| $k_c$ | 0 | $m\pi / a$ | $m\pi / a$ |
| $\omega_c$ | 0 | $m\pi / (a\sqrt{\mu\varepsilon})$ | $m\pi / (a\sqrt{\mu\varepsilon})$ |
| $\gamma$ | $\sqrt{-\omega^2\mu\varepsilon} = j\omega\sqrt{\mu\varepsilon}$ | $j\omega\sqrt{\mu\varepsilon}\sqrt{1-(\omega_c/\omega)^2}$ | $j\omega\sqrt{\mu\varepsilon}\sqrt{1-(\omega_c/\omega)^2}$ |
| Ez | 0 | $E_0 \sin \frac{m\pi x}{a}\ e^{-\gamma z}$ | 0 |
| Hz | 0 | 0 | $H_0 \cos \frac{m\pi x}{a}\ e^{-\gamma z}$ |
| **Et** | $E_0\,\hat{\mathbf{x}}\,e^{-\gamma z}$ | $\frac{-\gamma}{k_c} E_0 \cos\frac{m\pi x}{a}\hat{\mathbf{x}}\,e^{-\gamma z}$ | $\frac{j\omega\mu}{k_c} H_0 \sin\frac{m\pi x}{a}\hat{\mathbf{y}}\,e^{-\gamma z}$ |
| **Ht** | $\frac{E_0}{\sqrt{\mu/\varepsilon}}\,\hat{\mathbf{y}}\,e^{-\gamma z}$ | $\frac{-j\omega\varepsilon}{k_c} E_0 \cos\frac{m\pi x}{a}\hat{\mathbf{y}}\,e^{-\gamma z}$ | $\frac{j\beta}{k_c} H_0 \sin\frac{m\pi x}{a}\hat{\mathbf{x}}\,e^{-\gamma z}$ |
| $\mathbf{J_s}$(x=0) | $\frac{E_0}{\sqrt{\mu/\varepsilon}}\,\hat{\mathbf{z}}\,e^{-\gamma z}$ | $-\frac{j\omega\varepsilon a}{m\pi} E_0\,\hat{\mathbf{z}}\,e^{-\gamma z}$ | $H_0\,\hat{\mathbf{y}}\,e^{-\gamma z}$ |
| $\mathbf{J_s}$(x=a) | **- $\mathbf{J_s}$**(x=0) | $-(-1)^m\,\mathbf{J_s}(x=0)$ | $-(-1)^m\,\mathbf{J_s}(x=0)$ |

## 4.5 The rectangular waveguide

The geometry of the rectangular waveguide is shown in figure Fig. 4.6. It consists of a rectangular tube a*b, which is supposed to be infinite in the z direction.

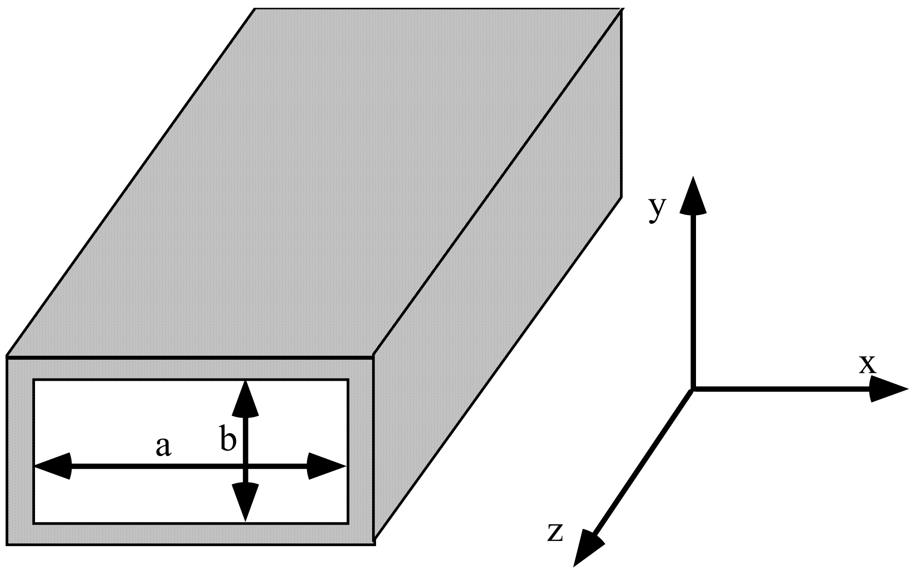



*Fig. 4.6 : Rectangular waveguide*

The guide is formed of four conducting walls, placed at x=0, x=a, y=0 and y=b. The propagation of the wave occurs along the z direction. As this type of guide is made of only one distinct conductor, it is not able to support a TEM wave: indeed, no static field can exist inside a hollow conductor made of a single conductor.

### *4.5.1 TM Modes*

Transverse magnetic modes (TM) have a non zero longitudinal component for the electric field ($E_z \neq 0$), whereas $H_z = 0$.

The wave equation will thus be solved for $e_z$ :

$$\left(\nabla_t^2 + k_c^2\right)e_z = \frac{\partial^2 e_z}{\partial x^2} + \frac{\partial^2 e_z}{\partial y^2} + k_c^2 e_z = 0 \qquad (4.42)$$

This equation can be solved using separation of variables. We suppose first that the solution can be written as

$$e_z = e_{zx}(x)e_{zy}(y) \qquad (4.43)$$

The wave equation becomes then

$$\frac{d^2 e_{zx}}{dx^2}\frac{1}{e_{zx}}+\frac{d^2 e_{zy}}{dy^2}\frac{1}{e_{zy}}=-k_c^2 \tag{4. 44}$$

This relation has to be valid for all values of x and y. It is thus necessary that both terms of the sums are constants :

$$\frac{d^2 e_{zx}}{dx^2}\frac{1}{e_{zx}}=-k_x^2 \ , \ \frac{d^2 e_{zy}}{dy^2}\frac{1}{e_{zy}}=-k_y^2 \ , \ k_x^2+k_y^2=k_c^2 \tag{4. 45}$$

The solutions to these differential equations are given by :

$$\begin{aligned} e_{zx} &= A\cos k_x x + B\sin k_x x \\ e_{zy} &= C\cos k_y y + D\sin k_y y \\ k_x^2 + k_y^2 &= k_c^2 \end{aligned} \tag{4. 46}$$

and

$$\begin{aligned} e_z &= \left(A\cos k_x x + B\sin k_x x\right)\left(C\cos k_y y + D\sin k_y y\right) \\ k_x^2 + k_y^2 &= k_c^2 \end{aligned} \tag{4. 47}$$

The six constants A, B, C, D, $k_x$ and $k_y$ are determined using the boundary conditions: A has to be equal to zero in order to satisfy the condition that the tangential electric field is zero at x=0, C has to be equal to zero in order to satisfy the condition that the tangential field has to be zero at y=0. In order to obtain zero tangential fields at x=a et y=b, we have two solutions : either B or D is equal to zero, and the solution is trivial, or

$$k_x=\frac{m\pi}{a} \ , \ k_y=\frac{n\pi}{b} \quad mn\neq 0 \tag{4. 48}$$

We get finally

$$
\begin{aligned}
e_z &= E_o \sin\frac{m\pi}{a}x \sin\frac{n\pi}{b}y \\
E_z &= E_o \sin\frac{m\pi}{a}x \sin\frac{n\pi}{b}y\, e^{-\gamma z} \\
\gamma &= \sqrt{k_c^2 - \omega^2 \varepsilon\mu}
\end{aligned}
\tag{4. 49}
$$

We note that these modes can propagate only for an imaginary γ, thus for

$$
\omega > \omega_c = \frac{1}{\sqrt{\varepsilon\mu}} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}
\tag{4. 50}
$$

For values blow this angular frequency, γ is real and the wave is attenuated in the guide.

### *4.5.2 TE modes*

Transverse electric modes have a non zero magnetic longitudinal component, and the characteristic equation to solve is :

$$
\left(\nabla_t^2 + k_c^2\right) h_z = \frac{\partial^2 h_z}{\partial x^2} + \frac{\partial^2 h_z}{\partial y^2} + k_c^2 h_z = 0
\tag{4. 51}
$$

Again, this equation is solved using the variable separation technique, to obtain:

$$
\begin{aligned}
h_z &= \left(A\cos k_x x + B\sin k_x x\right)\left(C\cos k_y y + D\sin k_y y\right) \\
k_x^2 + k_y^2 &= k_c^2
\end{aligned}
\tag{4. 52}
$$

The computation of the constants is a little less straight forward than in the TM case. We first derive the electric field components $e_x$ and $e_y$ from $h_z$ :

$$
\begin{aligned}
e_x &= -\frac{j\omega\mu}{k_c^2}\frac{\partial h_z}{\partial y} = -\frac{j\omega\mu k_y}{k_c^2}\left(A\cos k_x x + B\sin k_x x\right)\left(-C\sin k_y y + D\cos k_y y\right) \\
e_y &= -\frac{j\omega\mu}{k_c^2}\frac{\partial h_z}{\partial y} = -\frac{j\omega\mu k_y}{k_c^2}\left(-A\sin k_x x + B\cos k_x x\right)\left(-C\sin k_y y + D\cos k_y y\right)
\end{aligned}
\tag{4. 53}
$$

Two constants, B and D have to be equal to zero in order that $e_x$ is equal to zero at y=0 and $e_y$ at x=0. Moreover, $e_x$ has to be equal to zero at y=b and $e_y$ at x=a. The only non trivial solution is given by:

$$k_x = \frac{m\pi}{a} \, , \; k_y = \frac{n\pi}{b} \quad m + n \neq 0 \tag{4. 54}$$

and finally:

$$\begin{aligned} h_z &= H_o \cos\frac{m\pi}{a} x \cos\frac{n\pi}{b} y \\ H_z &= H_o \cos\frac{m\pi}{a} x \cos\frac{n\pi}{b} y \, e^{-\gamma z} \\ \gamma &= \sqrt{k_c^2 - \omega^2 \varepsilon\mu} \end{aligned} \tag{4. 55}$$

We note that these modes can propagate only for an imaginary $\gamma$ , thus for

$$\omega > \omega_c = \frac{1}{\sqrt{\varepsilon\mu}} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \tag{4. 56}$$

For values below this angular frequency, $\gamma$ is real and the wave is attenuated in the guide.

*4.5.3 Summary*

| | **TM Modes** | **TE Modes** |
|---|---|---|
| $k_c$ | $\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \; , \; mn \neq 0$ | $\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \; , \; m + n \neq 0$ |
| $\omega_c$ | $\frac{1}{\sqrt{\mu\varepsilon}} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$ | $\frac{1}{\sqrt{\mu\varepsilon}} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$ |
| $\alpha$ | $k_c \sqrt{1 - \left(\frac{\omega}{\omega_c}\right)^2} \; , \; \omega < \omega_c$ | $k_c \sqrt{1 - \left(\frac{\omega}{\omega_c}\right)^2} \; , \; \omega < \omega_c$ |

| | | |
|---|---|---|
| β | $k\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}$ , $\omega > \omega_c$ | $k\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}$ , $\omega > \omega_c$ |
| Ez | $E_0 \sin\frac{m\pi}{a}x \sin\frac{n\pi}{b}y\, \mathrm{e}^{-\gamma z}$ | 0 |
| Hz | 0 | $H_0 \cos\frac{m\pi}{a}x \cos\frac{n\pi}{b}y\, \mathrm{e}^{-\gamma z}$ |
| Ex | $-\frac{\gamma m\pi}{ak_c^2}E_0 \cos\frac{m\pi}{a}x \sin\frac{n\pi}{b}y\, \mathrm{e}^{-\gamma z}$ | $\frac{j\omega\varepsilon n\pi}{bk_c^2}H_0 \cos\frac{m\pi}{a}x \sin\frac{n\pi}{b}y\, \mathrm{e}^{-\gamma z}$ |
| Ey | $-\frac{\gamma n\pi}{bk_c^2}E_0 \sin\frac{m\pi}{a}x \cos\frac{n\pi}{b}y\, \mathrm{e}^{-\gamma z}$ | $-\frac{j\omega\varepsilon m\pi}{ak_c^2}H_0 \sin\frac{m\pi}{a}x \cos\frac{n\pi}{b}y\, \mathrm{e}^{-\gamma z}$ |
| Hx | $\frac{j\omega\varepsilon n\pi}{bk_c^2}E_0 \sin\frac{m\pi}{a}x \cos\frac{n\pi}{b}y\, \mathrm{e}^{-\gamma z}$ | $\frac{\gamma m\pi}{ak_c^2}H_0 \sin\frac{m\pi}{a}x \cos\frac{n\pi}{b}y\, \mathrm{e}^{-\gamma z}$ |
| Hy | $-\frac{j\omega\varepsilon m\pi}{ak_c^2}E_0 \cos\frac{m\pi}{a}x \sin\frac{n\pi}{b}y\, \mathrm{e}^{-\gamma z}$ | $\frac{\gamma n\pi}{bk_c^2}H_0 \cos\frac{m\pi}{a}x \sin\frac{n\pi}{b}y\, \mathrm{e}^{-\gamma z}$ |
| Z | $\sqrt{\frac{\mu}{\varepsilon}}\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}$ | $\frac{\sqrt{\frac{\mu}{\varepsilon}}}{\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}}$ |

## 4.6 The circular waveguide

*The geometry of a circular waveguide is described in*
Fig. 4.7. It consist of tube having a circular section of radius a, and supposed to be infinite in the propagating direction z.

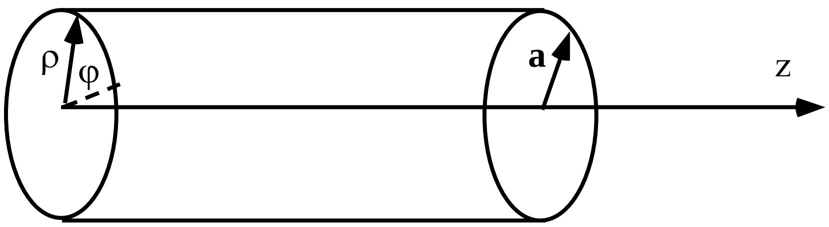



*Fig. 4.7 : Circular waveguide*

For the same reasons as in the case of the rectangular waveguide, this structure cannot support a TEM wave. Before studying the TM and TE modes, we will express the transverse fields (§4.1.4) in cylindrical coordinates:

$$\begin{aligned}
e_\rho &= \frac{-1}{k_c^2}\left[\gamma\frac{\partial e_z}{\partial\rho} + \frac{j\omega\mu}{\rho}\frac{\partial h_z}{\partial\varphi}\right] \\
e_\varphi &= \frac{1}{k_c^2}\left[-\frac{\lambda}{\rho}\frac{\partial e_z}{\partial\varphi} + j\omega\mu\frac{\partial h_z}{\partial\rho}\right] \\
h_\rho &= \frac{1}{k_c^2}\left[\frac{j\omega\varepsilon}{\rho}\frac{\partial e_z}{\partial\varphi} + -\gamma\frac{\partial h_z}{\partial\rho}\right] \\
h_\varphi &= \frac{-1}{k_c^2}\left[j\omega\varepsilon\frac{\partial e_z}{\partial\rho} + \frac{\gamma}{\rho}\frac{\partial h_z}{\partial\varphi}\right]
\end{aligned} \qquad (4.\ 57)$$

where

$$k_c^2 = \gamma^2 + k_0^2 = k_0^2 - \beta^2 \qquad (4.\ 58)$$

*4.6.1 TM modes*

We have to solve the wave equation for $e_z$ in cylindrical coordinates:

$$\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}+k_c^2\right)e_z=0 \tag{4. 59}$$

We apply again the variable separation technique:

$$e_z\left(\rho,\varphi\right)=R\left(\rho\right)P\left(\varphi\right) \tag{4. 60}$$

Which yields:

$$\frac{\rho^2}{R}\frac{d^2R}{d\rho^2}+\frac{\rho}{R}\frac{dR}{d\rho}+\rho^2k_c^2=-\frac{1}{P}\frac{d^2P}{d\varphi^2} \tag{4. 61}$$

The left hand side of this equation depends only on the radial coordinate ρ whereas the right hand side only on the azimuthal coordinate φ, thus both sides must be equal to a constant $k\varphi^2$ :

$$\begin{gathered}-\frac{1}{P}\frac{d^2P}{d\varphi^2}=k_\varphi^2\ ,\ \frac{d^2P}{d\varphi^2}+Pk_\varphi^2=0\\ \frac{\rho^2}{R}\frac{d^2R}{d\rho^2}+\frac{\rho}{R}\frac{dR}{d\rho}+\rho^2k_c^2=k_\varphi^2\ \ ,\ \ \rho^2\frac{d^2R}{d\rho^2}+\rho\frac{dR}{d\rho}+R\left(\rho^2k_c^2-k_\varphi^2\right)=0\end{gathered} \tag{4. 62}$$

The general solution for the equation in φ has the form :

$$P\left(\varphi\right)=A\sin k_\varphi\varphi+B\cos k_\varphi\varphi \tag{4. 63}$$

The solution has to be periodic in φ, meaning that $k_\varphi$ has to be an integer.

$$P\left(\varphi\right)=A\sin n\varphi+B\cos n\varphi \tag{4. 64}$$

The equation in ρ takes the form :

$$\rho^2 \frac{d^2R}{d\rho^2} + \rho \frac{dR}{d\rho} + R\left(\rho^2 k_c^2 - n^2\right) = 0 \tag{4. 65}$$

This equation is a Bessel equation, which has solutions of the following form:

$$R(\rho) = CJ_n(k_c\rho) + DY_n(k_c\rho) \tag{4. 66}$$

$J_n$ and $Y_n$ are Bessel function of order n of the first and second kind. Bessel functions of second kind become infinite at the origin, which would not have any physical meaning. Thus D has to be equal to zero. We get finally:

$$e_z(\rho,\varphi) = J_n(k_c\rho)(A\sin n\varphi + B\cos n\varphi) \tag{4. 67}$$

where the constant C has been absorbed in A and B. We have now to determine the cut off wave number $k_c$. The boundary conditions imply that $e_z(\rho,\varphi)$ becomes zero at $\rho=a$. We have thus:

$$J_n(k_c a) = 0 \;\; donc \;\; k_c = \frac{p_{nm}}{a} \tag{4. 68}$$

where $p_{nm}$ is the $m^{th}$ zero of the Bessel function of first kind and order n. The zeros of the Bessel functions are readily available in tables or numerical databases.

The propagation constant for the $TM_{nm}$ mode is given by:

$$\beta_{nm} = \sqrt{k_0^2 - k_c^2} = \sqrt{k_0^2 - \left(\frac{p_{nm}}{a}\right)^2} \tag{4. 69}$$

and the cut-off frequency by

$$f_{c_{nm}} = \frac{k_c}{2\pi\sqrt{\mu\varepsilon}} = \frac{p_{nm}}{2\pi a\sqrt{\mu\varepsilon}} \tag{4. 70}$$

The first TM mode to propagate is the $TM_{01}$ mode, obtained for the first zero of the Bessel function of order zero, $p_{01}$=2.405. There is no $TM_{10}$, as $m \geq 1$. All the components of the electric and magnetic fields are readily from $e_z(\rho,\varphi)$.

We notice that the solutions contain two independent variables A and B. The value of the latter will depend on the source exciting the waveguide. The fact that we have two constants comes from the circular symmetry of the problem, having solutions with either a sinusoidal or cosinusoidal dependency. It is possible to choose either A or B equal to zero, by a proper position and orientation of the coordinate system.

### *4.6.2 TE modes*

The calculus is the same as for the TM case. The wave equation to solve is:

$$\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}+k_c^2\right)h_z=0 \tag{4. 71}$$

Applying the same procedure as for the TM case, we get:

$$h_z(\rho,\varphi)=J_n(k_c\rho)(A\sin n\varphi+B\cos n\varphi) \tag{4. 72}$$

The wave number is obtained using the boundary conditions: $e_\varphi(\rho,\varphi)$ has to be zero at $\rho=a$. Using the beginning of §4.6, we have

$$\mathrm{e}_\varphi(\rho,\varphi)=\frac{j\omega\mu}{k_c}(A\sin n\varphi+B\cos n\varphi)J_n'(k_c\rho) \tag{4. 73}$$

where J'n is the derivative of $J_n$ with respect to its argument. The boundary condition imposes that :

$$J_n'(k_c a)=0 \text{ thus } k_{c_{nm}}=\frac{p_{nm}'}{a} \tag{4. 74}$$

where $p'_{nm}$ is the $m^{th}$ zero of the derivative of the Bessel function of the first kind of order n. The TE modes are thus defined by the cut-off wave number $k_{c_{nm}}$. The propagation constant is given by

$$\beta_{nm}=\sqrt{k^2-k_c^2}=\sqrt{k^2-\left(\frac{p_{nm}'}{a}\right)^2} \tag{4. 75}$$

and the cut-off frequency by:

$$f_{c_{nm}} = \frac{p'_{nm}}{2\pi a\sqrt{\mu\varepsilon}} \tag{4. 76}$$

Again, we have m≥1. It is interesting to note that the smallest cut-off frequency is the one of the mode $TE_{11}$, corresponding to $p'_{11}$=1.8141. Indeed, $p'_{01}$=3.832, which yields a higher cut-off frequency for the $TE_{01}$ mode.

The $TE_{11}$ mode is called the dominant mode of the circular waveguide, because it has the lowest cut-off frequency of all TE and TM modes.

*4.6.3 Summary*

| | **Modes TM** | **Modes TE** |
|---|---|---|
| $k_c$ | $\frac{p_{nm}}{a}$ | $\frac{p'_{nm}}{a}$ |
| $\omega_c$ | $\frac{p_{nm}}{a\sqrt{\mu\varepsilon}}$ | $\frac{p'_{nm}}{a\sqrt{\mu\varepsilon}}$ |
| α | $k_c\sqrt{1-\left(\frac{\omega}{\omega_c}\right)^2}$ , $\omega<\omega_c$ | $k_c\sqrt{1-\left(\frac{\omega}{\omega_c}\right)^2}$ , $\omega<\omega_c$ |
| β | $k\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}$ , $\omega>\omega_c$ | $k\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}$ , $\omega>\omega_c$ |
| Ez | $J_n(k_c\rho)(A\sin n\varphi+B\cos n\varphi)\mathrm{e}^{-\gamma z}$ | 0 |
| Hz | 0 | $J_n(k_c\rho)(A\sin n\varphi+B\cos n\varphi)\mathrm{e}^{-\gamma z}$ |
| Eρ | $\frac{-\gamma}{k_c}(A\sin n\varphi+B\cos n\varphi)J'_n(k_c\rho)\mathrm{e}^{-\gamma z}$ | $-\frac{j\omega\mu n}{k_c^2\rho}(A\cos n\varphi-B\sin n\varphi)J_n(k_c\rho)\mathrm{e}^{-\gamma z}$ |
| Eφ | $-\frac{\gamma n}{k_c^2\rho}(A\cos n\varphi-B\sin n\varphi)J_n(k_c\rho)\mathrm{e}^{-\gamma z}$ | $\frac{j\omega\mu}{k_c}(A\sin n\varphi+B\cos n\varphi)J'_n(k_c\rho)\mathrm{e}^{-\gamma z}$ |
| Hρ | $\frac{j\omega\varepsilon n}{k_c^2\rho}(A\cos n\varphi-B\sin n\varphi)J_n(k_c\rho)\mathrm{e}^{-\gamma z}$ | $\frac{-\gamma}{k_c}(A\sin n\varphi+B\cos n\varphi)J'_n(k_c\rho)\mathrm{e}^{-\gamma z}$ |
| Hφ | $\frac{j\omega\varepsilon}{k_c}(A\sin n\varphi+B\cos n\varphi)J'_n(k_c\rho)\mathrm{e}^{-\gamma z}$ | $-\frac{\gamma n}{k_c^2\rho}(A\cos n\varphi-B\sin n\varphi)J_n(k_c\rho)\mathrm{e}^{-\gamma z}$ |

| | | |
|---|---|---|
| Z | $\sqrt{\frac{\mu}{\varepsilon}}\sqrt{1-\left(\frac{\omega_C}{\omega}\right)^2}$ | $\frac{\sqrt{\frac{\mu}{\varepsilon}}}{\sqrt{1-\left(\frac{\omega_C}{\omega}\right)^2}}$ |

*4.6.4 Zeros of the Bessel functions of the first kind*

| n | $p_{n1}$ | $p_{n2}$ | $p_{n3}$ |
|---|---|---|---|
| 0 | 2.405 | 5.520 | 8.654 |
| 1 | 3.832 | 7.016 | 10.174 |
| 2 | 5.135 | 8.417 | 11.620 |

*4.6.5 Zeros of the derivative of the Bessel functions of first kind*

| n | $p'_{n1}$ | $p'_{n2}$ | $p'_{n3}$ |
|---|---|---|---|
| 0 | 3.832 | 7.016 | 10.174 |
| 1 | 1.841 | 5.331 | 8.536 |
| 2 | 3.054 | 6.706 | 9.970 |

## 4.7 Printed microwave transmission lines

**References:**

S. Ramo, J.R. Whinnery and T. van Duzer, “Fields and Waves in Communication Electronics”, Wiley, 1984, § 8.6.

F. E. Gardiol, "Hyperfréquences", volume XIII du Traité d'Electricité, Presses Polytechniques Romandes, § 2.11.

K.C. Gupta, R. Garg & I.J. Bahl, "Microstrip Lines and Slotlines", Artech House, Dedham MA, 1979

R.K Hoffmann, "Handbook of Microwave Integrated Circuits", Artech House, Dedham MA, 1987

### *4.7.1 Introduction*

Printed microwave circuits gradually replace since the early fifties the more conventional waveguides and transmission lines, especially in consumer products. Their advantages are their light weight and bulk and low production cost, while their drawbacks are relatively high losses and the fact they are dispersive. Indeed and like all printed circuits, they are manufactured using photolithographic processes, which ensure a high repeatability and easy mass production.

Several kinds of printed circuits devoted to microwaves exist, but the most popular one is without any doubt the microstrip circuit. In the frame of this course, we will first introduce the stripline, which is homogeneous and thus non dispersive, then the microstrip line and finally the coplanar waveguide, which is used manly at mm-wave frequencies, where the losses of microstrip lines become prohibitive.

### *4.7.2 Stripline*

*Definition*

The stripline structure is illustrated in Fig. 4.8. It consists of a conductive strip sandwiched between two ground planes.

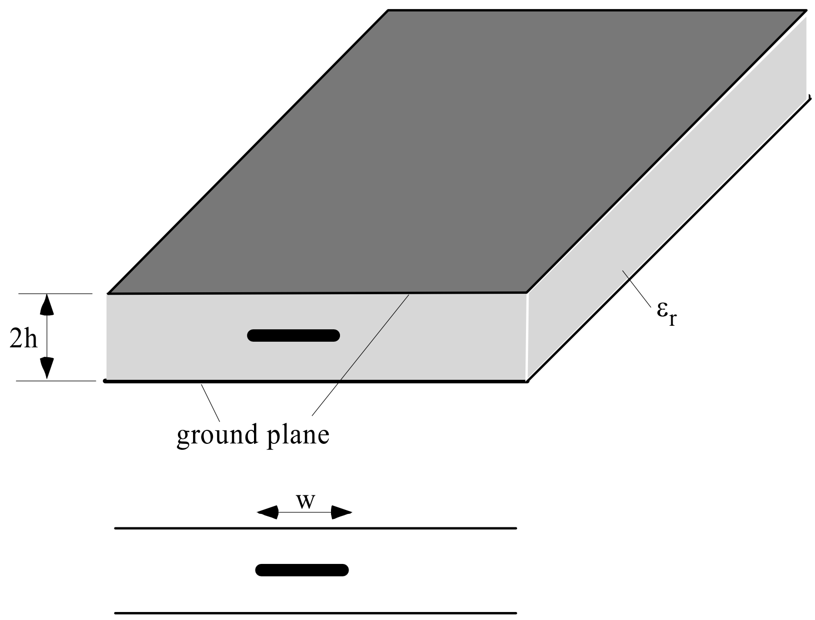



*Fig. 4.8: Stripline*

This structure is characterized by two conductive surfaces (ground planes) separated by a distance 2h. The volume between the ground planes is filled by a homogeneous dielectric medium of permittivity $\varepsilon_r$. A strip of width w is located between the two planes.

*Stripline propagation modes*

The analysis of this type of structure is unfortunately quite complex, and no analytic solution to Maxwell's equations exists for these specific boundary conditions. We will thus concentrate on the dominant mode only, as in most situations it will be the only propagating mode.

The dominant mode of this type of homogeneous structures is a TEM mode, as the structure is formed of two distinct conductors: the ground planes and the strip. We need thus to solve Laplace's equation for this structure, in order to characterize the TEM mode. An exact solution can be obtained using conformal transforms (see for instance "Stripline Circuit Design", by H.Howe Jr., Artech House, Dedham Ma, 1974), but this is a complicated procedure which yields results in a cumbersome form. We prefer here to give analytic expressions that are a good approximation of the rigorous solution, and which are much more convenient to use.

*Propagation constant*

In the case of a non magnetic medium, the phase velocity of a TEM wave is given by

$$v_\varphi = \frac{1}{\sqrt{\varepsilon_r \varepsilon_o \mu_o}} = \frac{c}{\sqrt{\varepsilon_r}} \tag{4. 77}$$

where c is the free space velocity of light. We deduce the propagation constant :

$$\beta = \frac{\omega}{v_\varphi} = \omega\sqrt{\varepsilon_r \varepsilon_o \mu_o} = \sqrt{\varepsilon_r}\, k_o \tag{4. 78}$$

*Characteristic impedance*

The characteristic impedance of a line supporting a TEM mode is given by :

$$Z_o = \sqrt{\frac{L}{C}} \tag{4. 79}$$

where L is the linear inductance of the line and C its linear capacitance. These two quantities are obtained solving Laplace's equations numerically, and doing a curve fitting of the obtained solution. We obtain finally the following approximation for the characteristic impedance :

$$Z_o = \frac{30\pi}{\sqrt{\varepsilon_r}} \frac{2h}{w_e + 0{,}441(2h)} \tag{4. 80}$$

where $w_e$ is the effective width of the central strip, given by :

$$\frac{w_e}{2h} = \frac{w}{2h} - \begin{cases} 0 & \text{for} \quad \dfrac{w}{2h} > 0.35 \\ \left(0.35 - \dfrac{w}{2h}\right)^2 & \text{for} \quad \dfrac{w}{2h} < 0.35 \end{cases} \tag{4. 81}$$

These expressions are valid for a central strip which is infinitesimally thin, and have an accuracy of about 1%. We note that the characteristic impedance becomes smaller when the strip becomes wider.

In a circuit conception process, we often want the inverse relation, yielding the width of the strip as a function of the characteristic impedance. This is obtained via the following approximation:

$$\frac{w}{2h} = \begin{cases} x & for \quad \sqrt{\varepsilon_r} Z_o < 120 \\ 0.85 - \sqrt{0.6 - x} & for \quad \sqrt{\varepsilon_r} Z_o > 120 \end{cases} \tag{4. 82}$$

with

$$x = \frac{30\pi}{\sqrt{\varepsilon_r} Z_o} - 0.441 \tag{4. 83}$$

*Attenuation of a stripline*

There are two kinds of losses in a stripline transmission line: dielectric losses and ohmic losses. The first are the same for all TEM lines and are given by:

$$\alpha_d = \frac{k \tan\delta}{2} \quad [Np/m] \tag{4. 84}$$

where k is the wave number in the medium and δ its loss angle :

$$k = \omega\sqrt{\varepsilon_r \varepsilon_o \mu_o} = \frac{\omega\sqrt{\varepsilon_r}}{c}$$
$$\tan\delta = \frac{\varepsilon''}{\varepsilon'} \tag{4. 85}$$

The ohmic losses are computed using a perturbation method. We get the following approximation:

$$\alpha_c = \begin{cases} \dfrac{0.0027 R_s \varepsilon_r Z_o}{30\pi(2h-t)} A & for \quad \sqrt{\varepsilon_r} Z_o < 120 \\ \dfrac{0.16 R_s}{Z_o 2h} B & for \quad \sqrt{\varepsilon_r} Z_o > 120 \end{cases} \quad [Np/m]$$
$$A = 1 + \frac{2w}{2h-t} + \frac{2h+t}{\pi(2h-t)} \ln\left(\frac{4h-t}{t}\right) \tag{4. 86}$$
$$B = 1 + \frac{2h}{(0.5w+0.7t)}\left(0.5 + \frac{0.414t}{w} + \frac{1}{2\pi}\ln\frac{4\pi w}{t}\right)$$

where t is the thickness of the strip and $R_S$ is the surface resistance of the conductor.

$$R_s = \sqrt{\frac{\omega \mu_o}{2\sigma}} \tag{4. 87}$$

where σ is the conductivity of the metal.

### 4.7.3 Microstrip

*Definition*

A microstrip line consists of a thin metallic conductor, the strip, placed on one face of a dielectric plate, the substrate. The other side of the plate is entirely covered by a conductor, the ground plane. This structure is illustrated in Fig. 4.9.

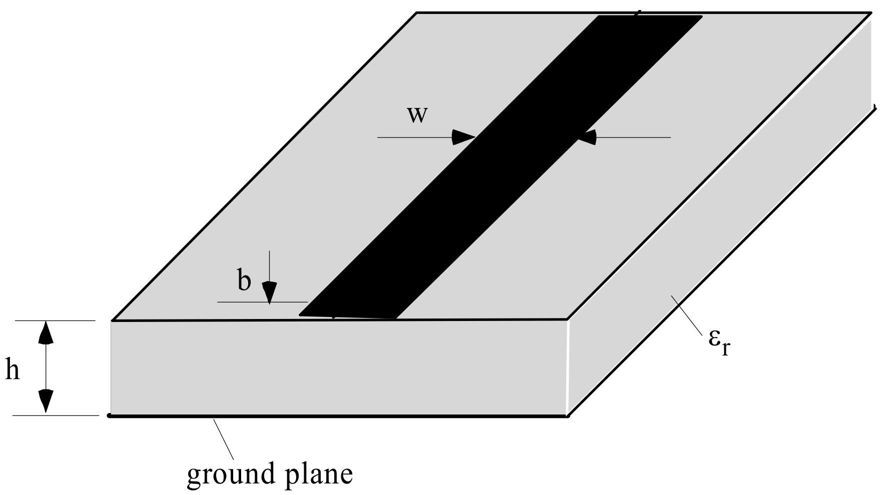



*Fig. 4.9 : Microstrip line*

The main characteristics of the line are:

- The relative permittivity of the substrate $\varepsilon_r$.
- The height of the substrate, in general some fractions of wavelength.
- The width w of the strip. This width has usually the same order of magnitude as the height of the substrate ($0.1\ h \leq w \leq 10\ h$).
- The thickness of the strip, usually small ($b/h \ll 1$).

These characteristics have an influence on :

- The concentration of the electric field in the substrate (no radiation). The higher the dielectric constant, the more the fields are concentrated in the substrate and the less the line will radiate.

- The characteristic impedance of the line, which depends mostly on the permittivity and the relation w/h.

*Propagation modes*

In first approximation, we can consider a microstrip structure as the half of a stripline structure (Fig. 4.10).

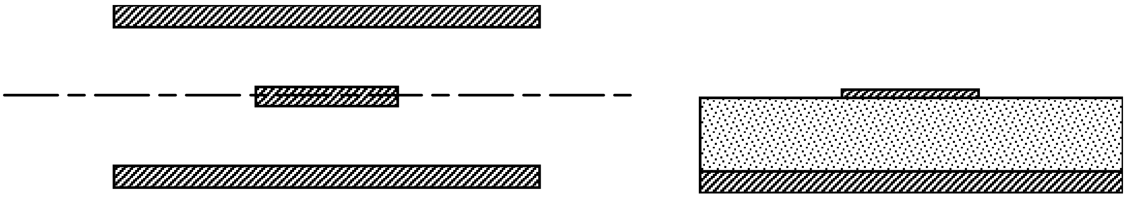


*Fig. 4.10*

In the absence of the dielectric substrate, a microstrip line could be viewed as a bifilar line, constituted by two conductors of width w separated by a distance 2h (the ground plane acting like a mirror). Such a structure would support a TEM wave.

The presence of the dielectric beneath the strip makes the structure inhomogeneous in the transverse plane. The propagating modes are thus hybrid modes. It is indeed easy to understand that such a structure cannot support a TEM wave : the phase velocity of this mode would indeed be equal to $c/\sqrt{\varepsilon_r}$ in the dielectric, while it should be equal to the velocity of light in the air. The resulting phase mismatch leads thus to the introduction of longitudinal components for the electric and magnetic fields, so that the boundary condition can be satisfied at the air dielectric interface.

In most practical applications of microstrip lines, the dielectric substrate is electrically thin:

$h < 0.05\ \lambda$. In consequence, the longitudinal components of the electromagnetic fields are very weak, and we have a quasi TEM mode. This means that the field distribution is very similar to the one obtained for the structure of Fig. 4.11, where the strip is placed in a homogeneous dielectric. .

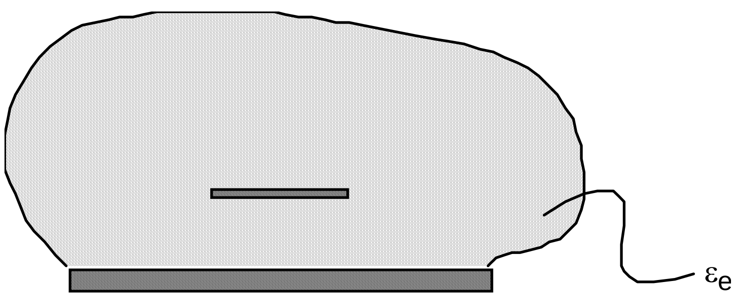


*Fig. 4.11 : Equivalent homogeneous structure*

This type of structure supports a TEM mode, with the following characteristics:

$$\begin{aligned} v_{\varphi} &= \frac{c}{\sqrt{\varepsilon_{\mathrm{e}}}} \\ \beta &= k_o \sqrt{\varepsilon_{\mathrm{e}}} \\ 1 &< \varepsilon_{\mathrm{e}} < \varepsilon_r \end{aligned} \qquad (4.\,88)$$

Microstrip lines have also been studied using numerical techniques, and the obtained results approximated by analytic expressions.

*Effective permittivity of a microstrip circuit*

The effective permittivity of a microstrip circuit is the equivalent homogeneous permittivity simulating best its characteristics. For a zero thickness strip, an approximation of this effective permittivity is given by :

$$\begin{aligned} \varepsilon_{\mathrm{e}} &= \frac{1}{2}(\varepsilon_r + 1) + \frac{1}{2}(\varepsilon_r - 1)\left[\left(1 + 12\frac{h}{w}\right)^{-0.5} + 0.04\left(1 - \frac{w}{h}\right)^2\right] \quad \text{for} \quad \frac{w}{h} \le 1 \\ \varepsilon_{\mathrm{e}} &= \frac{1}{2}(\varepsilon_r + 1) + \frac{1}{2}(\varepsilon_r - 1)\left(1 + 12\frac{h}{w}\right)^{-0.5} \quad \text{for} \quad \frac{w}{h} \ge 1 \end{aligned} \qquad (4.\,89)$$

The relative error of these approximations is smaller than 1 % when

$$0.05 \le \frac{w}{h} \le 20 \quad \text{and} \quad \varepsilon_r \le 16 \qquad (4.\,90)$$

We obtain for the phase velocity and the wave length :

$$v_\varphi = \frac{c}{\sqrt{\varepsilon_e}}$$
$$\lambda_g = \frac{\lambda_o}{\sqrt{\varepsilon_e}} \qquad (4.\ 91)$$

*Characteristic impedance*

For a strip of zero thickness, the following formulas yield a good approximation for the characteristic impedance of a microstrip line (relative error smaller than 1% for 0.05≤w/h≤20):

$$Z_c = \frac{Z_o}{2\pi\sqrt{\varepsilon_e}} \ln\left(\frac{8h}{w} + \frac{w}{4h}\right) \quad \mathit{for} \quad \frac{w}{h} \le 1$$
$$Z_c = \frac{Z_o}{\sqrt{\varepsilon_e}} \left(\frac{w}{h} + 1.393 + 0.667 \ln\left(\frac{w}{h} + 1.444\right)\right)^{-1} \quad \mathit{for} \quad \frac{w}{h} \ge 1 \qquad (4.\ 92)$$

where $Z_o = 120\pi$ is the characteristic impedance of the vacuum.

In a circuit conception process, we often want the inverse relation, yielding the width of the strip in function of the characteristic impedance. This is obtained via the following approximation:

$$\frac{w}{h} = 4\left[\frac{1}{2}\mathrm{e}^{A} - \mathrm{e}^{-A}\right]^{-1} \quad \mathit{for} \quad \frac{w}{h} \le 2$$
$$\frac{w}{h} = \frac{\varepsilon_r - 1}{\pi\varepsilon_r}\left(\ln(B-1) + 0.39 - \frac{0.61}{\varepsilon_r}\right) + \frac{2}{\pi}\left(B - 1 - \ln(2B-1)\right) \quad \mathit{for} \quad \frac{w}{h} \ge 2 \qquad (4.\ 93)$$

with

$$A = \frac{Z_c}{Z_o}\pi\sqrt{2(\varepsilon_r + 1)} + \frac{\varepsilon_r - 1}{\varepsilon_r + 1}\left(0.23 + \frac{0.11}{\varepsilon_r}\right)$$
$$B = \frac{\pi}{2\sqrt{\varepsilon_r}}\frac{Z_o}{Z_c} \qquad (4.\ 94)$$

We see that, as for striplines, the characteristic impedance of a microstrip line becomes smaller when the strip becomes wider.

*Attenuation in a microstrip line*

There are two type of losses in a microstrip line, dielectric losses and Ohmic losses. The dielectric losses are given by

$$\alpha_d = \frac{k_o \varepsilon_r (\varepsilon_e - 1) \tan\delta}{2\sqrt{\varepsilon_e}(\varepsilon_r - 1)} \quad [Np/m] \tag{4. 95}$$

where $k_o$ is the wave number in free space and δ the loss angle of the dielectric.

The Ohmic losses are approximated by :

$$\alpha_c = \frac{R_s}{wZ_c} \quad [Np/m] \tag{4. 96}$$

where $R_s$ is the surface resistance of the conductor.

$$R_s = \sqrt{\frac{\omega\mu}{2\sigma}} \tag{4. 97}$$

and σ is the conductivity of the metal.

*Radiation of microstrip lines*

Radiation in a microstrip is linked to the apparition of higher order non guided modes. These are excited at the vicinity of discontinuities, like a step in width, a bend or the end of the line. For a line having a characteristic impedance of 50 **Ω**, we can compute the frequency $f_m$ for which the proportion of the radiated power remains smaller than 1% of the total power :

$$f_m[GHz] = \frac{2.14\sqrt[4]{\varepsilon_r}}{h[mm]} \tag{4. 98}$$

For a high frequency application, we should thus select a high permittivity substrate, and/or use a thin substrate.

*Dispersion in a microstrip line*

The quasi-TEM approximation used in the sections above neglects the longitudinal components of the electromagnetic fields, and allows thus no prediction for the dispersion. The concentration of the electric fields in the substrate will increase with the frequency, which leads us to think that the effective permittivity, the propagation constant and the characteristic impedance of the line will be frequency dependent. The rigorous study of these phenomena is complex and out of the scope of this course. But for practical applications, we can use the following approximation for the effective permittivity:

$$\varepsilon_{ed}(f) = \varepsilon_r - \frac{\varepsilon_r - \varepsilon_e}{1 + \left(\frac{f}{f_p}\right)^2 G}$$

$$f_p = \frac{Z_c}{2h\mu_o} \qquad (4.\ 99)$$

$$G = 0.6 + 0.009 Z_c$$

We use $\varepsilon_{ed}$ rather than $\varepsilon_e$ in the computation of the characteristic impedance, the wavelength and the phase velocity. In the cases when $f \ll f_p$, this correction is not necessary.

### 4.7.4 coplanar waveguides

**References:**

K.C. Gupta, R. Garg & I.J. Bahl, "Microstrip Lines and Slotlines", Artech House, Dedham MA, 1979.

T.Q. Deng, M.S. Leong & P.S. Kooi, "Accurate formulas for coplanar waveguide synthesis", Electronics Letters, Vol. 31, 1995, pp. 2017-2019.

*Definition*

Coplanar waveguides are meeting a new interest since some years, mainly as transmission lines in the mm-wave domain (30GHz-300GHz). They are indeed much cheaper to manufacture than traditional waveguides, and have fewer losses than microstrip lines. A coplanar waveguide is depicted Fig. 4.12. It consists of a strip situated on the same substrate side as the ground plane. The strip, of width s, is separated from the ground plane by two slots, of width w. The dielectric substrate has a height h.

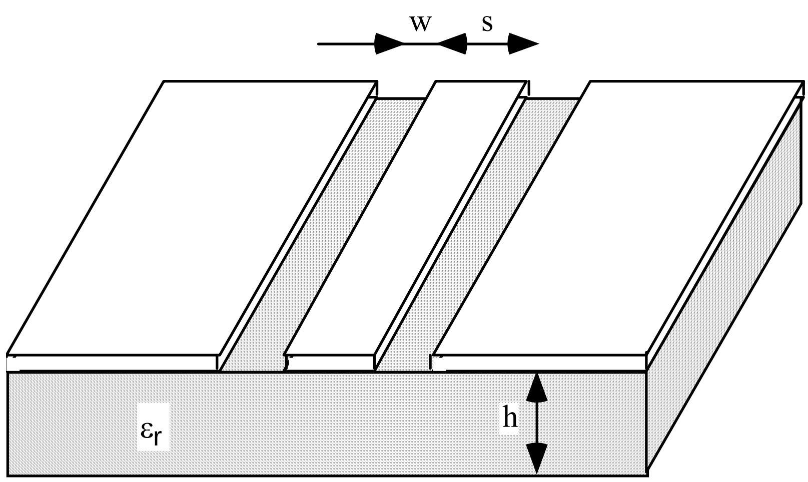



*Fig. 4.12: coplanar waveguide*

*Effective permittivity*

The effective permittivity of such a structure can be approximated by :

$$\varepsilon_e = \frac{\varepsilon_r + 1}{2}\left(\tanh\left(1.785\ln\frac{h}{w} + 1.75\right) + \frac{kw}{h}\left(0.04 - 0.7k + 0.01\left(1 - \frac{\varepsilon_r}{10}\right)(0.25 + k)\right)\right)$$

$$k = \frac{s}{s + w} \qquad (4.\ 100)$$

Again, the effective permittivity allows us to obtain the phase velocity, the propagation constant and the wavelength by :

$$\beta = \frac{\omega\sqrt{\varepsilon_e}}{c} \ , \ v_\varphi = \frac{c}{\sqrt{\varepsilon_e}} \ , \ \lambda_g = \frac{\lambda_o}{\sqrt{\varepsilon_e}} \tag{4. 101}$$

*Characteristic impedance*

The characteristic impedance is approximated by:

$$Z_c = \frac{30\pi}{\sqrt{\varepsilon_e}} \begin{cases} \dfrac{\pi}{\log\left(2\dfrac{1+\sqrt{k}}{1-\sqrt{k}}\right)} & 0 \le k \le 0.707 \\ \dfrac{\log\left(2\dfrac{1+\sqrt{k}}{1-\sqrt{k}}\right)}{\pi} & 0.707 \le k \le 1 \end{cases} \tag{4. 102}$$

### *4.7.6 Summary*

| Characteristic | coaxial cable | Waveguides | Stripline | microstrip |
|---|---|---|---|---|
| Dominant mode | TEM | $TE_{10}$ | TEM | Quasi TEM |
| Other modes | TM, TE | TM, TE | TM, TE | Hybrids |
| Dispersion | none | medium | none | weak |
| Bandwidth | high | low | high | high |
| Losses | medium | small | high | high |
| Max. power | medium | high | small | small |
| Size | big | big | medium | small |
| Ease of manufacturing. | medium | medium | easy | easy |

| | | | | |
|---|---|---|---|---|
| Integration of components | difficult | difficult | medium | easy |

# 5. Microwave network analysis

**References:**

F. E. Gardiol, "Hyperfréquences", volume XIII du Traité d'Electricité, Presses Polytechniques Romandes, Chap 6.

R.E. Collin, "Foundations for Microwave Engineering", Mc Graw Hill, 1992, Chap. 4.

## 5.1 Introduction

We will see in this chapter how the concepts of low-frequency circuit analysis can be extended to microwave circuits and networks. We will reconsider familiar concepts like current, voltage and impedance, find out if and when they can be used in microwave circuit analysis. We will learn to view currents and voltages as sums of incident and reflected waves. We will then introduce generalized waves and the scattering matrix as very efficient and practical tools for microwave circuit analysis.

## 5.2 Voltage, current and impedance

Currents and voltages are difficult to define in the microwave bands, excepted for the case of transmission lines supporting only a TEM wave. In all other cases, it is not possible to define these quantities in a univocal way. Moreover, they are extremely difficult to measure in a reliable way. Nevertheless, Kirchhoff's model is a very convenient tool for describing a circuit, and we would like to retain it. We will thus try to define equivalent currents and voltages on transmission line, remembering that excepted for the TEM case, these values are concepts without physical meaning and are not uniquely defined.

Each propagating mode will be described by a separate voltage current pair.

### *5.2.1 TEM Modes*

The measurement of currents and voltages is very difficult if not impossible at microwave frequencies, excepted when access ports can be clearly defined. This is the case only for TEM or quasi TEM modes.

Figure 5.1 illustrates the electric and magnetic fields for an arbitrary TEM transmission line.

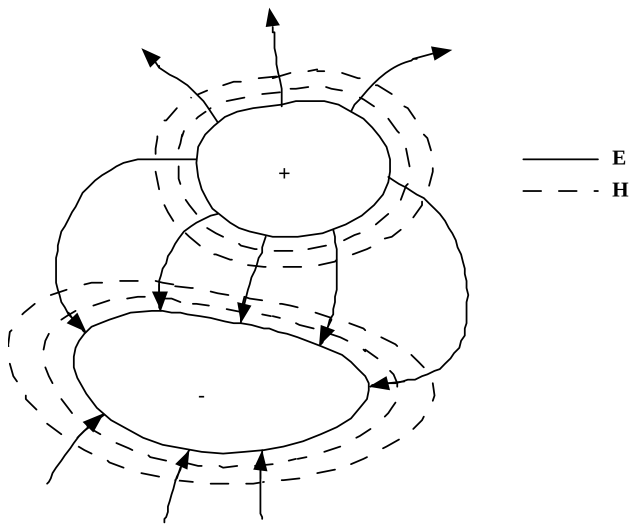



*Fig. 5. 1 : Arbitrary TEM line*

The voltage difference between the two conductors is defined as :

$$V = \int_{+}^{-} \mathbf{E} \cdot \mathbf{dl} \qquad (5.\ 1)$$

In the case of a TEM wave, the field has a static behaviour, and the voltage will not depend on the integration path, as long as the latter goes from conductor + to conductor -. Thus, the voltage is uniquely defined and there is no ambiguity.

The total current in conductor + is defined by Ampere's law :

$$I = \oint_{C+} \mathbf{H} \cdot \mathbf{dl} \qquad (5.\ 2)$$

where C+ is a closed integration path containing conductor +, but not conductor -. The characteristic impedance is the written as :

$$Z_C = \frac{V}{I} = \sqrt{\frac{L}{C}} \qquad (5.\ 3)$$

Where L is the inductance per unit length of the TEM line and C its capacitance per unit length.

### *5.2.2 Non TEM modes*

The situation is less clear for non-TEM modes, as a simple example can show :

The transverse fields of the $TE_{10}$ mode of a rectangular waveguide are given by :

$$\begin{aligned} E_y(x,y,z) &= E_0 \frac{j\omega\mu a}{\pi} \sin\frac{\pi x}{a} e^{-j\beta z} = E_0 \, e_y(x,y) e^{-j\beta z} \\ H_x(x,y,z) &= E_0 \frac{j\beta a}{\pi} \sin\frac{\pi x}{a} e^{-j\beta z} = E_0 h_y(x,y) e^{-j\beta z} \end{aligned} \tag{5. 4}$$

The voltage should thus be defined as

$$V = E_0 \frac{-j\omega\mu a}{\pi} \sin\frac{\pi x}{a} e^{-j\beta z} \int_y dy \tag{5. 5}$$

This voltage would depend on the x position we place the integration path in the guide, and of the geometry of this path. The result is clearly different if we choose a path $0<y<b$ at $x=a/2$ or at $x=0$. So what is the voltage ?

The answer is that in this case there is no "correct" voltage, which could be measured. We may however define a voltage and a current in many different ways for a non-TEM mode.

In order to obtain useful results, we will follow the following rules in our definition :

- The voltage and current are defined for one mode only. We decide (arbitrarily) that the voltage has to be proportional to the amplitude of the transverse electric field, while the current has to be proportional to the amplitude of the transverse magnetic field.
- In order to enable the use of Kirchhoff's model, the product of the current and the voltage should yield the power flux of the considered mode.
- The voltage divided by the current should be equal to the characteristic impedance of the line. The latter should also be equal to the mode impedance of the considered mode.

In an arbitrary guide, the transverse fields can be expressed as a function of an incident and a reflected wave. The voltage and current must thus be expressed in the same way :

$$
\begin{aligned}
\mathbf{E_t}(x,y,z) &= \mathbf{e_t}(x,y)\left(E_o^+\,\mathrm{e}^{-j\beta z} + E_o^-\,\mathrm{e}^{j\beta z}\right) \\
&= \frac{\mathbf{e_t}(x,y)}{C_1}\left(V^+\,\mathrm{e}^{-j\beta z} + V^-\,\mathrm{e}^{j\beta z}\right) \\
\mathbf{H_t}(x,y,z) &= \mathbf{h_t}(x,y)\left(E_o^+\,\mathrm{e}^{-j\beta z} - E_o^-\,\mathrm{e}^{j\beta z}\right) \\
&= \frac{\mathbf{h_t}(x,y)}{C_2}\left(I^+\,\mathrm{e}^{-j\beta z} - I^-\,\mathrm{e}^{j\beta z}\right)
\end{aligned}
\tag{5. 6}
$$

We write thus :

$$
\begin{aligned}
V(z) &= V^+\,\mathrm{e}^{-j\beta z} + V^-\,\mathrm{e}^{j\beta z} \\
I(z) &= I^+\,\mathrm{e}^{-j\beta z} - I^-\,\mathrm{e}^{j\beta z}
\end{aligned}
\tag{5. 7}
$$

The characteristic impedance of this wave is defined (by analogy to the TEM case) as

$$
Z_c = \frac{V^+}{I^+} = \frac{V^-}{I^-} = \frac{C_1 E_o^+}{C_2 E_o^+} = \frac{C_1}{C_2} \tag{5. 8}
$$

If we want moreover that the characteristic impedance is equal to the wave impedance of the mode, we get :

$$
\frac{C_1}{C_2} = Z_{\mathrm{mod}} \tag{5. 9}
$$

### *5.2.3 Impedance concepts*

It is important to make the difference between :

- The characteristic impedance of the medium. It depends only on the material constituting the medium :

$$
Z_o = \sqrt{\frac{\mu}{\varepsilon}} \tag{5. 10}
$$

- The wave impedance of a mode. It will depend on the type of the mode (TE, TM, TEM), on the guide, and on the materials used. It is also dependent on the frequency and the geometry :

$$Z_{\mathrm{mod}} = \frac{|\mathbf{E_t}|}{|\mathbf{H_t}|} \tag{5. 11}$$

- The characteristic impedance, defined as the voltage divided by the current. It is univocally defined only for a TEM transmission line :

$$Z_c = \frac{V^+}{I^+} = \frac{V^-}{I^-} = \sqrt{\frac{L}{C}} \tag{5. 12}$$

## 5.3 The impedance matrix

The concepts of voltage, current and impedance defined for transmission lines above can also be used to characterize microwave components, circuits and systems. The latter will then be defined by an impedance matrix, obtained from the voltage and current waves flowing on the transmission lines which are linked to the ports of the element.

**References:**

R.E. Collin, "Foundations for Microwave Engineering", Mc Graw Hill, 1992.

### *5.3.1 Impedance of a single port element*

The simplest possible microwave component has only one access. Its impedance matrix reduces to a scalar, defined as the voltage divided by the current, both "measured" at the access of the component, the *reference plane*.

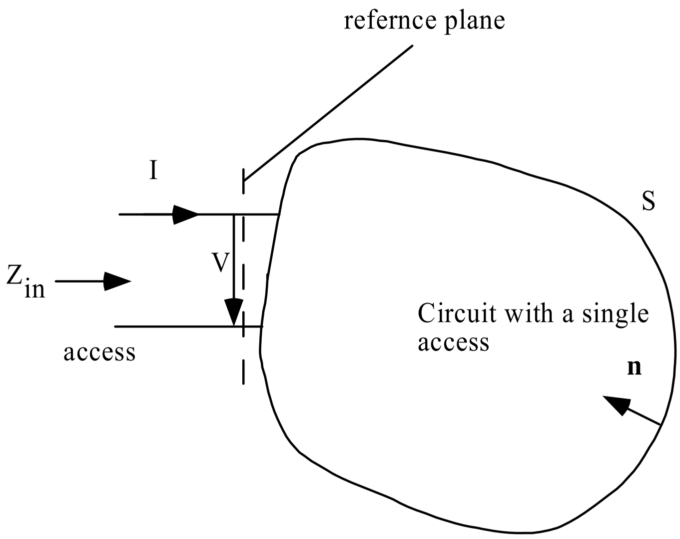



*Fig. 5. 2 : Single port element*

$$Z_{in} = \frac{V}{I} \tag{5. 13}$$

*5.3.2 Impedance characteristics of a single port element*

The complex power supplied to the element is given by Poynting's vector :

$$P = \oint_s \mathbf{E} \times \mathbf{H}^* \cdot \mathbf{ds} = P_r + 2j\omega\left(W_m - W_e\right) \tag{5. 14}$$

The E and H fields on the transmission line are by definition linked to the voltage and the current :

$$\begin{aligned} \mathbf{E_t}(x,y,z) &= V(z)\frac{\mathbf{e_t}(x,y)}{C_1}\mathrm{e}^{-j\beta z} \\ \mathbf{H_t}(x,y,z) &= I(z)\frac{\mathbf{h_t}(x,y)}{C_2}\mathrm{e}^{-j\beta z} \end{aligned} \tag{5. 15}$$

Thus, with the chosen definition for voltage and current :

$$\frac{1}{C_1 C_2}\int_s \mathbf{e_t} \times \mathbf{h_t} \cdot \mathbf{ds} = 1 \tag{5. 16}$$

Thus

$$P = \frac{1}{C_1 C_2}\int_s VI^* \mathbf{e_t} \times \mathbf{h_t} \cdot \mathbf{ds} = VI^* \tag{5. 17}$$

Moreover, the input impedance can be written as a function of the mean power :

$$Z_{in} = R + jX = \frac{V}{I} = \frac{VI^*}{|I|^2} = \frac{P}{|I|^2}$$
$$= \frac{\left(P_r + 2j\omega\left(W_m - W_e\right)\right)}{|I|^2} \tag{5. 18}$$

Where $P_r$ is the real mean power, $W_m$ is the stored magnetic energy and $W_e$ the stored electric energy. We can deduce from the above relation :

- R is proportional to the real power dissipated in the system (losses)
- X is proportional to the mean reactive energy stored in the system

### *5.3.3 Impedance and admittance matrices*

Let us consider the generic element depicted in figure 5.3. It is characterized by a certain number of accesses defined by reference planes located on the transmission lines linking the component to the outside world. These planes, noted $t_n$, are the *reference planes* between which the component is defined.

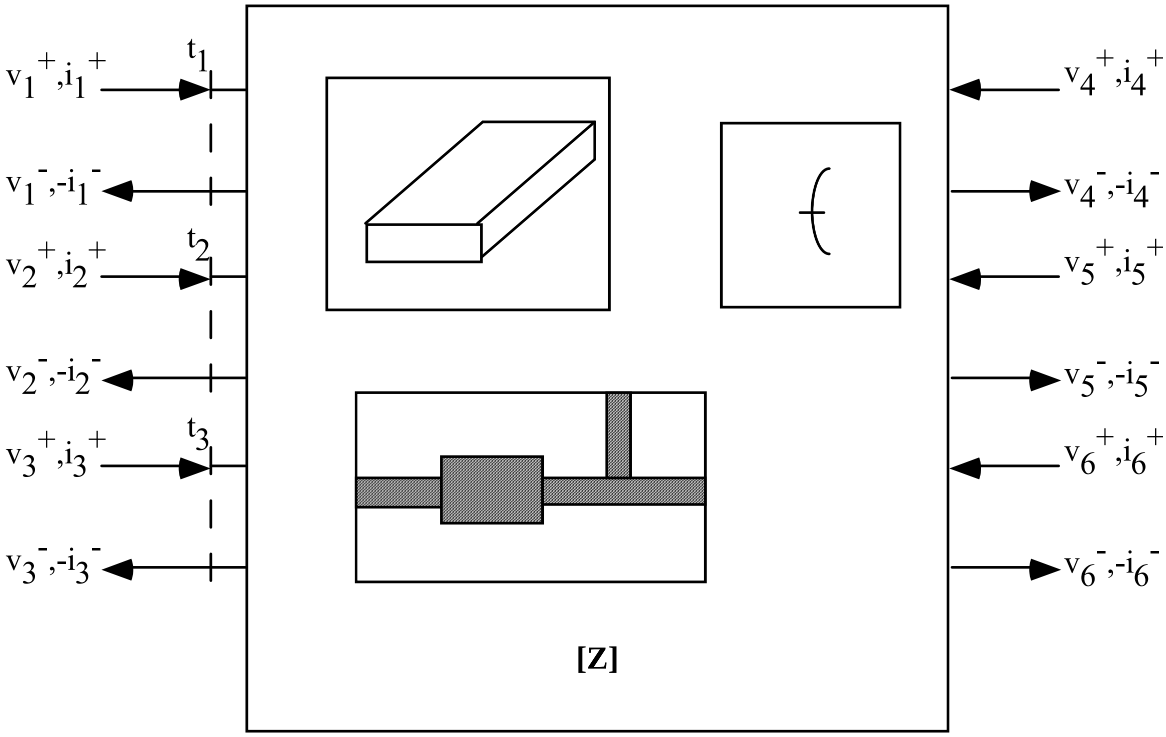



*Fig. 5. 3 : Multi-port microwave component and its access ports*

An axis of coordinates $z_i$ is linked to each transmission line i. By definition, the origin of this axis is located in the reference plane. We have thus at ports $t_1$, $t_2$, ..., $t_n$

$$V_n = V_n^+ + V_n^- \qquad I_n = I_n^+ - I_n^- \tag{5. 19}$$

The impedance and admittance matrices characterizing the component are defined by :

$$[V] = [Z][I] \qquad [I] = [Y][V] \tag{5. 20}$$

with

$$Z_{ij} = \left.\frac{V_i}{I_j}\right|_{I_k=0 \ for\ k\neq j}$$
$$Y_{ij} = \left.\frac{I_i}{V_j}\right|_{V_k=0 \ for\ k\neq j} \qquad (5.\ 21)$$

In consequence,

- The impedance matrix is obtained in open circuit conditions.
- The admittance matrix is obtained in short circuit conditions.

The impedance matrix is the inverse of the admittance matrix

$$[Y]=[Z]^{-1} \qquad (5.\ 22)$$

### *5.3.4 Properties of the impedance and admittance matrix*

#### *5.3.4.1 Reciprocity*

Let us consider the case where the basic conditions for Lorentz' reciprocity theorem are respected, thus the case where the component is isotropic, linear and passive. Consider the component depicted in figure 5.4, where all the accesses excepted for two are short circuited. Consider now $E_a$, $H_a$, $E_b$, et $H_b$ which are due to independent sources located somewhere in the circuit. Lorentz' reciprocity theorem states that :

$$\oint_s \mathbf{E_a} \times \mathbf{H_b} \cdot \mathbf{ds} = \oint_s \mathbf{E_b} \times \mathbf{H_a} \cdot \mathbf{ds} \qquad (5.\ 23)$$

where s is a closed integration surface enclosing the component.

We select the closed surface s as the external limit of the component passing through the reference planes, such that $E_{tan}$=0, excepted for reference planes 1 and 2. (If the transmission lines are made of conductors, this is always true. Otherwise, we can always select a surface sufficiently far away so that $E_{tan}$ is negligible).

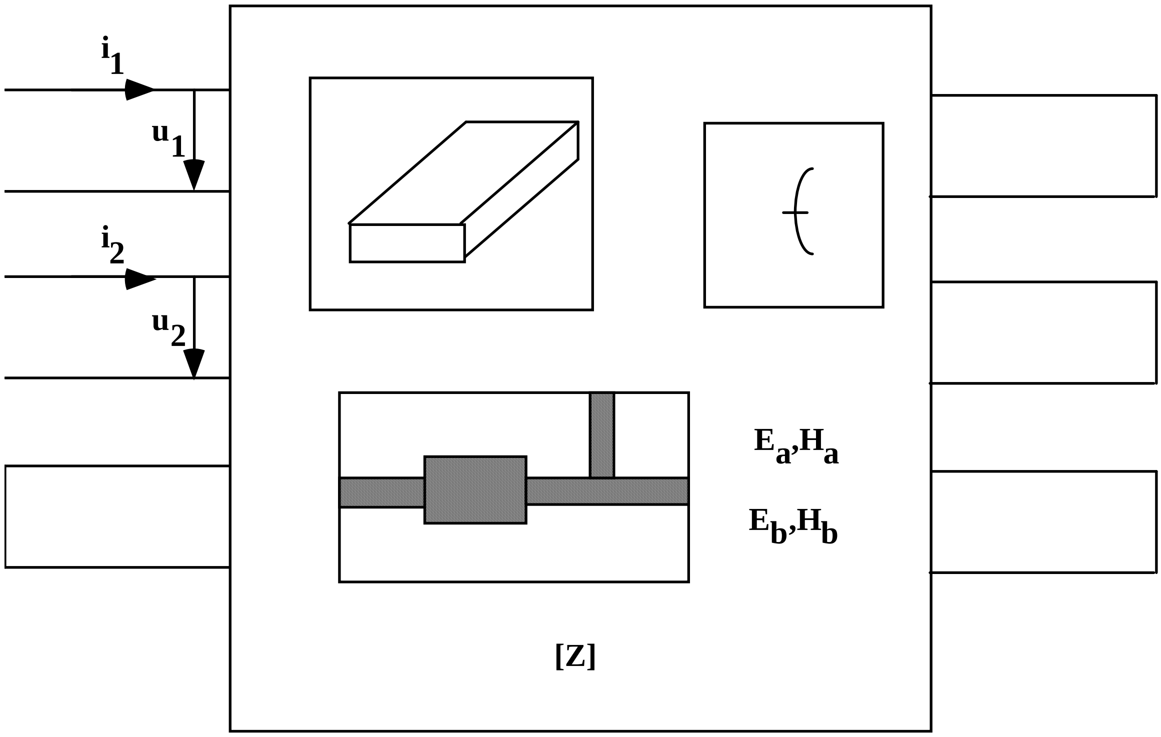



*Fig. 5. 4 : Illustration of the reciprocity principle*

The only contributions to the integrals come then from reference planes 1 and 2, the only ones which are not short circuited.

We write on these planes :

$$
\begin{aligned}
&\mathbf{E_{1a}} = V_{1a}\frac{\mathbf{e}_1}{C_1} \quad \mathbf{H_{1a}} = I_{1a}\frac{\mathbf{h_1}}{K_1}\\
&\mathbf{E_{1b}} = V_{1b}\frac{\mathbf{e}_1}{C_1} \quad \mathbf{H_{1b}} = I_{1b}\frac{\mathbf{h_1}}{K_1}\\
&\mathbf{E}_{2\mathbf{a}} = V_{2a}\frac{\mathbf{e}_2}{C_2} \quad \mathbf{H}_{2\mathbf{a}} = I_{2a}\frac{\mathbf{h}_2}{K_2}\\
&\mathbf{E}_{2\mathbf{b}} = V_{2b}\frac{\mathbf{e}_2}{C_2} \quad \mathbf{H}_{2\mathbf{b}} = I_{2b}\frac{\mathbf{h}_2}{K_2}
\end{aligned}
\tag{5. 24}
$$

And the reciprocity theorem becomes :

$$\left(V_{1a}I_{1b} - V_{1b}I_{1a}\right)\int_{s_1} \frac{1}{C_1K_1}\mathbf{e_1} \times \mathbf{h_1} \cdot \mathbf{ds} + \left(V_{2a}I_{2b} - V_{2b}I_{2a}\right)\int_{s_2} \frac{1}{C_2K_2}\mathbf{e}_2 \times \mathbf{h}_2 \cdot \mathbf{ds} = 0 \tag{5. 25}$$

But, by definition

$$\int_{s_1} \frac{1}{C_1K_1}\mathbf{e}_1 \times \mathbf{h}_1 \cdot \mathbf{ds} = \int_{s_1} \frac{1}{C_2K_2}\mathbf{e}_2 \times \mathbf{h}_2 \cdot \mathbf{ds} = 1 \tag{5. 26}$$

Thus

$$V_{1a}I_{1b} - V_{1b}I_{1a} + V_{2a}I_{2b} - V_{2b}I_{2a} = 0 \tag{5. 27}$$

We have

$$\begin{aligned} I_1 &= Y_{11}V_1 + Y_{12}V_2 \\ I_2 &= Y_{21}V_1 + Y_{22}V_2 \end{aligned} \tag{5. 28}$$

Thus

$$\left(V_{1a}V_{2b} - V_{1b}V_{2a}\right)\left(Y_{12} - Y_{21}\right) = 0 \tag{5. 29}$$

This relation has to hold for any sources, thus for any voltage. This means that :

$$Y_{12} = Y_{21} \tag{5. 30}$$

This relation can be generalized to all the ports of the component. We can thus write in a general way, for a circuit or component having neither active elements, plasmas or ferrites that :

$$\begin{aligned} Y_{ij} &= Y_{ji} \\ Z_{ij} &= Z_{ji} \end{aligned} \tag{5. 31}$$

Thus the impedance and admittance matrices are symmetric for a reciprocal component.

### *5.3.4.2 Lossless circuit*

Let us consider a lossless component with N ports. We can write that for this component the average power consumed by the circuit is zero

$$\operatorname{Re}\{P_{av}\} = 0 \qquad (5.\ 32)$$

By definition of the voltages and the currents at the ports, the mean power delivered to the component is given by :

$$P_{av} = [V]^t\,[I]^* \qquad (5.\ 33)$$

Which can be written in term of the impedance matrix as

$$\begin{aligned} P_{av} &= \left([Z][I]\right)^t [I]^* \\ &= [I]^t [Z][I]^* \\ &= \sum_{n=1}^{N} \sum_{m=1}^{N} I_m Z_{mn} I_n^* \end{aligned} \tag{5. 34}$$

The currents $I_n$ are independent, thus the real part of each m=n term has to be zero :

$$\mathrm{Re}\left\{I_n Z_{nn} I_n^*\right\} = \left|I_n\right|^2 \mathrm{Re}\left\{Z_{nn}\right\} = 0 \tag{5. 35}$$

We deduce from this that the diagonal terms of the impedance matrix of a lossless circuit must be purely imaginary.

$$\mathrm{Re}\left\{Z_{nn}\right\} = 0 \tag{5. 36}$$

We suppose now that all the currents flowing into the circuit are equal to zero, excepted for

$I_n$ and $I_m$. We write

$$\mathrm{Re}\left\{\left(I_n I_m^* + I_m I_n^*\right) Z_{mn}\right\} = \left(I_n I_m^* + I_m I_n^*\right) \mathrm{Re}\left\{Z_{mn}\right\} = 0 \tag{5. 37}$$

From which we deduce that

$$\mathrm{Re}\left\{Z_{mn}\right\} = 0 \tag{5. 38}$$

We have thus shown that the impedance (and admittance) matrix of a lossless component has to be <u>purely imaginary</u>

### *5.3.5 Examples of impedance matrices*

#### *1) Transmission line*

Consider the transmission line section depicted in figure 5.5

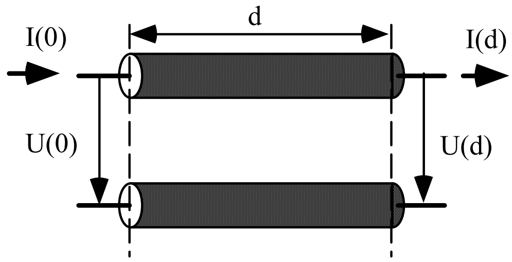



*Fig. 5. 5 : Transmission line of length d*

Its equivalent two-port is given by

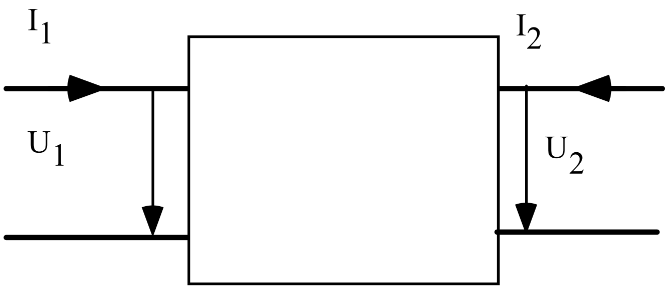



*Fig. 5. 6 : Equivalent two-port*

Where, by definition

$$U_1 = U(0), I_1 = I(0),$$
$$U_2 = U(d), I2 = -I(d) \tag{5. 39}$$

Knowing that

$$U(z) = U_+ \,\mathrm{e}^{-\gamma z} + U_- \,\mathrm{e}^{+\gamma z}$$
$$I(z) = I_+ \,\mathrm{e}^{-\gamma z} - I_- \,\mathrm{e}^{+\gamma z} \tag{5. 40}$$

We write

$$U(0)=U_+ + U_- \quad U(d)=U_+\,\mathrm{e}^{-\gamma d}+U_-\,\mathrm{e}^{+\gamma d}$$
$$I(0)=I_+ - I_- \quad I(d)=I_+\,\mathrm{e}^{-\gamma d}-I_-\,\mathrm{e}^{+\gamma d} \tag{5. 41}$$

The impedance and admittance matrices are then written as

$$\begin{bmatrix}U_1\\U_2\end{bmatrix}=\begin{bmatrix}Z_{11}&Z_{12}\\Z_{21}&Z_{22}\end{bmatrix}\begin{bmatrix}I_1\\I_2\end{bmatrix} = Z_c\begin{bmatrix}\coth(\gamma d)&\dfrac{1}{\sinh(\gamma d)}\\\dfrac{1}{\sinh(\gamma d)}&\coth(\gamma d)\end{bmatrix}\begin{bmatrix}I_1\\I_2\end{bmatrix}$$
$$\begin{bmatrix}I_1\\I_2\end{bmatrix}=\begin{bmatrix}Y_{11}&Y_{12}\\Y_{21}&Y_{22}\end{bmatrix}\begin{bmatrix}U_1\\U_2\end{bmatrix} = Y_c\begin{bmatrix}\coth(\gamma d)&\dfrac{-1}{\sinh(\gamma d)}\\\dfrac{-1}{\sinh(\gamma d)}&\coth(\gamma d)\end{bmatrix}\begin{bmatrix}U_1\\U_2\end{bmatrix} \tag{5. 42}$$

*2) Equivalent T circuit of a reciprocal two-port*

A reciprocal two-port has the following impedance matrix :

$$\begin{bmatrix}Z_{11}&Z_{12}\\Z_{12}&Z_{22}\end{bmatrix} \tag{5. 43}$$

It can be represented by an equivalent T circuit

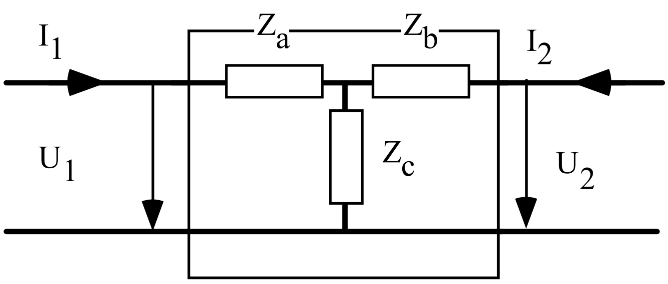


*Fig. 5. 7 : Equivalent T circuit of a reciprocal two-port*

where

$$\begin{aligned} Z_a &= Z_{11} - Z_{12} \\ Z_b &= Z_{22} - Z_{12} \\ Z_c &= Z_{12} \end{aligned} \tag{5. 44}$$

**example : equivalent T circuit of a transmission line section**

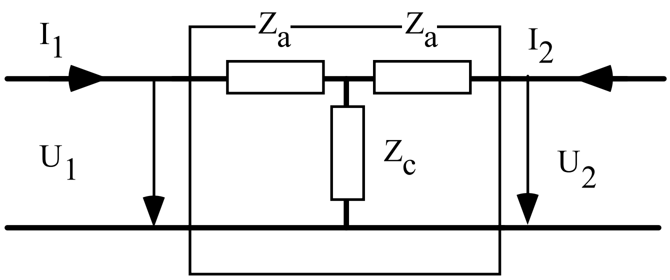



*Fig. 5. 8 : Equivalent T circuit of a transmission line section*

with

$$\begin{aligned} Z_a &= Z_{caract}\left(\coth(\gamma d) - \frac{1}{\sinh(\gamma d)}\right) \\ &= Z_{caract}\left(\frac{ch(\gamma d) - 1}{\sinh(\gamma d)}\right) = Z_{caract}\tanh\left(\frac{\gamma d}{2}\right) \\ Z_c &= \frac{Z_{caract}}{\sinh(\gamma d)} \end{aligned} \tag{5. 45}$$

*3) Equivalent Π circuit of a reciprocal two-port*

A reciprocal two-port has the following admittance matrix :

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12} & Y_{22} \end{bmatrix} \tag{5. 46}$$

Such a two port can be represented by an equivalent Π circuit

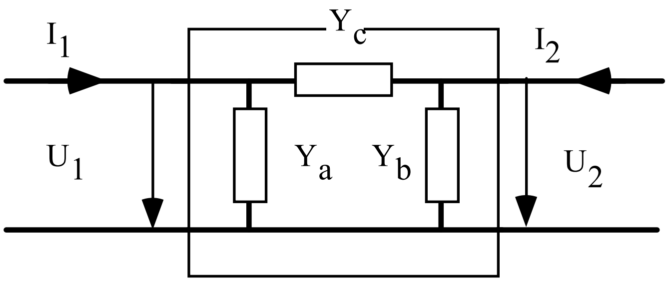



*Fig. 5. 9 : Equivalent Π circuit of a reciprocal two-port*

with

$$\begin{aligned} Y_a &= Y_{11} + Y_{12} \\ Y_b &= Y_{22} + Y_{12} \\ Y_c &= -Y_{12} \end{aligned} \tag{5. 47}$$

**Example : equivalent Π circuit of a transmission line section**

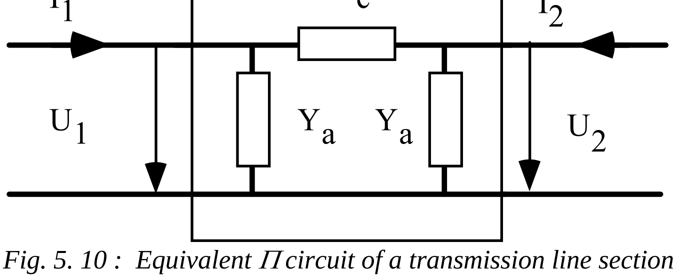



*Fig. 5. 10 : Equivalent Π circuit of a transmission line section*

With

$$
\begin{aligned}
Y_a &= Y_{caract}\left(\coth(\gamma d) - \frac{1}{\sinh(\gamma d)}\right) \\
&= Y_{caract}\left(\frac{ch(\gamma d) - 1}{\sinh(\gamma d)}\right) = Y_{caract}\tanh\left(\frac{\gamma d}{2}\right) \qquad (5.\ 48) \\
Y_c &= \frac{Y_{caract}}{\sinh(\gamma d)}
\end{aligned}
$$

## 5.4 The scattering matrix

**References:**

R.E. Collin, "Foundations for Microwave Engineering", Mc Graw Hill, 1992.

F. E. Gardiol, "Hyperfréquences", volume XIII du Traité d'Electricité, Presses Polytechniques Romandes, chap. 6

We have seen in the preceding sections that voltages and currents are not really well suited for the microwave range. One of the direct consequences of the non uniqueness of these values are that they are often not measurable. They can thus be used for the theoretical characterization of circuits and components, as we have seen above, but these theoretical impedances and admittances cannot be corroborated by measured results. This is why we introduce normalized wave amplitudes, which are linked to power, in order to characterize microwave circuits.

### *5.4.1 Normalized wave amplitudes*

We define the normalized waves amplitudes **a** and **b** as

$$a_i = \frac{v_i + Z_{ci} i_i}{2\sqrt{Z_{ci}}} \ , \ b_i = \frac{v_i - Z_{ci} i_i}{2\sqrt{Z_{ci}}} \qquad (5.\ 49)$$

<u>Note</u> : These normalized wave amplitudes have the dimension of the square root of the power, and power is easily measurable in microwaves.

The inverse relation is given by

$$v_i = \sqrt{Z_{ci}}\left(a_i + b_i\right) \ , \ i_i = \frac{\left(a_i - b_i\right)}{\sqrt{Z_{ci}}} \qquad (5.\ 50)$$

These normalizes amplitudes are defined on the transmission lines linking the ports of a component. But on these transmission lines, we have :

$$v_i = v_i^+ \,\mathrm{e}^{-j\beta z} + v_i^- \,\mathrm{e}^{+j\beta z}$$
$$i_i = i_i^+ \,\mathrm{e}^{-j\beta z} + i_i^- \,\mathrm{e}^{+j\beta z} \qquad (5.\ 51)$$

From which we deduce

$$a_i = \frac{v_i^+}{\sqrt{Z_{ci}}}\mathrm{e}^{-j\beta z}$$
$$b_i = \frac{v_i^-}{\sqrt{Z_{ci}}}\mathrm{e}^{+j\beta z} \qquad (5.\ 52)$$

Thus

- $a_i$ : is a purely progressive (incident) wave giving the signal (square root of the power) flowing into the port i
- $b_i$ : is a purely retrograde (reflected) wave, giving the signal flowing out of the port i.

### *5.4.2 Reference planes*

A microwave component is defined between its ports, which are planes transverse to the transmission lines linking the component to the outside world. On these planes are located the origin of the longitudinal coordinate $z_i$ related to the transmission line i (figure 5.11)

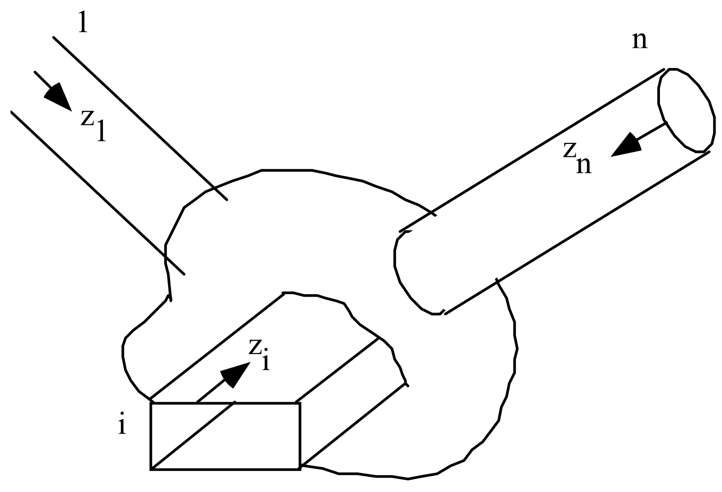



*Fig. 5. 11 : Microwave component with its reference planes*

By definition, the reference planes have to satisfy the following criteria :

- The reference planes are sufficiently far away from the component, to ensure that all evanescent modes have decayed.

- The transmission lines support only the dominant mode.
- The transmission lines are lossless

The active power at port i is given by

$$P_i = \mathrm{Re}\left[v_i i_i^*\right] = \mathrm{Re}\left[\left(a_i + b_i\right)\left(a_i^* - b_i^*\right)\right] = |a_i|^2 - |b_i|^2 \qquad (5.\ 53)$$

$|a_i|^2$ is thus the active power flowing into the component at port i, while $|b_i|^2$ is the active power flowing out of the component at port i.

### *5.4.3 Scattering matrix of a component*

A microwave component is characterized as a function of the generalized wave amplitudes flowing on the transmission lines at the reference planes (figure 5.12). It is then characterized by its scattering matrix as :

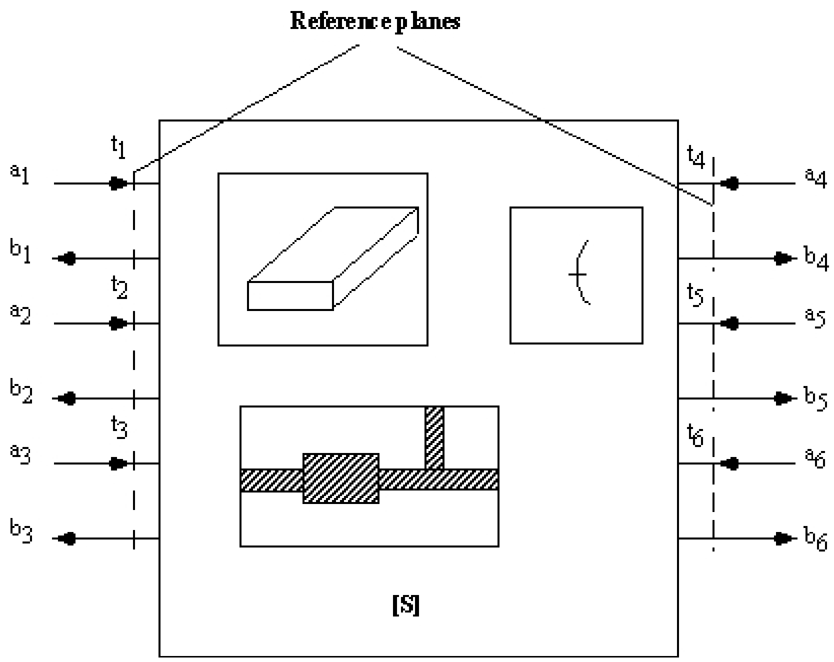


*Fig. 5. 12 : Microwave component with its reference planes*

$$[b] = [S][a] \qquad (5.\ 54)$$

with

$$s_{ij} = \left.\frac{b_i}{a_j}\right|_{a_k=0,\, k\neq j} \qquad (5.\ 55)$$

*5.4.4 Properties of the scattering matrix*

- The impedance matrix characterizes a component between open-circuits ($Z_{ij} = v_i/i_j$, $i_k$=0 for k≠j), while the admittance matrix characterizes a component between short-circuits ($Y_{ij} = i_i/v_j$, $i_k$=0 for k≠j). The scattering matrix characterizes a component between matched loads ($S_{ij} = b_i/a_j$, $a_k$=0 for k≠j).
- The term $s_{ij}$ is the transfer function of the signal between port j and port i.
- The scattering matrix depends on the component itself, but also on the environment of the component through the transmission lines.
- Changing the characteristic impedance of the transmission lines means changing also the scattering matrix.

*5.4.4.1 Reciprocity*

In the case of a passive, linear and isotropic component, we have seen that

$$z_{ij} = z_{ji} \qquad (5.\ 56)$$

It is easy to show through matrix transforms that for a reciprocal circuit

$$s_{ij} = s_{ji} \qquad (5.\ 57)$$

Thus, a reciprocal circuit has a symmetrical scattering matrix.

### 5.4.4.2 Lossless circuit

A lossless circuit is circuit where no active power is dissipated. This means that for such a circuit, the sum of the active power flowing into the circuit must be equal to the active power flowing out of the circuit :

$$\sum |a_i|^2 = \sum |b_i|^2 \tag{5. 58}$$

In a matrix notation, this is equivalent to

$$[\tilde{a}][a] - [\tilde{b}][b] = 0 \tag{5. 59}$$

Where the tilde sign means the transpose complex conjugate of a matrix :

$$[\tilde{a}] = \left[a^*\right]^t \tag{5. 60}$$

Moreover, by definition,

$$\begin{aligned} [b] &= [S][a] \\ [\tilde{b}] &= [\tilde{a}][\tilde{S}] \end{aligned} \tag{5. 61}$$

Thus

$$\begin{aligned} &[\tilde{a}][a] - [\tilde{a}][\tilde{S}][S][a] = 0 \\ &[\tilde{a}]\left\{[1] - [\tilde{S}][S]\right\}[a] = 0 \\ &[\tilde{S}][S] = [1] \end{aligned} \tag{5. 62}$$

This can be written as

$$\sum_{i=1}^{N} s_{ij}^* s_{ik} = \delta_{jk} \qquad \delta_{jk} = \begin{cases} 1 & si \quad j = k \\ 0 & si \quad j \neq k \end{cases} \tag{5. 63}$$

### *5.4.4.3 Moving the reference plane*

The origins of the axes zi, thus the position of the reference plane, are arbitrarily defined, as long as we are in single mode propagation. It can thus be interesting to study the effect of a translation of the reference plane along the axis on the scattering matrix (figure 5.13).

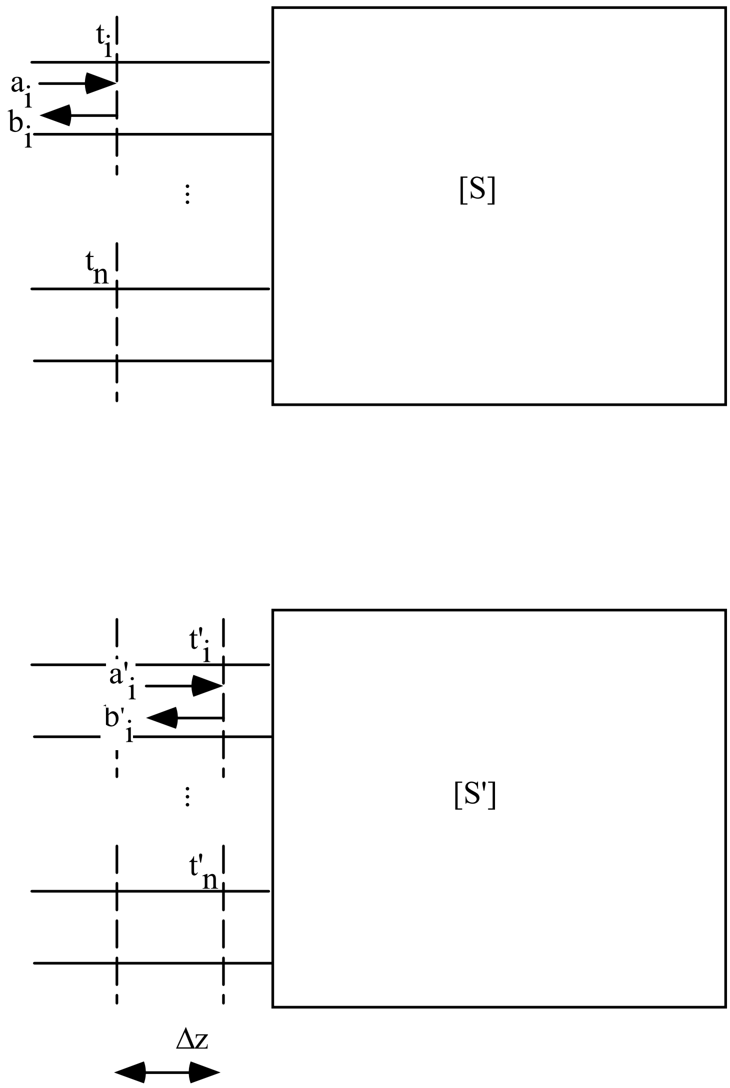



*Fig. 5. 13 : Translation of the reference plane*

The normalized wave amplitudes $a'_i$ et $b'_i$ linked to the translated coordinates system can be expressed in term of the normalized wave amplitudes $a_i$ et $b_i$, linked to the original coordinate system, by

$$a'_i = a_i \, e^{-j\varphi_i}$$
$$b'_i = b_i \, e^{j\varphi_i} \qquad (5.\ 64)$$
$$\varphi_i = -\beta_i \Delta z_i$$

But we have also

$$[b'] = [S'][a'] \text{ et } [b] = [S][a] \tag{5. 65}$$

We write

$$s'_{ii} = s_{ii}\, e^{j2\varphi_i} \tag{5. 66}$$

And in general

$$\begin{aligned} [a] &= \left[diag\, e^{j\varphi}\right][a'] \\ [a'] &= \left[diag\, e^{-j\varphi}\right][a] \\ [b] &= \left[diag\, e^{-j\varphi}\right][b'] \\ [b'] &= \left[diag\, e^{j\varphi}\right][b] \end{aligned} \tag{5. 67}$$

With

$$\left[diag\, e^{j\varphi}\right] = \begin{bmatrix} e^{j\varphi_1} & 0 & \ldots & 0 \\ 0 & e^{j\varphi_2} & & : \\ : & & & \\ 0 & \ldots & & e^{j\varphi n} \end{bmatrix} \tag{5. 68}$$

Thus

$$\begin{aligned} [b'] &= \left[diag\, e^{j\varphi}\right][S]\left[diag\, e^{j\varphi}\right][a'] \\ [S'] &= \left[diag\, e^{j\varphi}\right][S]\left[diag\, e^{j\varphi}\right] \\ s'_{ij} &= s_{ij}\, e^{j\left(\varphi_i + \varphi_j\right)} \end{aligned} \tag{5. 69}$$

### *5.4.4.4 Relation between impedance matrix and scattering matrix*

We define the two diagonal matrices

$$[G] = [diag\ Z_{ci}] = \begin{bmatrix} Z_{c1} & 0 & \dots & 0 \\ 0 & Z_{c2} & & : \\ : & & & \\ 0 & \dots & & Z_{cn} \end{bmatrix}$$

$$[F] = \left[ diag\ \frac{1}{2\sqrt{Z_{ci}}} \right] = \begin{bmatrix} \frac{1}{2\sqrt{Z_{c1}}} & 0 & \dots & 0 \\ 0 & \frac{1}{2\sqrt{Z_{c2}}} & & : \\ : & & & \\ 0 & \dots & & \frac{1}{2\sqrt{Z_{cn}}} \end{bmatrix} \tag{5. 70}$$

and use the definition of the normalized wave amplitudes to write

$$\begin{aligned}[S] &= [F]\big[[Z]-[G]\big]\big[[F]\big[[Z]+[G]\big]\big]^{-1} \\ &= [F]\big[[Z]-[G]\big]\big[[Z]+[G]\big]^{-1}[F]^{-1}\end{aligned} \tag{5. 71}$$

and

$$[Z] = [F]^{-1}\big[[1]+[s]\big]\big[[1]-[s]\big]^{-1}[F][G] \tag{5. 72}$$

where [1] is the identity matrix.

### *5.4.5 Flow charts*

The terms of the scattering matrix are transfer functions, linking an input port to an output port. They can be represented graphically by flow charts.

example 1 : two-port

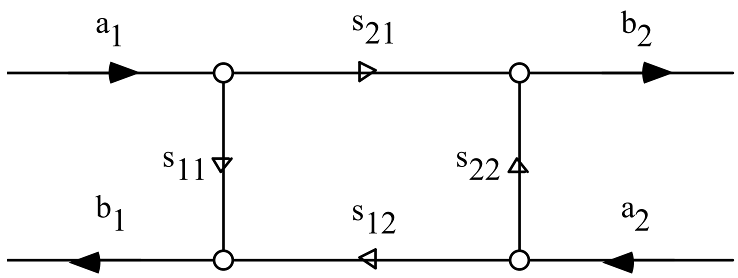



*Fig. 5. 14 : Flow chart of a two-port*

example 2 : three-port

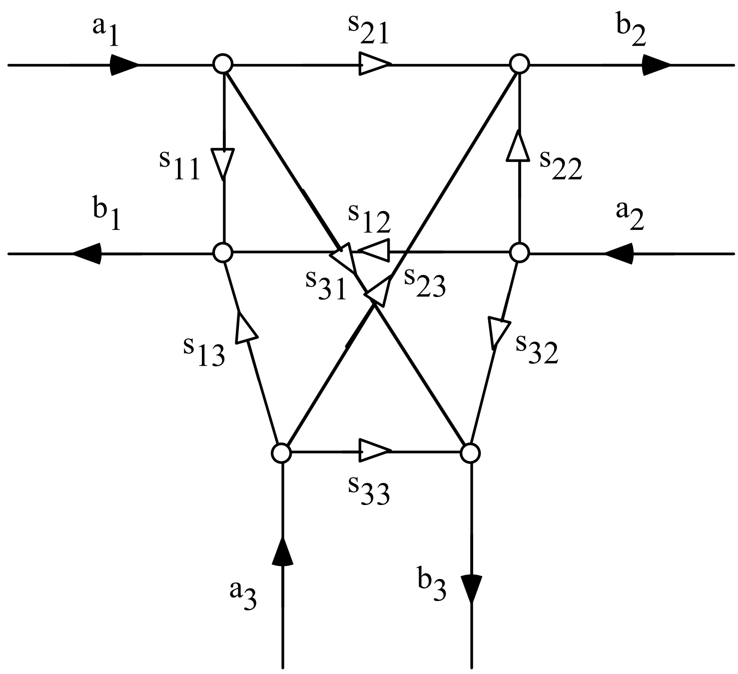



*Fig. 5. 15 : Flow chart of a three-port*

example 3 : two two-ports cascaded

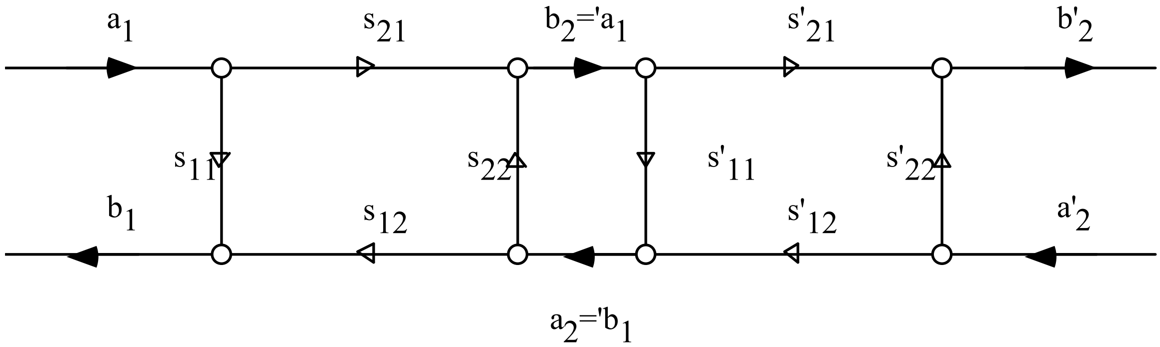



*Fig. 5. 16 : Cascaded two-ports*

### 5.4.5.1 Flow chart reduction rules

1) multiplication

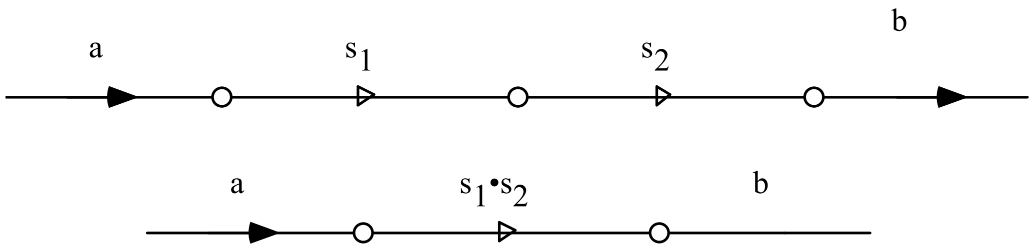



*Fig. 5. 17 : Two flow chart in series*

2) addition

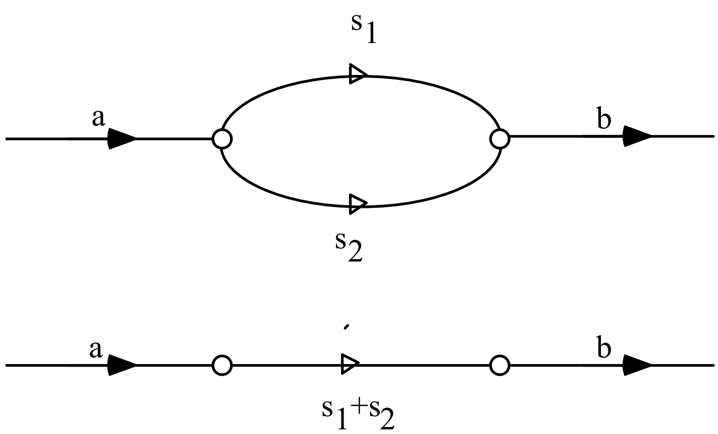



*Fig. 5. 18: Two flow charts in parallel*

3) retroaction

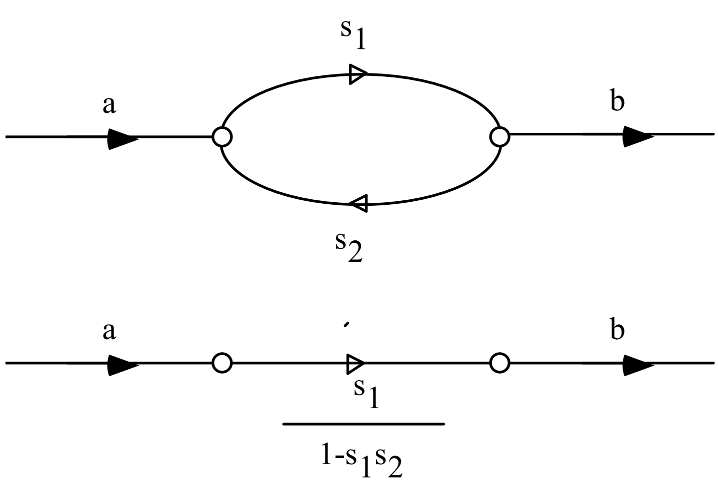


*Fig. 5. 19 : Retroaction of two flow charts*

### *5.4.5.2 Example*

Find the reflection coefficient at the input of a reciprocal two-port terminated by a short circuit

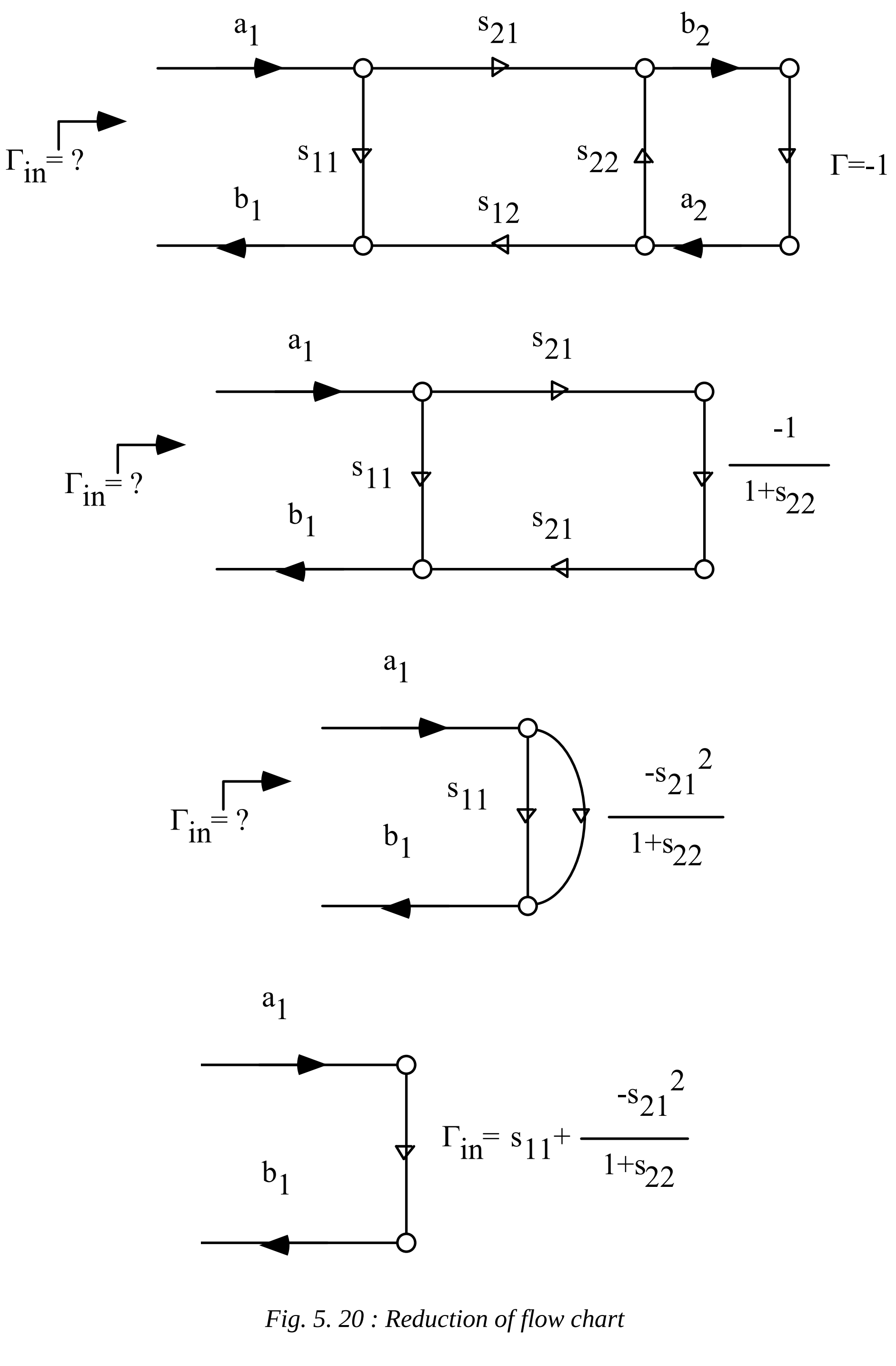
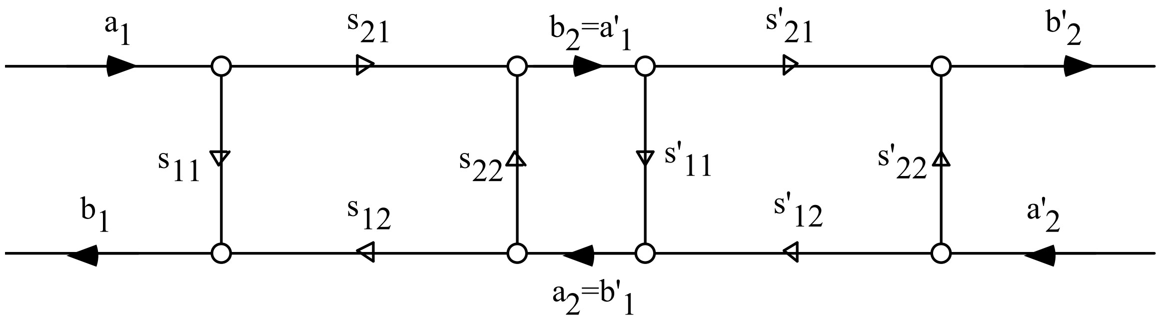
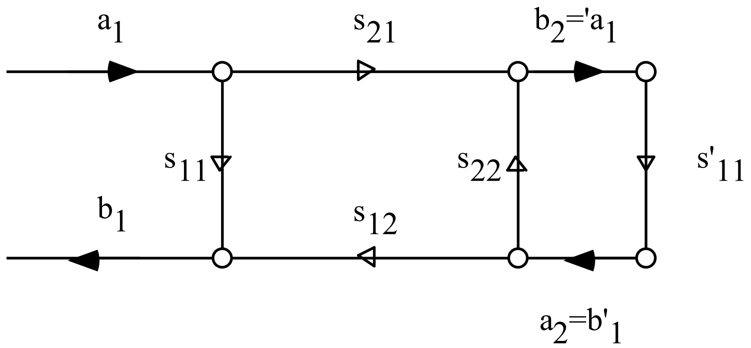



*Fig. 5. 20 : Reduction of flow chart*

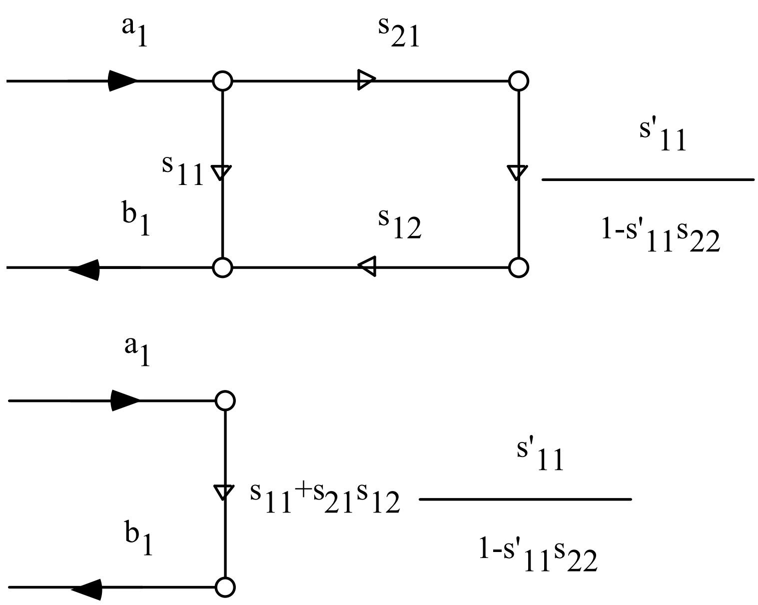
### *5.4.5.3 Example : two cascaded two-ports*



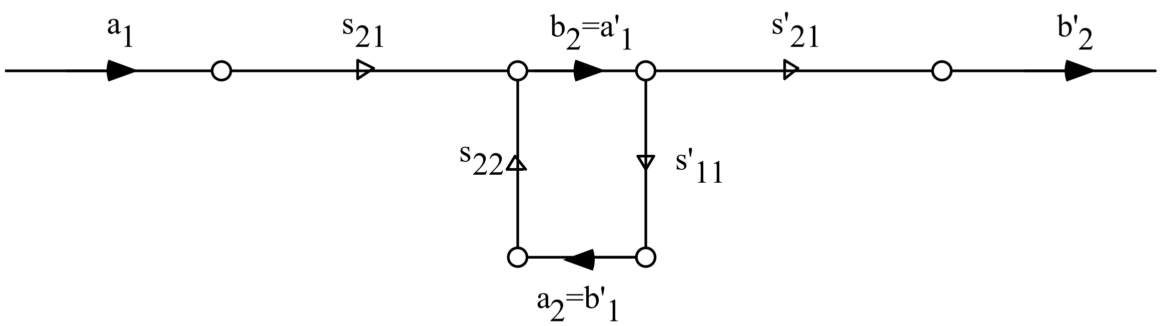

First stage : we look for the possible paths going from $a_1$ to $b_1$ :



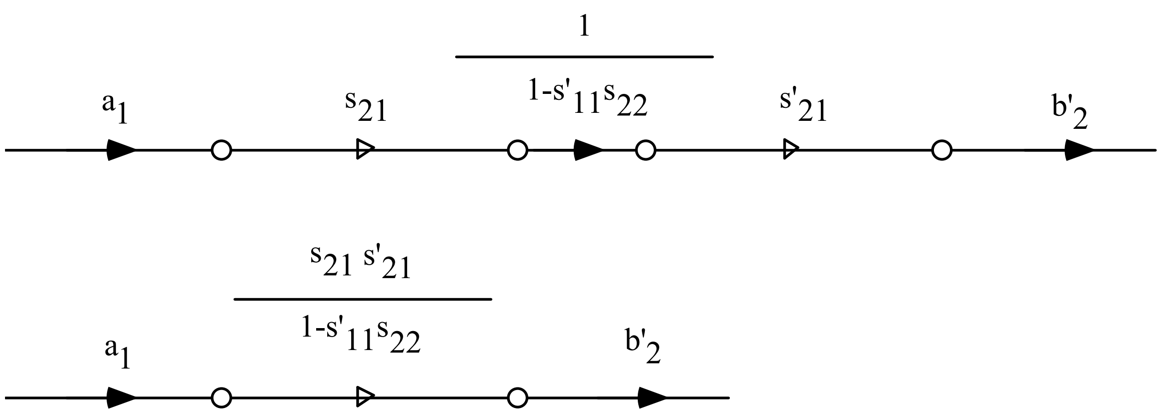

Stage two : we reduce



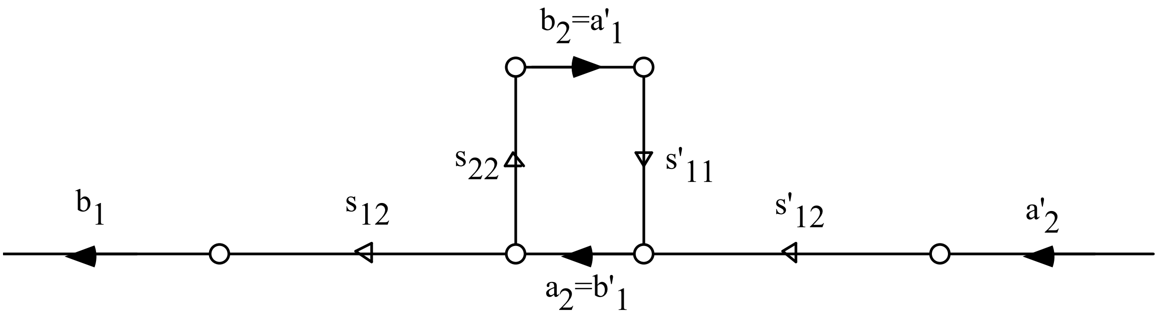

Stage three : we look for the possible paths going from $a_1$ to $b'_2$ :



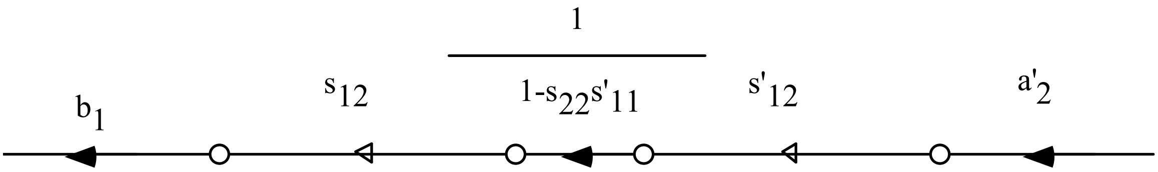

Stage 4 : we reduce

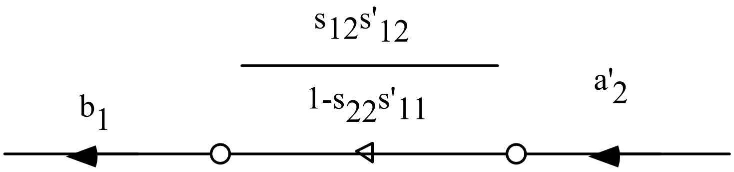


Stage 5 : we look for the possible paths going from $a'_2$ to $b_1$ :

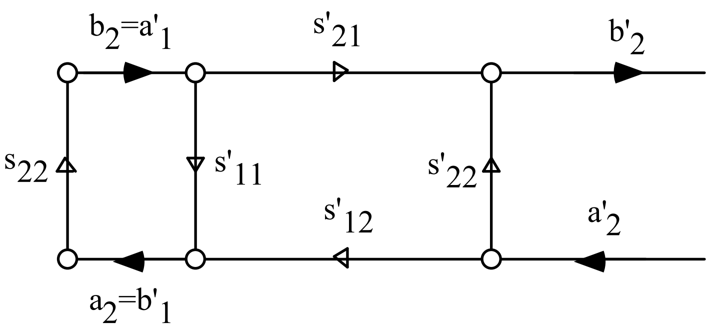



Stage 6: we reduce

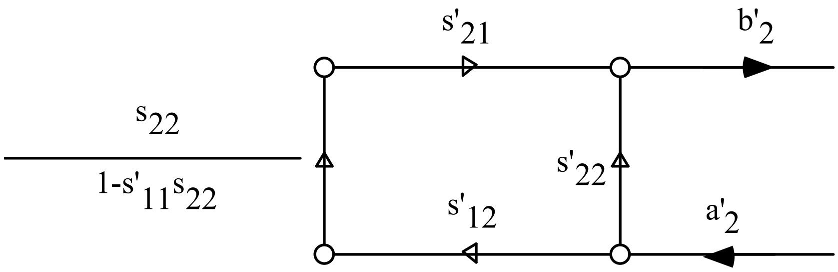


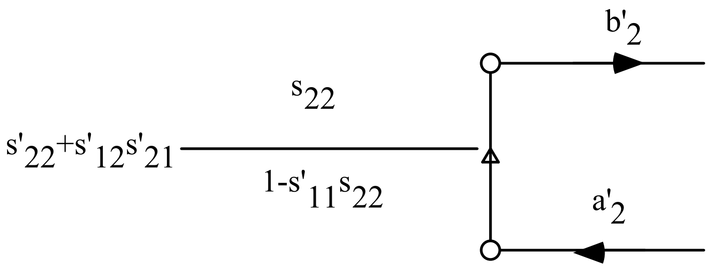




Stage 7 : we look for the possible paths going from $a'_2$ to $b_2'$ :




Stage 8 : we reduce




And we get finally the scattering matrix of two cascaded two-ports :

$$\begin{bmatrix} b_1 \\ b'_2 \end{bmatrix} = \begin{bmatrix} s_{11} + \dfrac{s_{21}s_{12}s'_{11}}{1-s_{11}'s_{22}} & \dfrac{s_{12}s'_{12}}{1-s_{22}s'_{11}} \\ \dfrac{s_{21}s'_{21}}{1-s'_{11}\,s_{22}} & s'_{22} + \dfrac{s'_{12}\,s'_{21}\,s_{22}}{1-s'_{11}\,s_{22}} \end{bmatrix} \begin{bmatrix} a_1 \\ a'_2 \end{bmatrix} \tag{5. 73}$$

*5.4.6 Summary of the general characteristics of the scattering matrix*

- $s_{ij}$ : transfer function between port j and i
- $s_{ii}$ : reflection coefficient at port i
- $\left|s_{ij}\right|^2 = \dfrac{P_i}{P_j}$ normalized transferred power from j to i
- The scattering of a reciprocal network is symmetric
- The scattering matrix of a lossless network is of the type $[S]\left[\tilde{S}\right] = [1]$
- The scattering matrix of a matched network has zeros on its main diagonal.

## 5.5 Voltage standing wave ratio

The voltage standing wave ration, or VSWR, is another useful mean to characterize the refflection coefficient at the ports of a device. It results from the fact that a reflection at a port will induced a reflected wave along the feeding line. This reflected wave will combine itself to the incident wave, in order to form a standing wave (fig. 5.21)

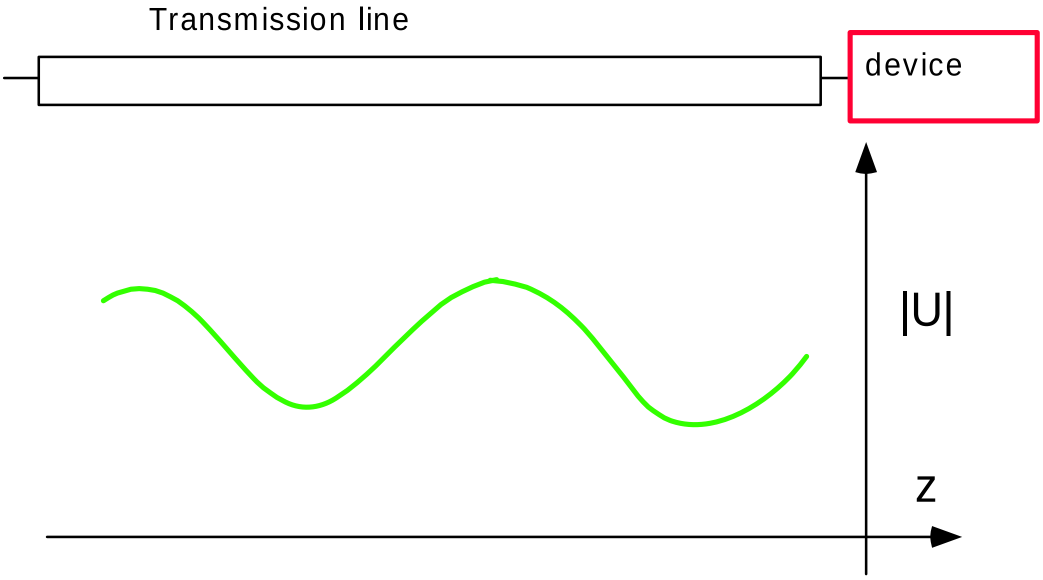



*Fig. 5. 21 : standing wave*

The voltage in the transmission line is given by :

$$U_i = \sqrt{Z_{ci}}\left(a_i + b_i\right) = \sqrt{Z_{ci}}\left(a_i + s_{ii}a_i\right) \tag{5. 74}$$

Thus

$$U_i(z) = \sqrt{Z_{ci}}\,a_i(z)\left[1 + s_{ii}\,e^{2j\beta z}\right] \tag{5. 75}$$

The modulus of the voltage can be written as

$$\begin{aligned}\left|U_i(z)\right| &= \sqrt{Z_{ci}}\left|a_i(z)\right|\sqrt{\left[1+\left|s_{ii}\right|\cos(\varphi+2\beta z)\right]^2 + \left|s_{ii}\right|^2\sin^2(\varphi+2\beta z)} \\ \left|U_i(z)\right| &= \sqrt{Z_{ci}}\left|a_i(z)\right|\sqrt{1+\left|s_{ii}\right|^2 + 2\left|s_{ii}\right|\cos(\varphi+2\beta z)}\end{aligned} \tag{5. 76}$$

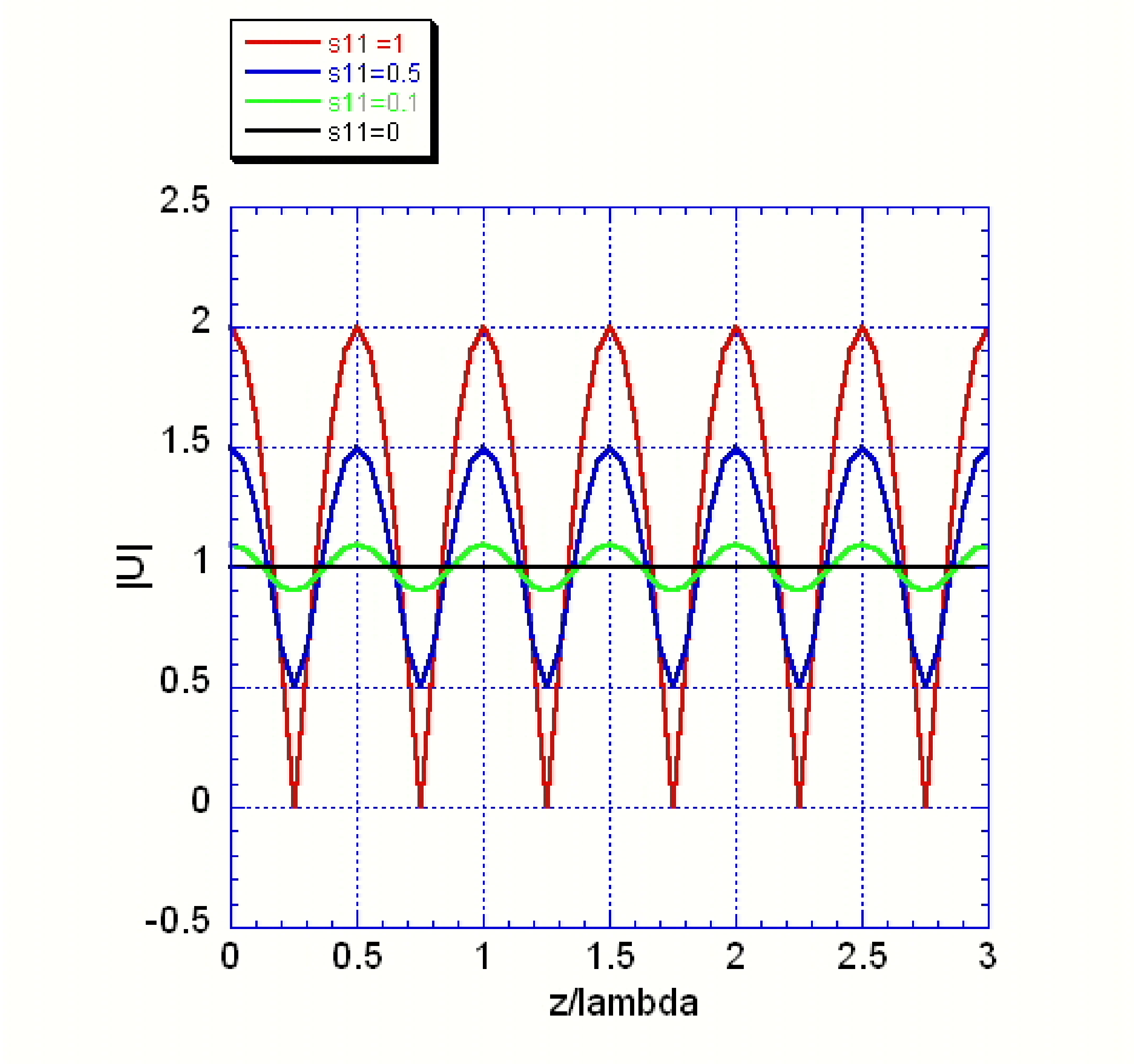
Let us now consiider th eminimum and the maximum values of the modulus of the voltage :

$$\begin{aligned}U_{\max} &= \sqrt{Z_{ci}}\left|a_i\right|\left(1+\left|s_{ii}\right|\right) \quad \text{en } \varphi + 2\beta z_i = 2n\pi \\ U_{\min} &= \sqrt{Z_{ci}}\left|a_i\right|\left(1-\left|s_{ii}\right|\right) \quad \text{en } \varphi + 2\beta z_i = (2n+1)\pi\end{aligned} \tag{5. 77}$$

and take the ratio between these values, which is called the voltage standing wave ration :

$$ROS = VSWR = \frac{|U_{\max}|}{|U_{\min}|} = \frac{1+|s_{ii}|}{1-|s_{ii}|} \qquad (5.\ 78)$$

For a matched load, the reflected vave is equal to zero, and thus the VSWR is equal to 1. For a total reflection the VSWR is infinite. Examples of standing ewaves for different reflection coefficients are depicted in figure 5.22.



# 6. Microwave components

## 6.1 Single port element

The scattering parameter of a single port element is a scalar, the reflection coefficient.

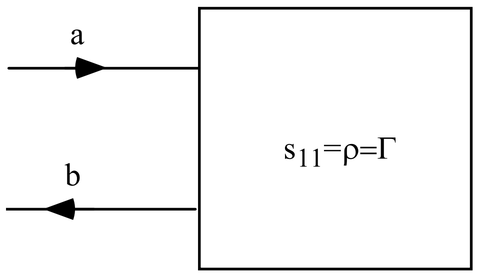



*Fig. 6. 1 : Component with a single access*

Its flow chart is elementary (Fig. 6. 2)

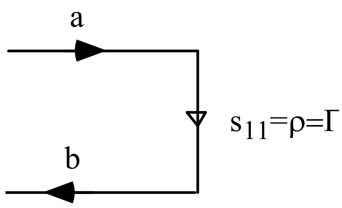



*Fig. 6. 2 : Flow chart of a single port element*

### *6.1.1 Lossless single port element*

To be lossless, a single port element must have

$$|a|^2 = |b|^2 \tag{6. 1}$$

which is equivalent to

$$\frac{|b|}{|a|} = |s| = 1 \tag{6. 2}$$

Thus, a lossless single port element is an element giving a total reflection

$$s = e^{j\varphi} \tag{6. 3}$$

There are two particular cases, the short circuit with

$$b = -a \;\Rightarrow\; s = -1 \tag{6. 4}$$

and the open circuit, where

$$b = a \;\Rightarrow\; s = 1 \tag{6. 5}$$

The short circuit is a very important device in microwave measurements, as it is used to set the reference planes of the devices under test. Mobile short circuits are also of interest for microwave measurements, as this allows presenting a reflection with a controlled amplitude and phase. Fig. 6. 3 illustrates some waveguide mobile short-circuits.

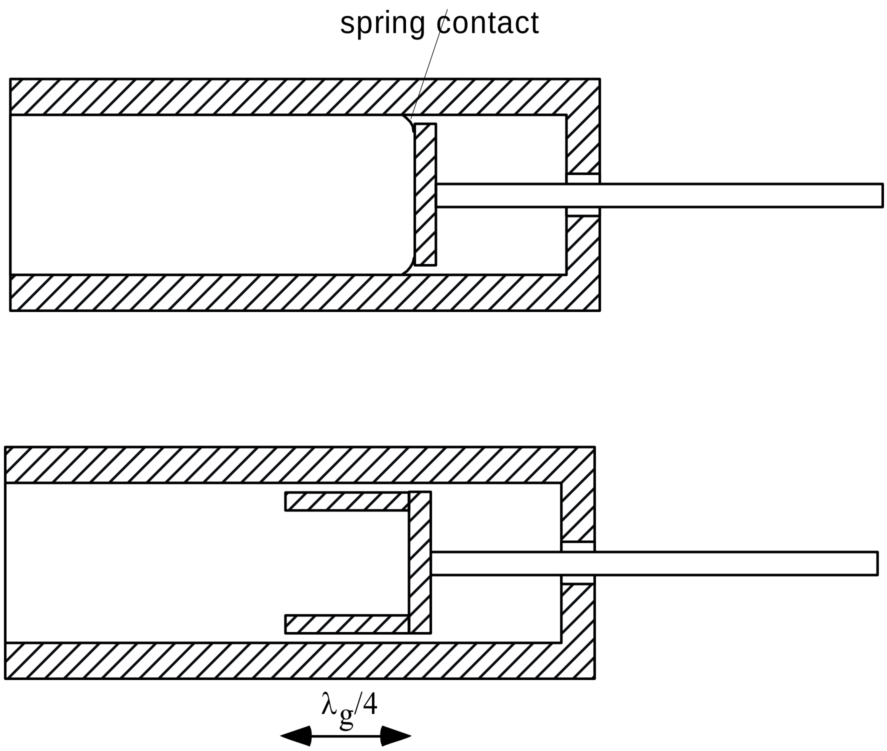



*Fig. 6. 3 : waveguide short circuits*

### *6.1.2 Matched single port element*

A single port element absorbing all the incident power is characterized by a zero reflection coefficient:

$$s_{ii} = 0 \qquad (6.\ 6)$$

Note: it is manifestly not possible to match a lossless single port element.

Matched loads are again very useful for microwave measurements. Indeed, as the s parameters are defined between matched loads, all the accesses which are not concerned by a measurement have to be terminated by a matched load during measurement.

There are different technologies to manufacture matched loads: For relatively low frequencies, a resistor of the right impedance will generally do the job. At microwave frequencies, loads made using absorbing materials are preferred, as they give a better match. Examples for waveguides are shown in Fig. 6. 4

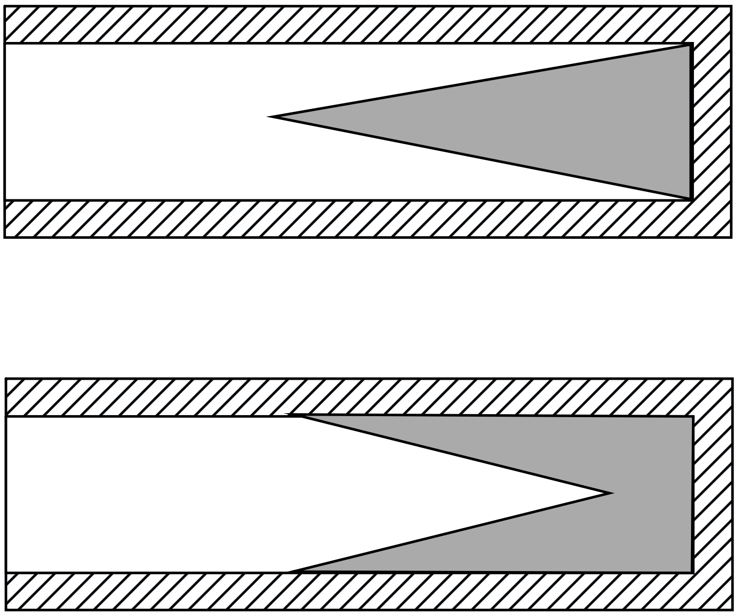


*Fig. 6. 4 : Wavguide loads*

## 6.2 Two-ports

The scattering matrix of a two-port has four terms:

$$[S] = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \qquad (6.\ 7)$$

where the terms on the diagonal are the reflection coefficient at the ports, and the terms outside the diagonal the transfer function between the ports. The flow chart is shown in Fig. 6. 5.

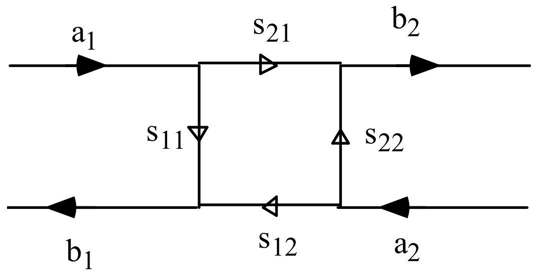


*Fig. 6. 5 : Flow chart of a two-port circuit*

### *6.2.1 Characteristics of two-ports*

- For a reciprocal two-port $s_{21}=s_{12}$
- For a lossless two-port

$$|s_{11}|^2+|s_{21}|^2=1$$
$$|s_{12}|^2+|s_{22}|^2=1 \tag{6. 8}$$
$$s_{11}^* s_{12}+s_{21}^* s_{22}=0$$

- For a lossless reciprocal two-port

$$|s_{11}|=|s_{22}| \tag{6. 9}$$

- In the case where

$$s_{11}=s_{22} \tag{6. 10}$$

  the two-port is said to be symmetric
- For a matched two-port

$$s_{11}=s_{22}=0 \tag{6. 11}$$

### *6.2.2 Matched attenuator*

$$[S]=\begin{bmatrix} 0 & s_{12} \\ s_{12} & 0 \end{bmatrix} \tag{6. 12}$$

It is a reciprocal component that attenuates the power between the input and the output. The attenuation level is defined as:

$$LA=10\log\frac{P_1}{P_2}=10\log\frac{|a_1|^2}{|b_2|^2}=10\log\frac{1}{|s_{12}|^2}=-20\log|s_{12}| \tag{6. 13}$$

A matched attenuator is absorbing power, and is thus a lossy component. Attenuators may be fixed or variable. The first is used only when a fixed amount of attenuation has to be provided, for instance in order to protect a device. In most of measurement setups however, variable attenuators are used, and two examples are shown below.

*Example one: rotary waveguide attenuator.*

This attenuator design is shown in Fig. 6. 6. It consists of three section of circular waveguide, all loaded by a thin resistive sheet in their central plane. The middle section can rotate around the central axe of the cylinder.

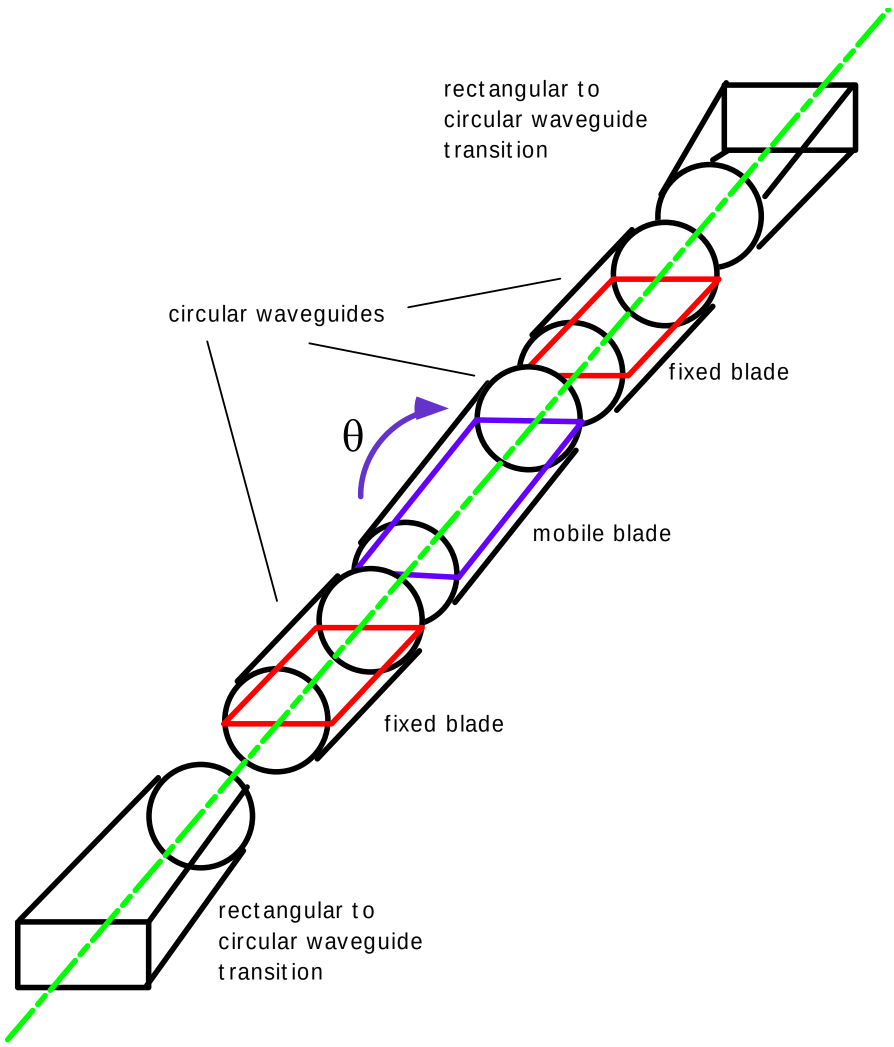



*Fig. 6. 6 : waveguide attenuator*

The principle of attenuation is explained in Fig. 6. 7.

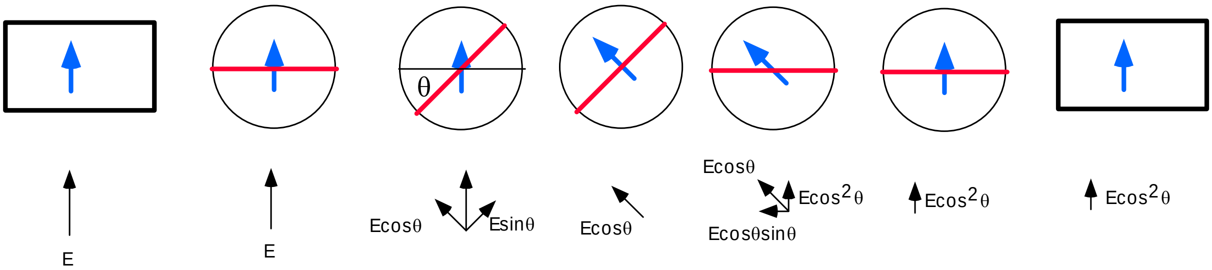



*Fig. 6. 7: rotary waveguide attenuator*

At the input of the device, the electric field is vertical (corresponding to the dominant mode of the rectangular waveguide $TE_{10}$). At the input of the fixed cylindrical waveguide, the field is vertical, thus orthogonal to the absorbing blade. The field will thus not be affected by the latter. At the input of the mobile section, the field is vertical, making an angle θ with the absorbing blade. The field can be decomposed in a component parallel to the blade and in a component orthogonal to the blade. The former will be absorbed travelling through the waveguide section, while the latter will not be affected. The field encounters now the last absorbing blade in the third circular section. It is again decomposed in two componsnt, parallel and perpendicular to the blade. Again, the former will be absorbed, the latter unaffected. Thus, at the output of this third circular waveguide, the field is again vertical, but has been attenuated by a factor of $\cos^2\theta$.

*Example two: T attenuator.*

An attenuator working in the lower frequencies of microwaves can also be made of lumped resistors. In this case, a T circuit is often chosen (Fig. 6. 8)

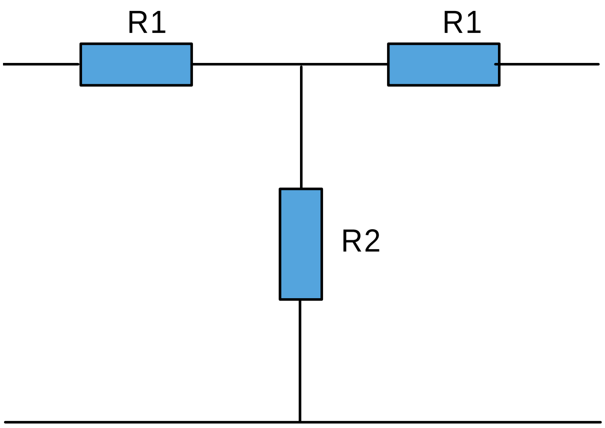



*Fig. 6. 8 : T attenuator.*

It can easily be shown that if the attenuator is matched, we have:

$$R_2 = \frac{Z_c^2 - R_1^2}{2R_1} \qquad (6.\ 14)$$

and that the attenuation is given by

$$-20\log s_{21} = -20\log\left(\frac{Z_c - R_1}{Z_c + R_1}\right) \qquad (6.\ 15)$$

where Zc is the characteristic impedance at the ports of the attenuator

### *6.2.3 Phase shifters*

The scattering matrix of a reciprocal lossless phase shifter is given by :

$$[S] = \begin{bmatrix} 0 & e^{j\varphi} \\ e^{j\varphi} & 0 \end{bmatrix} \qquad (6.\ 16)$$

The easiest way to manufacture a phase shifter is just to insert a length of transmission line. Indeed, the scattering matrix a length of matched transmission line s given by :

$$[S] = \begin{bmatrix} 0 & e^{-j\beta L} \\ e^{-j\beta L} & 0 \end{bmatrix} \qquad (6.\ 17)$$

where β is the propagation coefficient in the line and L the length of the line.

Many measurement setups require for elements with variable phase shifts. An example of realisation is the Fox phase shifter. Like the variable attenuator, it consists of three sections of circular waveguides, terminated at each end by a circular to rectangular waveguide transmission. The central section is mobile, and the three circular waveguides are loaded by a slab of dielectric in their centre, as depicted in Fig. 6. 9.

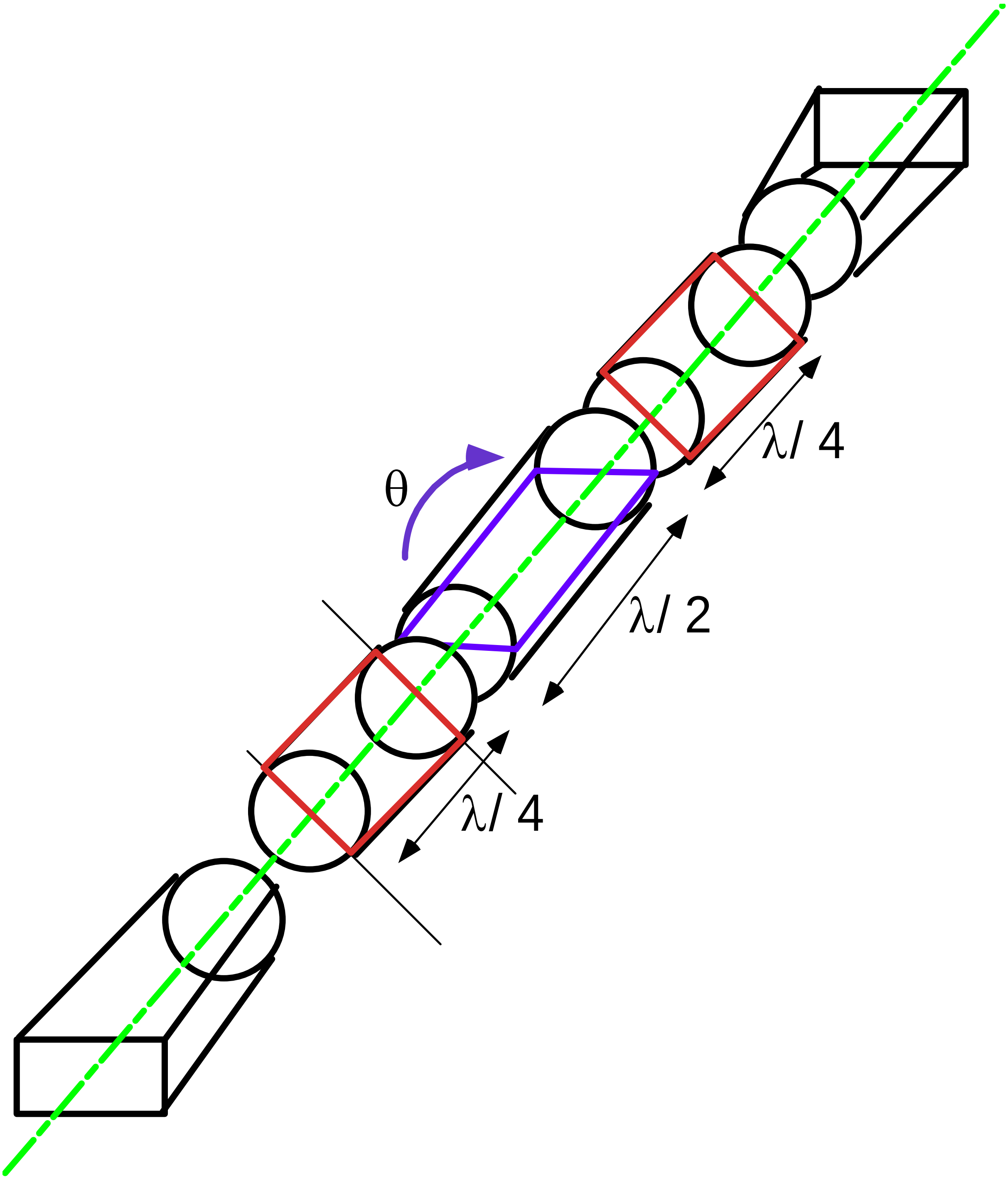



*Fig. 6. 9 : Fox ' phase shifter*

Again, the electric field is vertical at the input of the device. It will thus arrive vertically at the input of the first circular waveguide, which is called a quarter wave line. It effect is explained in Fig. 6. 9.

Quarter wave line

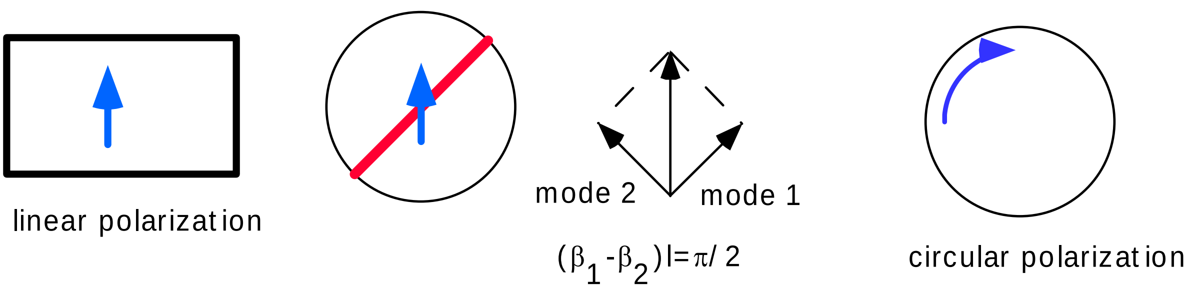



*Fig. 6. 10 : Quarter wave line*

A dielectric slab is located at the centre of the waveguide, making an angle of 45° with the field. The field is thus decomposed in two components, one parallel and the other perpendicular to the slab. The former will not be affected by the dielectric, whereas the latter will be slowed down due to the dielectric material. The length of the guide is selected in such a way that the two components will have a 90° phase shift at its end. The field is thus circularly polarized at the end of this guide.

The signals enters then in the second waveguide, which is a half wave line (see Fig. 6. 11)

Half wave line

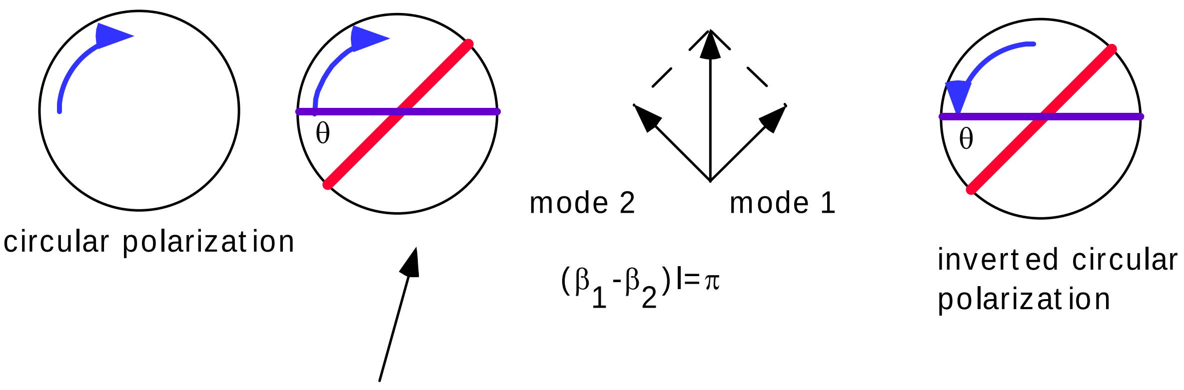



*Fig. 6. 11 : Half wave line*

This time the length of the line is such that the two field components will have a phase difference of 180° at its end. The rotation sense of the circular polarization will thus be inversed. Moreover, as the slab makes an angle θ with respect to the slab in the quarter wave line, a phase shift of θ will be added to both components of the field. The signal goes then trough a fixed quarter wave line, where the dielectric slab makes an angle of –θ with respect to the slab of the half wave line. Due to this quarter wave line, a phase shift of 90° will be added between the two components of the field, thus leading to two fields having the same phase (90°+180°+90°=360°), and the two components are recomposed to a vertical linearly polarized field. Moreover, another phase shift of θ due to the rotation is added to the field (fig. 6.12)

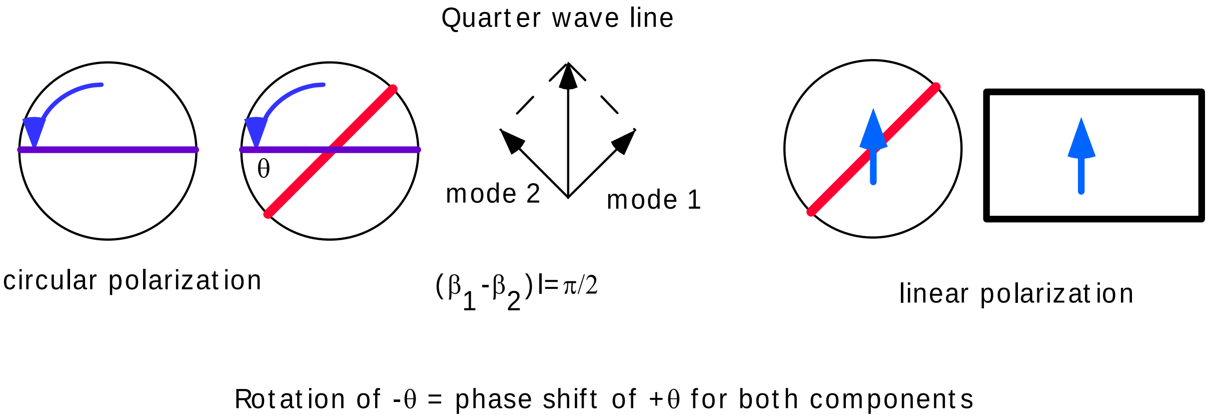



*Fig. 6. 12 : Quarter wave line*

Thus finally, the signal 's phase is shifted by a factor of 2θ (plus the phase shift due to the length of the transmission lines, which is fixed).

### *6.2.4 Non reciprocal two ports*

An isolator is a non reciprocal element, which lets the signal flow in one direction but blocks it in the other direction. It is a very useful element to protect components from parasitic reflections (sources for instance, which are very sensitive). The scattering matrix of an ideal isolator is given by :

$$[S]=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix} \tag{6. 18}$$

A gyrator is a particular case of a non reciprocal phase shifter, where difference of phase shift in the transfer function of the two directions in 180°. If the reference planes are chosen properly, the scattering matrix of a gyrator is

$$[s]=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} \tag{6. 19}$$

### *6.2.5 Frequency depending two-ports (filters, etc.)*

The scattering matrix of a frequency depending two port is given by

$$[S]=\begin{bmatrix}s_{11}(\omega) & s_{12}(\omega)\\ s_{12}(\omega) & s_{22}(\omega)\end{bmatrix} \tag{6. 20}$$

The specific case of the filter will be treated in chapter 7.

## 6.3 Three-ports

The general scattering matrix of a three-port is given by

$$[S]=\begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} \qquad (6.\ 21)$$

and its flow chart by

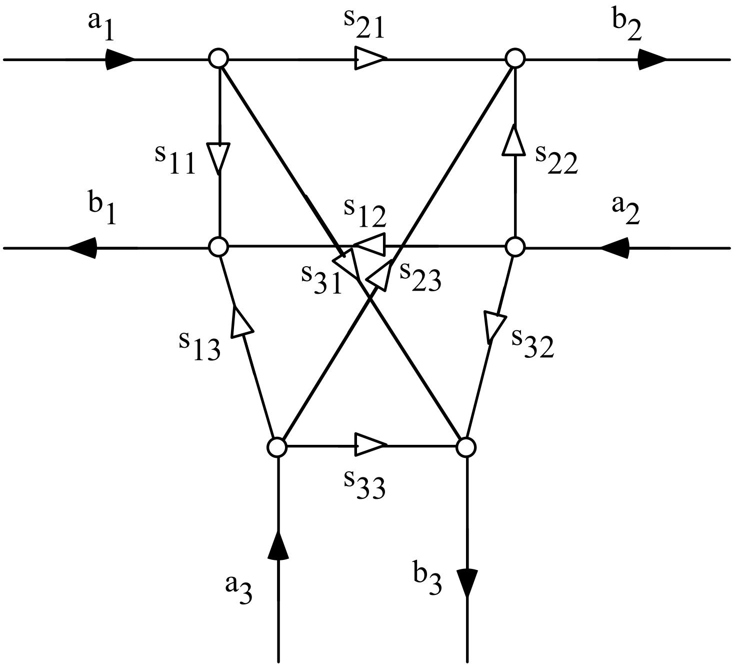



*Fig. 6. 13 : Flow chart of a three-port*

### *6.3.1 Characteristics of a three-port*

- For a reciprocal three-port, $s_{21}=s_{12}$, $s_{23}=s_{32}$, $s_{13}=s_{31}$
- For a lossless three-port

$$\sum_{i=1}^{3} s_{ij}^{*} s_{ik} = \delta_{ik} \quad j,k = 1,2,3 \qquad (6.\ 22)$$

- For a matched three-port

$$s_{11} = s_{22} = s_{33} = 0 \qquad (6.\ 23$$

- A three port cannot be at the same time lossless, reciprocal and matched, as is easily shown as follows: let us imagine that it would be possible. The losslessness relations would then be written as:

$$\begin{aligned}
&\left|s_{12}\right|^{2} + \left|s_{13}\right|^{2} = 1 \\
&\left|s_{12}\right|^{2} + \left|s_{23}\right|^{2} = 1 \\
&\left|s_{23}\right|^{2} + \left|s_{13}\right|^{2} = 1 \\
&s_{13}^{*} s_{23} = 0 \\
&s_{12}^{*} s_{23} = 0 \\
&s_{12}^{*} s_{13} = 0
\end{aligned} \qquad (6.\ 24)$$

Suppose that $s_{13}$ is non zero. We deduce immediately that $s_{23}=0$ and $s_{12}=0$, which is incompatible with the second relation above. The same contradiction is obtained when we start with s23 or $s_{12}$ different from zero.

As it is not possible to have a three port device which is matched, reciprocal and lossless, we want to check if we can at least have a three port device which is lossless, reciprocal and matched at two of its ports, that could for instance work as a power combiner. It can however easily be shown that the only lossless reciprocal three-port matched at two of its ports is not very interesting, as the non matched port is entirely decoupled from the two other ports. Its scattering parameters are given by:

$$\left|s_{ij}\right| = \left|s_{ji}\right| = \left|s_{kk}\right| = 1 \ \ with \quad i,j,k = 1,2,3 \qquad (6.\ 25)$$

and its flow chart in Fig. 6. 14.

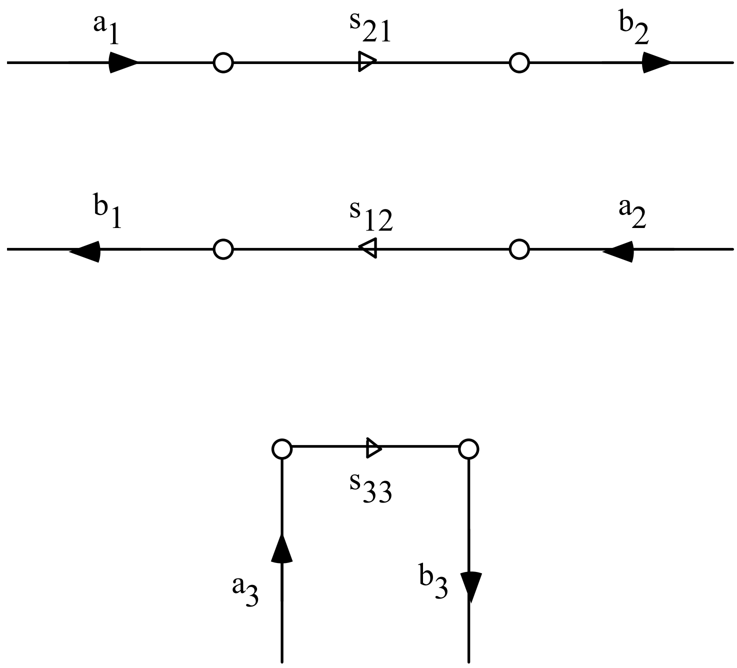



*Fig. 6. 14 : Lossless reciprocal three-port matched at two ports*

We may however have a three port which is lossless, reciprocal and "nearly" matched at two of its ports. Its scattering matrix is given by

$$[s] = \begin{bmatrix} \varepsilon & s_{12} & s_{13} \\ s_{12} & \varepsilon & s_{23} \\ s_{13} & s_{23} & s_{33} \end{bmatrix} \qquad (6.\ 26)$$

where ε << 1, and reference planes 1 and 2 have been selected in way that ε is real. Energy conservation rules gives us that

$$\varepsilon^2 + |s_{12}|^2 + |s_{13}|^2 = 1 \qquad (6.\ 27)$$

$$\varepsilon^2 + |s_{12}|^2 + |s_{23}|^2 = 1 \qquad (6.\ 28)$$

$$\varepsilon\left(s_{12} + s_{12}^*\right) + s_{13}s_{23}^* = 0 \qquad (6.\ 29)$$

$$\varepsilon s_{13} + s_{12}^* s_{23} + s_{13}^* s_{33} = 0 \qquad (6.\ 30)$$

from the two first, we find easily that

$$|s_{12}| = |s_{23}| \qquad (6.\ 31)$$

Using this result and $\varepsilon\left(s_{12} + s_{12}^*\right) + s_{13}s_{23}^* = 0$ (6. 29), we write that

$$\left|s_{13}\right| = \sqrt{2\varepsilon \operatorname{Re}\left(s_{12}\right)} \quad \sim \sqrt{\varepsilon} \tag{6. 32}$$

Using $\varepsilon^2 + \left|s_{12}\right|^2 + \left|s_{13}\right|^2 = 1$ (6. 27), we can then write

$$\left|s_{12}\right| = \sqrt{1 - 2\varepsilon \operatorname{Re}\left(s_{12}\right) - \varepsilon^2} \cong 1 - \varepsilon \operatorname{Re}\left(s_{12}\right) \approx 1 \tag{6. 33}$$

Moreover, $\varepsilon^2 + \left|s_{12}\right|^2 + \left|s_{13}\right|^2 = 1$ (6. 27) set the amplitude of $s_{33}$ :

$$\left|s_{33}\right| \cong \left|s_{12}\right| \tag{6. 34}$$

In conclusion, it is possible to have a three port which is lossless, reciprocal and nearly matched at two of its port. The third port will however be heavily mismatched. An example of this type of device is the slotted line, used in waveguide measurements.

### *6.3.2 The circulator*

A non reciprocal three-port can be lossless and matched. In this case, the losslessness relations are written as :

$$\begin{aligned}
&\left|s_{21}\right|^2 + \left|s_{31}\right|^2 = 1 \\
&\left|s_{12}\right|^2 + \left|s_{32}\right|^2 = 1 \\
&\left|s_{23}\right|^2 + \left|s_{13}\right|^2 = 1 \\
&s_{12}^* s_{13} = 0 \\
&s_{21}^* s_{23} = 0 \\
&s_{31}^* s_{32} = 0
\end{aligned} \tag{6. 35}$$

We suppose again that $s_{13}$ is different from zero. We get

$$s_{13} \neq 0 \Rightarrow s_{12} = 0 \Rightarrow \left|s_{32}\right| = 1 \Rightarrow s_{31} = 0 \Rightarrow \left|s_{21}\right| = 1 \Rightarrow s_{23} = 0 \Rightarrow \left|s_{13}\right| = 1 \tag{6. 36}$$

We may choose the reference planes in a way that the non zero terms are real, and we get the following scattering matrix :

$$[S]=\begin{bmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix} \tag{6. 37}$$

This element is a circulator, and its flow chart is depicted in Fig. 6. 15.

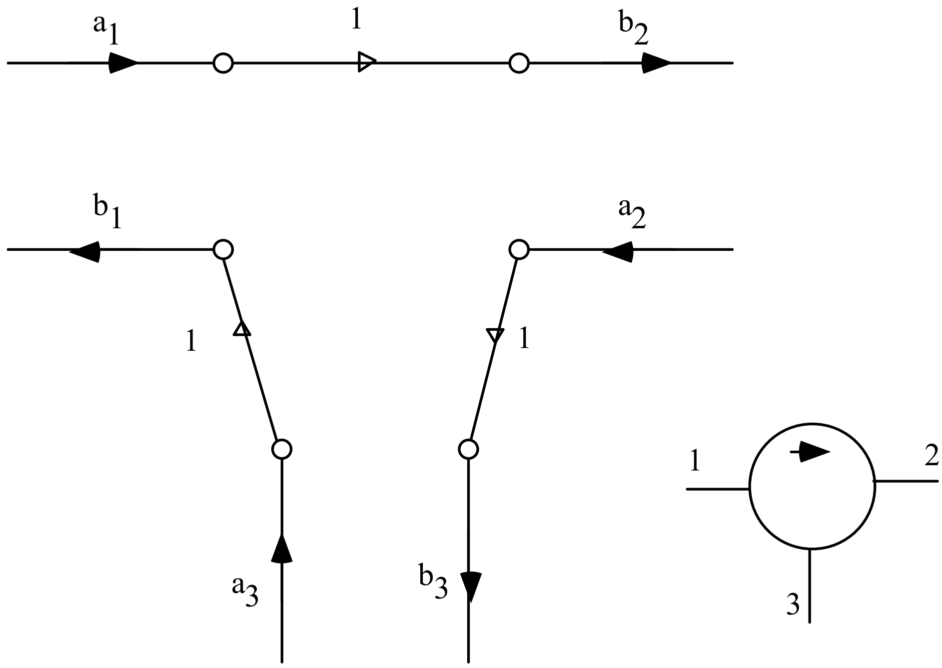



Fig. 6. 15 : *Flow chart of an ideal circulator*

### *6.3.3 Power splitters and combiners*

An useful application of a three port device is the ability of splitting the power from one input into two outputs, or to combine the power from two inputs into one output. Unfortunately, as we have seen above, we will not be able to design a device which is matched and lossless and able to act at the same time as splitter and divider.

Let us consider first the problem of splitting the signal (or power) into two branches. In this situation, as we do have only one input, we can realise a device which is reciprocal, lossless and matched at one of its ports, the input. This can be done using three sections of transmission lines, as depicted in Fig. 6. 16.

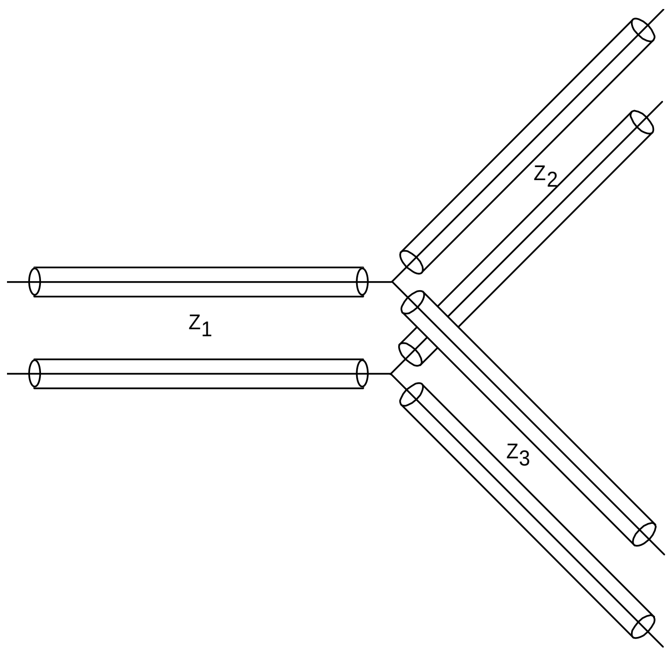


*Fig. 6. 16 : power splitter made by transmission lines*

The condition that the input (port 1) is matched is given by:

$$Z_1 = \frac{Z_2 Z_3}{Z_2 + Z_3} \qquad (6.\ 38)$$

provided that output 2 and 3 are terminated by matched loads.

In cases where all three ports can serve alternatively as inputs and output, we have seen can we cannot have losslessness, reciprocity and all ports matched. Thus, we have the choice of either adding losses, or allow a mismatch at the ports.
An example of the latter is shown below. It consists of three identical sections of transmission lines, connected in a way to have a perfect central symmetry of the structure (Y junction).

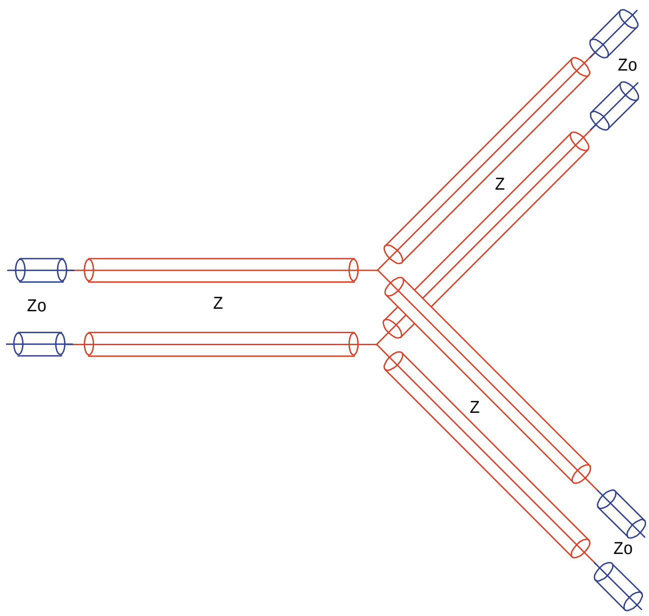



*Fig. 6. 17: non matched symmetrical power splitter (Y junction)*

In this case, the reflection coefficient at all the ports is identical, and we can place the reference planes in a way that this coefficient is purely real. All the transmission coefficients between the ports will also be identical, due to the symmetry of the structure and the reference planes. Thus, we can write :

$$s_{11} = s_{22} = s_{33} = A \qquad (6.\ 39)$$
$$s_{12} = s_{13} = s_{23} = B + jC$$

As the device is lossless, the energy conservation relations can be written as :

$$A^2 + 2B^2 + 2C^2 = 1 \qquad (6.\ 40)$$
$$2AB + B^2 + C^2 = 0$$

These two equations define a closed curve in the A,B,C space (see Fig. 6. 18). The maximal reflection coefficient is given for uncoupled ports (A=1, B=0, C=0) and the smallest possible reflection is obtained for (C=0, B=-2/3, A=1/3). In this latter case, the VSWR is equal to

$$VSWR = \frac{1+\frac{1}{3}}{1-\frac{1}{3}} = 2 \qquad (6.\ 41)$$

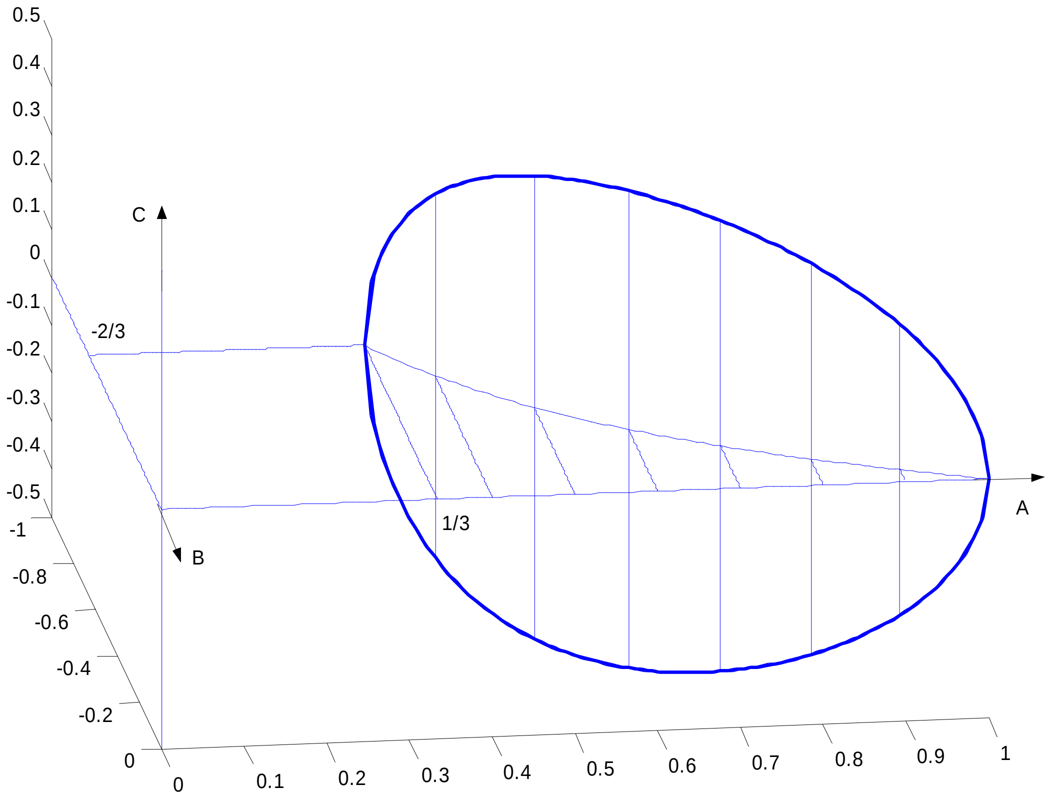



*Fig. 6. 18: Solutions for the s parameters of a symmetrical three port junction (Y junction)*

As mentioned above, another solution to the problem of a three port serving as power splitter and power combiner is to allow for losses. A very simple crude solution is the resistive matched power splitter, shown in Fig. 6. 19.

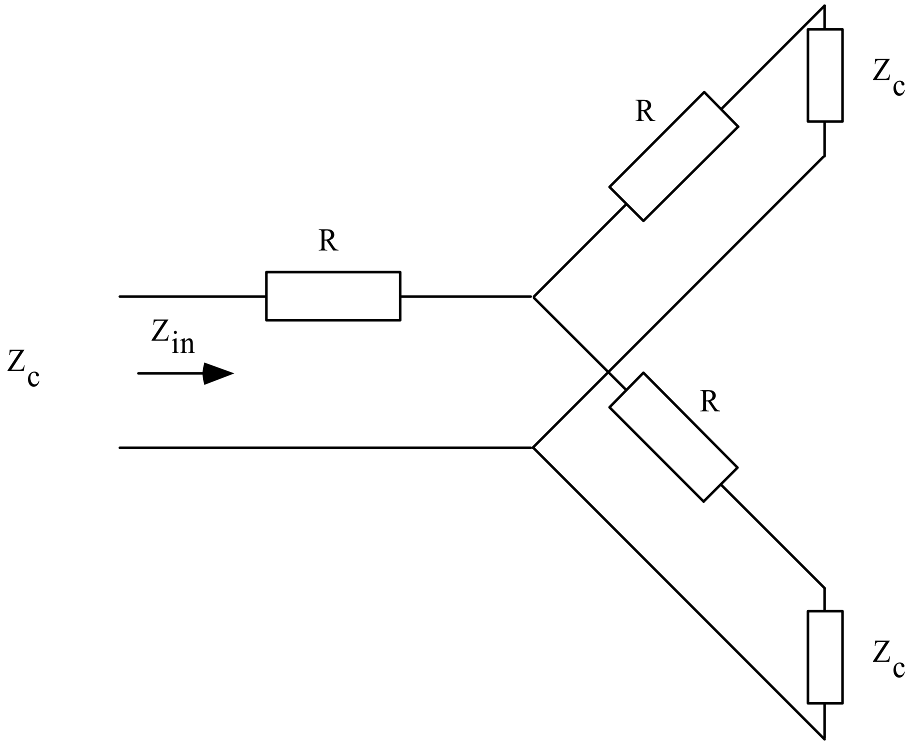


*Fig. 6. 19: resistive matched power splitter*

It is easy to show if we want this circuit to be matched at its three port

$$R = \frac{Z_C}{3} \tag{6. 42}$$

The scattering matrix of this device is given by

$$[S] = \frac{1}{2}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \tag{6. 43}$$

Thus half the power is lost in the resistors!!

A more clever device, which can work as power splitter and power combiner is the Wilkinson divider. An example of Wilkinson divider realised in microstrip technology is illustrated in Fig. 6. 20.

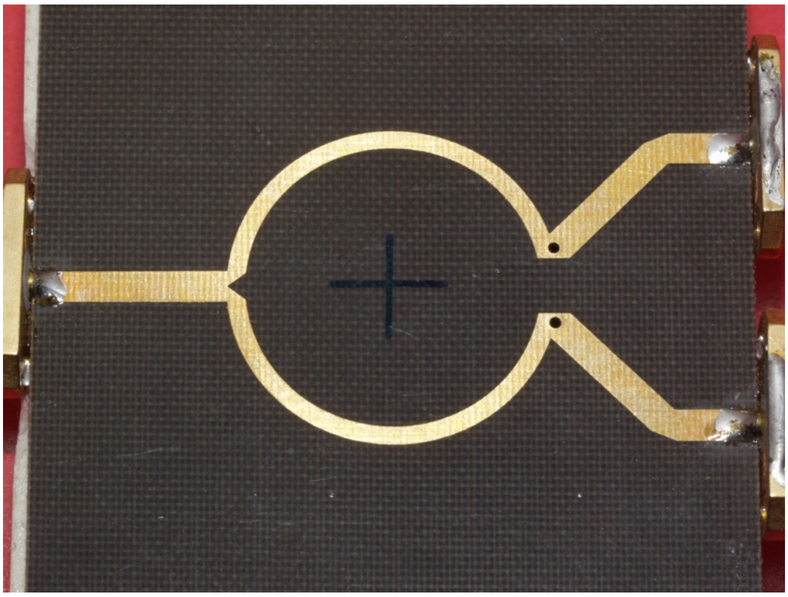


*Fig. 6. 20 : Microstrip Wilkinson divider*

To understand the way this devices works, let's start from its layout shown in Fig. 6. 21.

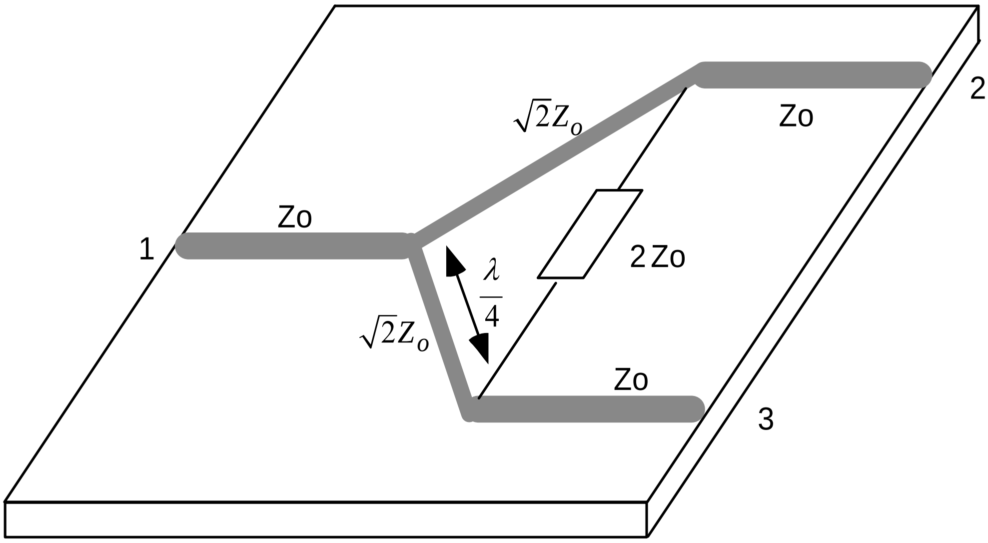



*Fig. 6. 21 : Wilkinson divider layout*

We see that it consists of three transmission lines at the three port, where port 1 and 2 and ports 1 and 3 are linked by a quarter wave line of impedance $\sqrt{2}Z_O$ . Moreover, a resistor of value 2Zo is located between ports 2 and 3. A transmission line equivalence to this circuit is depicted in Fig. 6. 22.

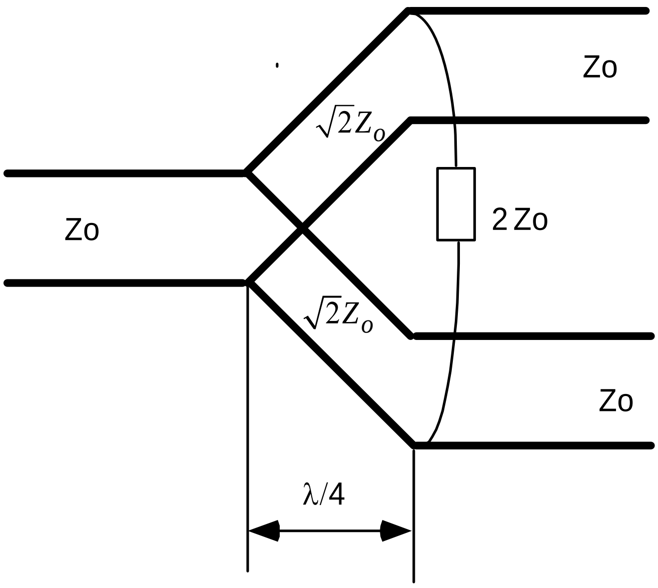



*Fig. 6. 22: Wilkinson divider*

We notice that this equivalent circuit has an horizontal axis of symmetry. We normalise all the impedance to Zo, we enhance the symmetry by splitting the resistor and the impedance at port 1, and we add two voltage sources at ports two and three Fig. 6. 23)., where Z and r are unknowns to be determined in order to achieve the match at all three ports.

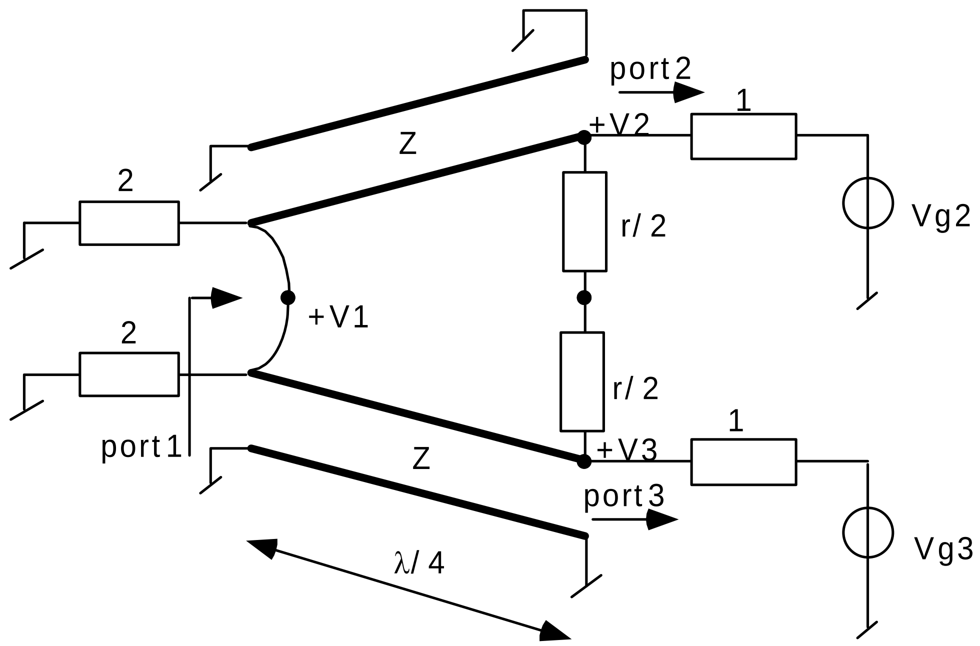



*Fig. 6. 23 : Symmetric equivalent circuit of Wilkinson divider*

We will now excite ports 2 and 3 first using an even mode (Vg2=Vg3=2 V) then using an odd mode (Vg2=—Vg3=2 V). The superposition of these two modes yields Vg2=4 V et Vg3=0.

**Even mode :**

In this case, Vg2=Vg3 = 2 V. As a consequence, V2=V3 and no current flows through the two resistors r/2, or through the short circuit between the transmission lines at port 1. The circuit of fig 6.23 can thus be divided by introducing two open circuits, as depicted in Fig. 6. 24 (where the grounded parts of the lines are not shown).

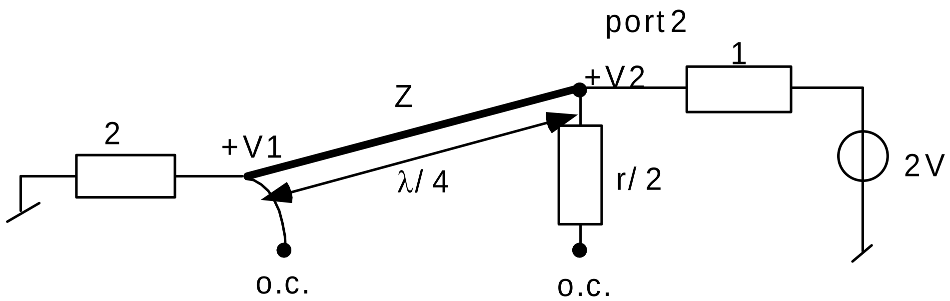



*Fig. 6. 24 : even excitation*

Looking into the circuit from port 2, we see an impedance equal to :

$$Z_{in}^{e} = \frac{Z^2}{2} \tag{6. 44}$$

as the transmission line acts as a quarter wave transformer. Thus, port 2 will be matched for $Z = \sqrt{2}$ and all the power will be delivered to port one, as no current flows through the resistor. In order to determine $s_{21}$, we need to compute V1, that we can find using transmission lines' equations. If x=0 at port 2 and x=λ/4 at port 1, the voltage on the line can be written as :

$$\begin{aligned} V(x) &= V^+\left(e^{-j\beta x} + \Gamma e^{j\beta x}\right) \\ V(0) &= V^+(1+\Gamma) = V_2 = V \\ V_1 &= V(\lambda/4) = jV^+(1-\Gamma) = jV\frac{\Gamma-1}{\Gamma+1} \end{aligned} \tag{6.45}$$

The reflection coefficient Γ gives the reflection seen at port 1, looking towards the normalised resistor of 2, thus :

$$\Gamma = \frac{2-\sqrt{2}}{2+\sqrt{2}} \tag{6. 46}$$

and

$$V_1 = jV\frac{-1}{\sqrt{2}} \tag{6. 47}$$

Thus

$$S_{12} = S_{21} = \frac{V_1}{V_2} = \frac{-j}{\sqrt{2}} = -j0.707 \tag{6. 48}$$

and by symmetry

$$S_{13} = S_{31} = \frac{V_1}{V_2} = \frac{-j}{\sqrt{2}} = -j0.707 \tag{6. 49}$$

## Odd mode

For the odd excitation mode, Vg2=–Vg3=2 V, and V2=–V3. Thus, the voltage is equal to zero along the symmetry line of the circuit, et we obtain the following equivalent circuit :

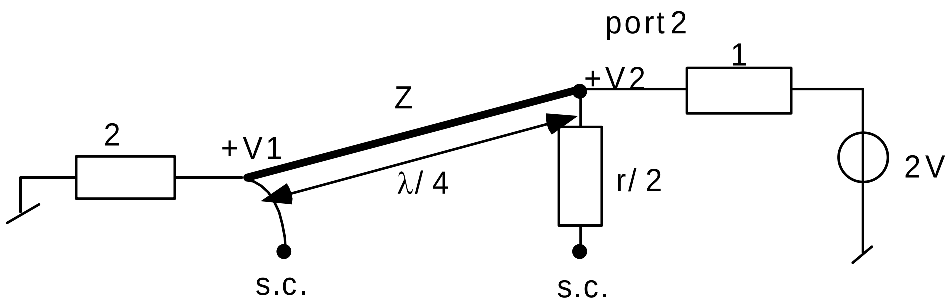



*Fig. 6. 25: odd excitation*

Looking into port 2, we see an impedance equal to r/2. Indeed, the short circuit at port 1 is viewed as an open circuit through the quarter wave line. Thus, port 2 will be matched if r=2. In this excitation mode, all the power is delivered to the resistor, and is thus lost.

In summary, all the following S parameters can be obtained from the above results using symmetry and reciprocity considerations:

$$s_{33} = s_{22} = 0 \text{ (ports 2 and 3 are matched for even and odd excitations )}$$
$$s_{12} = s_{21} = -j0.707 \text{ (the component is recprocal)}$$
$$s_{13} = s_{31} = -j0.707 \text{ (the component is recprocal)}$$
$$s_{23} = s_{32} = 0 \text{ (due to the presence of open and short circuits on the symmetry line)}$$

We must still compute the reflection coefficient at port 1, $s_{11}$. In order to do this, we determine the impedance seen at port 1, when ports 2 and 3 are terminated by matched loads. The equivalent circuit is show in figure 6.26, the latter being identical to the even case, as V2=V3.

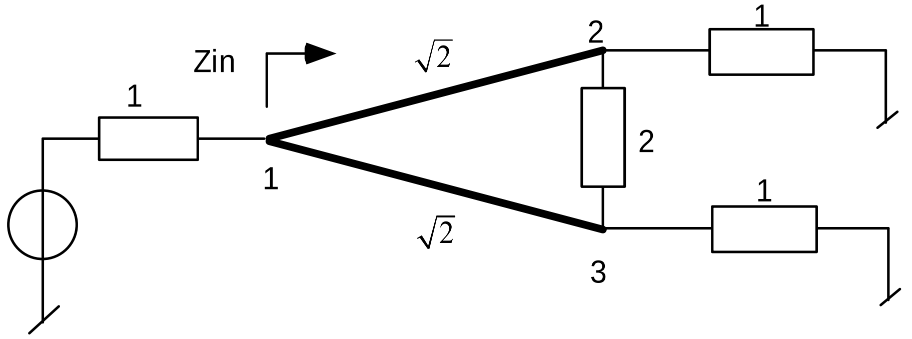



*Fig. 6. 26: Match at port 1*

Thus, there will be no current flowing from port 2 to 3, and no current through the resistor. We have the following equivalent circuit:

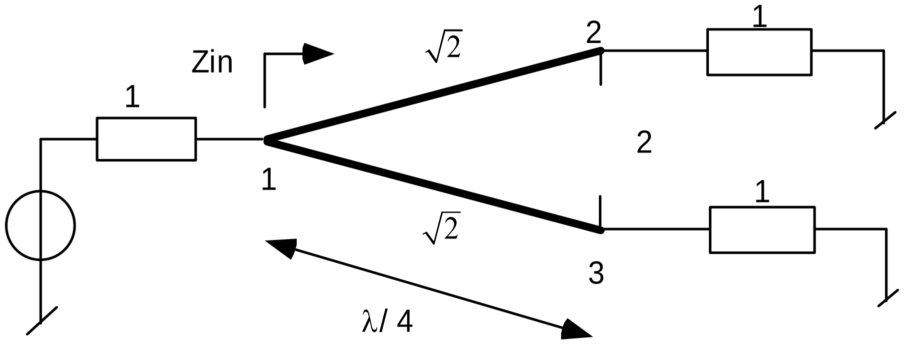



*Fig. 6. 27 : computation of Zin*

We have thus two quarter wave transformers in parallel, terminated by a matched load (normalized impedance of 1). The normalized input impedance at port 1 of the Wilkinson divider is thus given by

$$Z_{in} = \frac{1}{2}\frac{\left(\sqrt{2}\right)^2}{1} = 1$$

Port 1 is thus also matched ($s_{11}$ =0).

## 6.4 The four-port

The scattering matrix of a four-port is given by :

$$[S] = \begin{bmatrix} s_{11} & s_{12} & s_{12} & s_{14} \\ s_{21} & s_{22} & s_{23} & s_{24} \\ s_{31} & s_{32} & s_{33} & s_{34} \\ s_{41} & s_{42} & s_{43} & s_{44} \end{bmatrix} \tag{6. 50}$$

and its flow chart by :

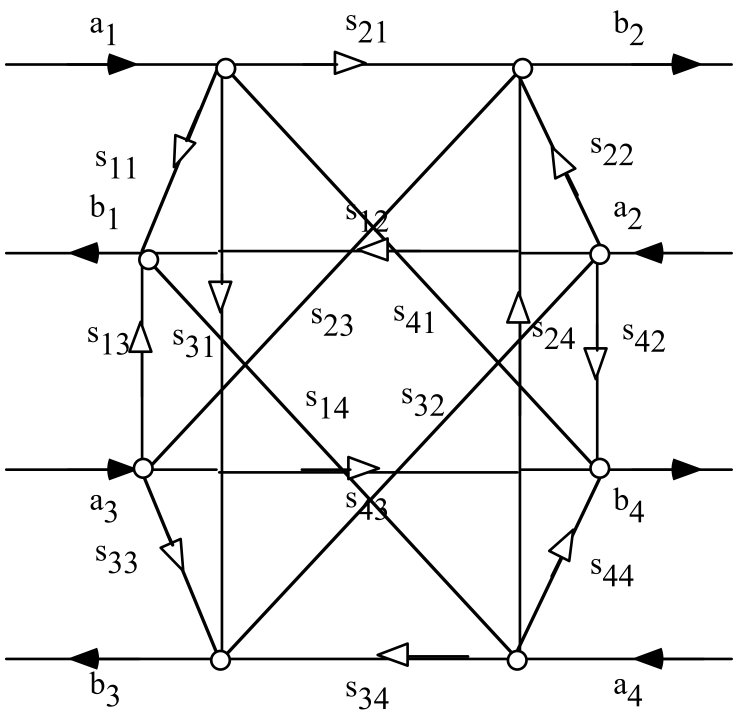



*Fig. 6. 28 : Flow chart of a four-port*

### *6.4.1 The directional coupler*

We can show using the power conservation relations that the only lossless, reciprocal and matched four-port has the following scattering matrix :

$$[S]=\begin{bmatrix} 0 & s_{12} & 0 & s_{14} \\ s_{12} & 0 & s_{23} & 0 \\ 0 & s_{23} & 0 & s_{34} \\ s_{14} & 0 & s_{34} & 0 \end{bmatrix} \tag{6. 51}$$

where we have moreover

$$\begin{aligned} &|s_{12}|^2+|s_{14}|^2=1 \\ &|s_{12}|^2+|s_{23}|^2=1 \\ &|s_{23}|^2+|s_{34}|^2=1 \\ &|s_{14}|^2+|s_{34}|^2=1 \\ &s_{12}s_{23}^*+s_{14}s_{34}^*=0 \\ &s_{12}s_{23}^*+s_{14}s_{34}^*=0 \end{aligned} \tag{6. 52}$$

This element, which links an input to two outputs, the last being isolated, is called a directional coupler. If we choose the reference planes in a judicious way, we get the following scattering matrix:

$$[S]=\begin{bmatrix} 0 & \alpha & 0 & \beta e^{j\psi} \\ \alpha & 0 & \beta e^{j\theta} & 0 \\ 0 & \beta e^{j\theta} & 0 & \alpha \\ \beta e^{j\psi} & 0 & \alpha & 0 \end{bmatrix} \tag{6. 53}$$

with

$$\begin{aligned} &\alpha^2+\beta^2=1 \\ &\psi+\theta=\pi+2n\pi \end{aligned} \tag{6. 54}$$

**Proof:**

The losslessness equations give

$$\begin{bmatrix} 0 & s_{12} & s_{13} & s_{14} \\ s_{12} & 0 & s_{23} & s_{24} \\ s_{13} & s_{23} & 0 & s_{34} \\ s_{14} & s_{23} & s_{34} & 0 \end{bmatrix}^* \begin{bmatrix} 0 & s_{12} & s_{13} & s_{14} \\ s_{12} & 0 & s_{23} & s_{24} \\ s_{13} & s_{23} & 0 & s_{34} \\ s_{14} & s_{23} & s_{34} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6. 55}$$

The third line * the first column and the first line * the second column yield :

$$s_{13}^{*}s_{14} + s_{23}^{*}s_{24} = 0$$
$$s_{13}^{*}s_{23} + s_{14}^{*}s_{24} = 0 \qquad (6.56)$$

We multiply the first expression by $s_{14}^{*}$ and the second by $s_{23}^{*}$ and we subtract the second from the first to obtain:

$$s_{13}^{*}\left(|s_{14}|^2 - |s_{23}|^2\right) = 0 \qquad (6.57)$$

This equation has two solutions:

a) $s_{13} = 0$

From this and the energy conservation relation, we get easily that

$$s_{14} \neq 0, s_{23} \neq 0, \ s_{24} = 0 \qquad (6.58)$$

and we obtain the following scattering matrix

$$[S] = \begin{bmatrix} 0 & s_{12} & 0 & s_{14} \\ s_{12} & 0 & s_{23} & 0 \\ 0 & s_{23} & 0 & s_{34} \\ s_{14} & 0 & s_{34} & 0 \end{bmatrix} \qquad (6.59)$$

b) $|s_{14}| = |s_{23}|$

We choose the reference plane such as :

$$s_{14} = s_{23} = j\delta \qquad (6.60)$$

We use two other energy conservation relations: the first line 1 * the first column and the third line * the third column to get :

$$\begin{aligned} |s_{12}|^2 + |s_{13}|^2 + |s_{14}|^2 &= 1 \\ |s_{13}|^2 + |s_{23}|^2 + |s_{34}|^2 &= 1 \end{aligned} \qquad (6.\ 61)$$

Thus

$$|s_{12}| = |s_{34}| \qquad (6.\ 62)$$

We define the two last reference planes to obtain:

$$s_{12} = s_{34} = \gamma \qquad (6.\ 63)$$

We take two new energy conservation relations: line 1 * column 4 and line 3 * column 4

$$\begin{aligned} s_{12}^{*} s_{24} + s_{13}^{*} s_{34} &= 0 = \gamma\left(s_{24} + s_{31}^{*}\right) \\ s_{13}^{*} s_{14} + s23 s_{24} &= 0 = \delta\left(s_{31}^{*} - s_{24}\right) \end{aligned} \qquad (6.\ 64)$$

This system of equations admits two solutions:

a) $s_{13} = s_{24} = 0$

we get in this case the same scattering matrix as before :

$$[S] = \begin{bmatrix} 0 & s_{12} & 0 & s_{14} \\ s_{12} & 0 & s_{23} & 0 \\ 0 & s_{23} & 0 & s_{34} \\ s_{14} & 0 & s_{34} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \gamma & 0 & j\delta \\ \gamma & 0 & j\delta & 0 \\ 0 & j\delta & 0 & \gamma \\ j\delta & 0 & \gamma & 0 \end{bmatrix} \qquad (6.\ 65)$$

b) $\gamma = \delta = 0$

This solution does not correspond to a four-port anymore, but to two decoupled two-ports :

$$[S]=\begin{bmatrix} 0 & 0 & s_{13} & 0 \\ 0 & 0 & 0 & s_{24} \\ s_{13} & 0 & 0 & 0 \\ 0 & s_{24} & 0 & 0 \end{bmatrix} \tag{6. 66}$$

Thus, the only possible solution for a lossless, reciprocal and matched four-port has the following scattering matrix:

$$[S]=\begin{bmatrix} 0 & s_{12} & 0 & s_{14} \\ s_{12} & 0 & s_{23} & 0 \\ 0 & s_{23} & 0 & s_{34} \\ s_{14} & 0 & s_{34} & 0 \end{bmatrix} \tag{6. 67}$$

We apply now again the energy conservation equations to get the characteristics of this four-port :

$$\begin{aligned} &|s_{12}|^2+|s_{14}|^2=1 \\ &|s_{12}|^2+|s_{23}|^2=1 \\ &|s_{23}|^2+|s_{34}|^2=1 \\ &|s_{14}|^2+|s_{34}|^2=1 \\ &s_{12}s_{23}^*+s_{14}s_{34}^*=0 \\ &s_{12}s_{14}^*+s_{23}s_{34}^*=0 \end{aligned} \tag{6. 68}$$

From which we obtain

$$\begin{aligned} &|s_{12}|=|s_{34}|=\alpha \\ &|s_{14}|=|s_{23}|=\beta \\ &\alpha^2+\beta^2=1 \end{aligned} \tag{6. 69}$$

We write these terms in polar form

$$
\begin{aligned}
s_{12} &= \alpha\, e^{j\varphi} \\
s_{34} &= \alpha\, e^{j\eta} \\
s_{14} &= \beta\, e^{j\psi} \\
s_{23} &= \beta\, e^{j\theta}
\end{aligned}
\qquad (6.\ 70)
$$

The two last energy conservation relations yield

$$
(\varphi + \eta) = (\psi + \theta) + \pi + 2n\pi \qquad (6.\ 71)
$$

We choose the reference planes such as

$$
\varphi = \eta = 0 \qquad (6.\ 72)
$$

And we finally obtain for a lossless, reciprocal and matched four-port :

$$
[S] = \begin{bmatrix} 0 & \alpha & 0 & \beta\, e^{j\psi} \\ \alpha & 0 & \beta\, e^{j\theta} & 0 \\ 0 & \beta\, e^{j\theta} & 0 & \alpha \\ \beta\, e^{j\psi} & 0 & \alpha & 0 \end{bmatrix} \qquad (6.\ 73)
$$

with

$$
\begin{aligned}
\alpha^2 + \beta^2 &= 1 \\
\psi + \theta &= \pi + 2n\pi
\end{aligned}
\qquad (6.\ 74)
$$

Its flow chart is illustrated in Fig. 6. 29.

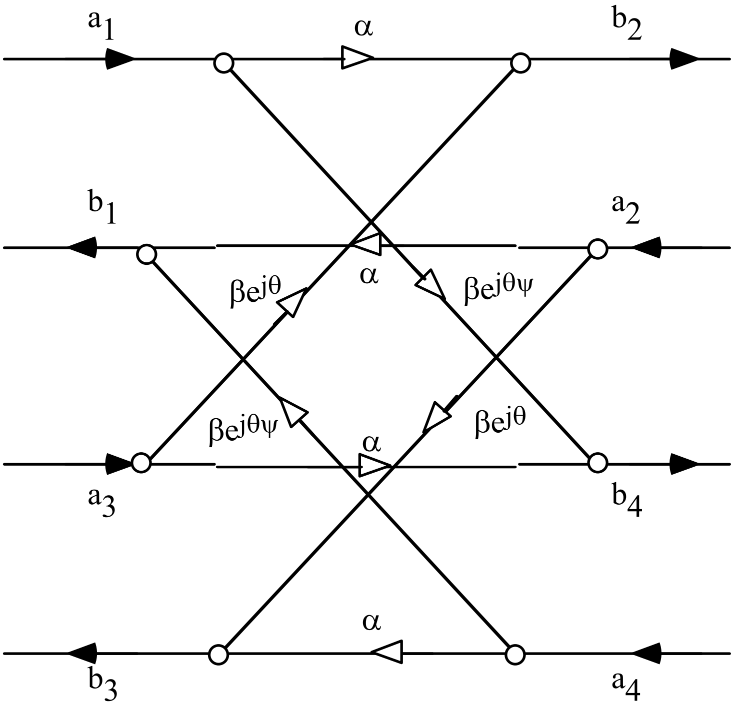


*Fig. 6. 29 : Flow chart of a directional coupler*

### *6.4.2 Particular case: the symmetric coupler*

We choose

$$\psi = \theta = \frac{\pi}{2} \tag{6. 75}$$

and we obtain

$$[S] = \begin{bmatrix} 0 & \alpha & 0 & j\beta \\ \alpha & 0 & j\beta & 0 \\ 0 & j\beta & 0 & \alpha \\ j\beta & 0 & \alpha & 0 \end{bmatrix} \tag{6. 76}$$

Its flow chart is depicted in Fig. 6. 30.

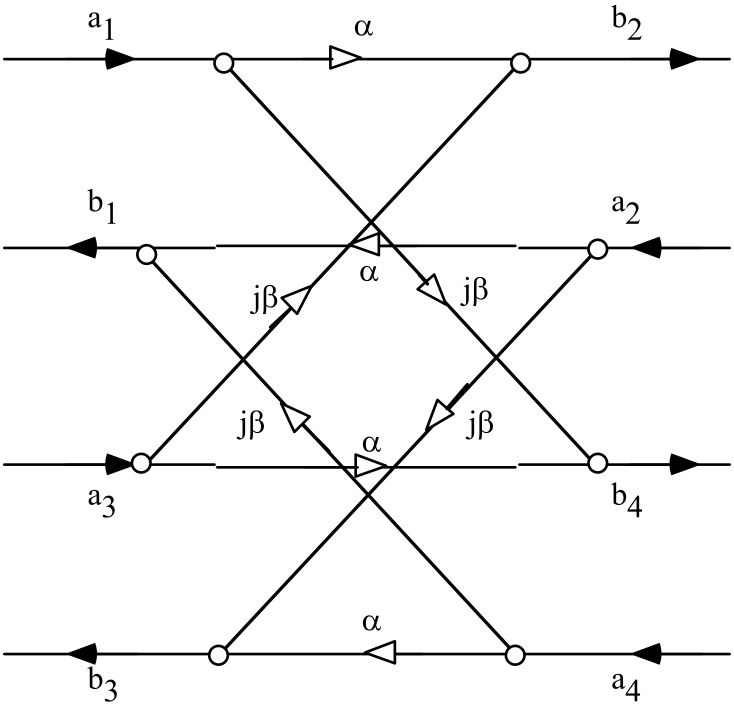



*Fig. 6. 30 : Flow chart of a symmetric coupler*

Example: the hybrid coupler $\alpha = \beta = \dfrac{1}{\sqrt{2}}$

$$[S] = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 & j \\ 1 & 0 & j & 0 \\ 0 & j & 0 & 1 \\ j & 0 & 1 & 0 \end{bmatrix} \tag{6. 77}$$

An example of hybrid coupler realized in microstrip technology is illustrate in Fig. 6. 31.

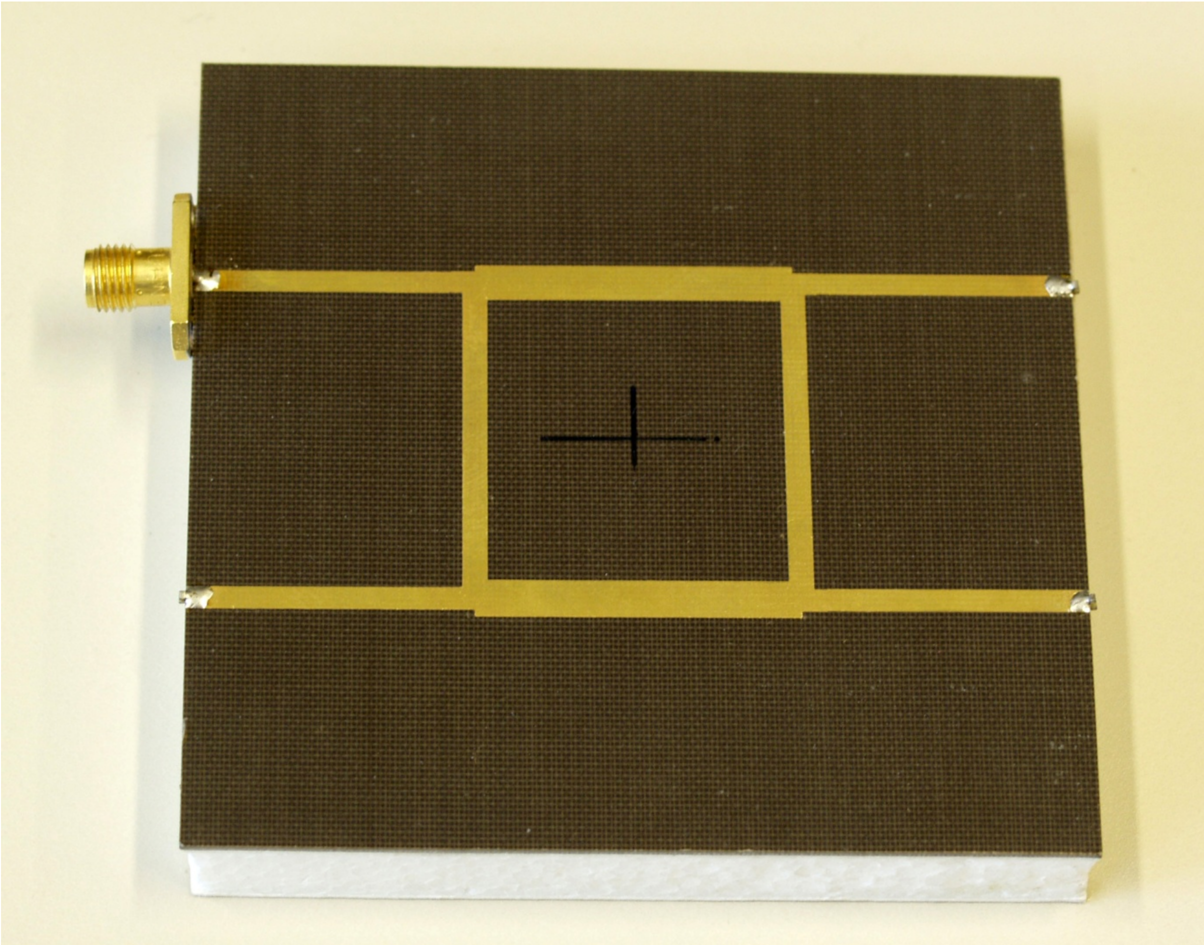


*Fig. 6. 31 : Microstrip Hybrid coupler, or branch coupler*

In order to understand and design these devices, we notice first that they have two axes of symmetry, one horizontal and the other vertical. We will use this fact, by introducing a general analysis theory for circuits with double symmetry:

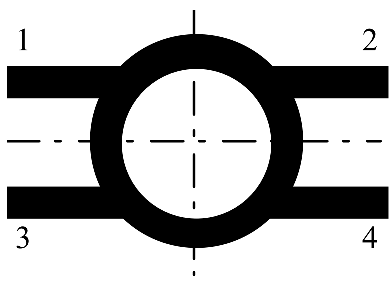



*Fig. 6. 32   Four-port with a horizontal and vertical symmetry*

When a four port component has two symmetry planes, and when its reference planes are also symmetrical with respect to those symmetry planes, its scattering matrix has only four independent terms:

$$
\begin{aligned}
s_{11} &= s_{22} = s_{33} = s_{44} = s_1 \\
s_{12} &= s_{21} = s_{34} = s_{43} = s_2 \\
s_{13} &= s_{31} = s_{24} = s_{42} = s_3 \\
s_{14} &= s_{41} = s_{23} = s_{32} = s_4
\end{aligned}
\tag{6. 78}
$$

Its matrix takes thus the following form

$$
\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} s_1 & s_2 & s_3 & s_4 \\ s_2 & s_1 & s_4 & s_3 \\ s_3 & s_4 & s_1 & s_2 \\ s_4 & s_3 & s_2 & s_1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}
\tag{6. 79}
$$

**Symmetric and antisymmetric excitations**

The application of a symmetric excitation (signals with same amplitude and phase on the 4 ports) yields:

$$
\begin{aligned}
& a_1 = a_2 = a_3 = a_4 = a_{ss} \\
& b_1 = b_2 = b_3 = b_4 = b_{ss} = \\
& \left(s_1 + s_2 + s_3 + s_4\right) a_{ss} = \rho_{ss} a_{ss}
\end{aligned}
\tag{6. 80}
$$

Four a double antisymmetric excitation, we get:

$$
\begin{aligned}
& a_1 = -a_2 = -a_3 = a_4 = a_{aa} \\
& b_1 = -b_2 = -b_3 = b_4 = b_{aa} = \\
& \left(s_1 - s_2 - s_3 + s_4\right) a_{aa} = \rho_{aa} a_{aa}
\end{aligned}
\tag{6. 81}
$$

When we have a symmetric up-down excitation (excitation at ports 1 and 3, respectively 2 and 4 are symmetric) and an antisymmetric left-right excitation, we get :

$$
\begin{aligned}
& a_1 = -a_2 = a_3 = -a_4 = a_{as} \\
& b_1 = -b_2 = b_3 = -b_4 = -b_{as} = \\
& \left(s_1 - s_2 + s_3 - s_4\right) a_{as} = \rho_{as} a_{as}
\end{aligned}
\tag{6. 82}
$$

And finally, when the excitation is symmetric left-right, but antisymmetric up-down, we have:

$$
\begin{aligned}
& a_1 = a_2 = -a_3 = -a_4 = a_{sa} \\
& b_1 = b_2 = -b_3 = -b_4 = -b_{sa} = \\
& \left(s_1 + s_2 - s_3 - s_4\right) a_{as} = \rho_{sa} a_{sa}
\end{aligned}
\tag{6. 83}
$$

In summary, the four reflection coefficients are linked to the four terms of the scattering matrix through:

$$\begin{bmatrix} \rho_{ss} \\ \rho_{as} \\ \rho_{sa} \\ \rho_{aa} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} \tag{6. 84}$$

If we invert this relation, we obtain:

$$\begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \rho_{ss} \\ \rho_{as} \\ \rho_{sa} \\ \rho_{aa} \end{bmatrix} \tag{6. 85}$$

### Meaning of symmetric and antisymmetric excitations

When two ports are excited symmetrically, the same currents flow into both ports. The directions of these currents are thus opposite, and in the symmetry plane, they cancel each other. This plane becomes thus an open circuit plane.

Inversally, when two ports are antisymmetrically excited, the voltage at both ports have the same amplitude but the opposite sign. In the symmetry plane, these voltages thus cancel each other. We can then consider that the symmetry plane is a short circuit plane.

.

We obtain thus the four situations of Fig. 6. 33 :

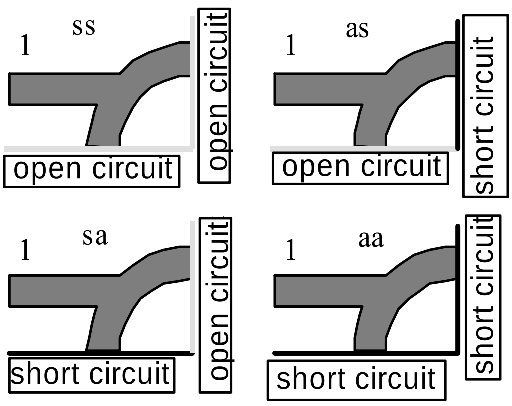



*Fig. 6. 33 symmetric* and *antisymmetric excitations*

The complete analysis of a four-port with a double symmetry can thus be simplified into the analysis of four one port circuits, made of one quarter of the initial structure terminated by open or short circuits in the symmetry planes. When the element is lossless, we know moreover that the amplitude of the four reflection coefficients is equal to unity, thus only their phase need to be determined.

**Matched and directive four port.**

A four port is matched when $\underline{s}_1 = 0$, and we get:

$$\rho_{ss} + \rho_{as} + \rho_{sa} + \rho_{aa} = 0 \qquad (6.86)$$

Moreover, one of the ports is isolated from the input port. Let us consider that access 3 is isolated from port 1, we have $\underline{s}_3 = 0$, which yields :

$$\rho_{ss} + \rho_{as} - \rho_{sa} - \rho_{aa} = 0 \qquad (6.87)$$

These two conditions are simultaneously satisfied for:

$$\rho_{ss} = -\rho_{as} \qquad \rho_{sa} = -\rho_{aa} \qquad (6.88)$$

The two non zero terms of the scattering matrix are then given by:

$$s_2 = \frac{1}{2}\left(\rho_{ss} - \rho_{aa}\right) \qquad S_4 = \frac{1}{2}\left(\rho_{ss} + \rho_{aa}\right) \qquad (6.89)$$

In the case of a symmetric coupler, α and β are given by:

$$\alpha = s_2 = \frac{1}{2}\left(\rho_{ss} - \rho_{aa}\right) \qquad j\beta = s_4 = \frac{1}{2}\left(\rho_{ss} + \rho_{aa}\right) \qquad (6.90)$$

The four reflection coefficients become:

$$\rho_{ss} = -\rho_{as} = \alpha + j\beta \qquad \rho_{aa} = -\rho_{sa} = -\alpha + j\beta \qquad (6.91)$$

The four reflection coefficients are situated on the four corners of a rectangle inscribed in the unit circle of the complex plane:

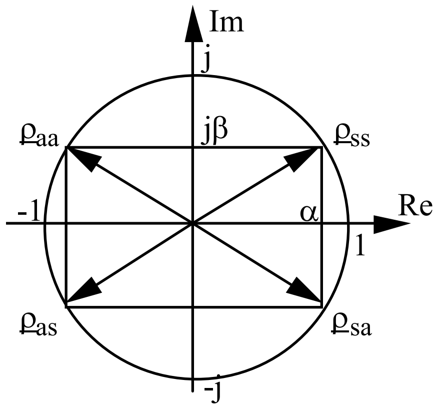



*Fig. 6. 34: Position of the four reflection coefficients*

**Branch coupler**

A typical branch coupler is depicted in Fig. 6. 35.
This geometry has a double symmetry, and in the case the coupler is matched, it is a symmetric coupler.

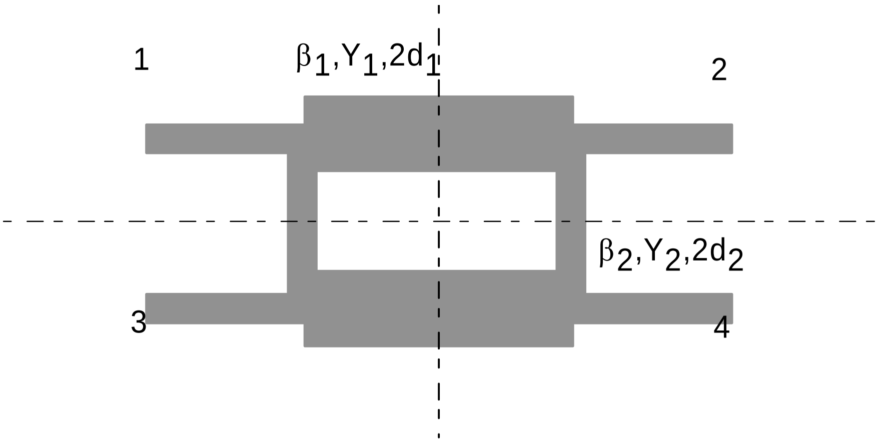



*Fig. 6. 35: Microstrip branch coupler.*

The length and characteristic impedance of the different line sections determine the power division between the two coupled ports. The four reflection coefficients are obtained using transmission line theory, using the relations for open circuited or short circuited transmission lines. We neglect the reactive effects at the discontinuities, and we get:

$$\begin{aligned}
\rho_{ss} &= \frac{Y_c - jY_1\tan(\beta_1 d_1) - jY_2\tan(\beta_2 d_2)}{Y_c + jY_1\tan(\beta_1 d_1) + jY_2\tan(\beta_2 d_2)} e^{j\varphi} \\
\rho_{as} &= \frac{Y_c + jY_1\cot(\beta_1 d_1) - jY_2\tan(\beta_2 d_2)}{Y_c - jY_1\cot(\beta_1 d_1) + jY_2\tan(\beta_2 d_2)} e^{j\varphi} \\
\rho_{sa} &= \frac{Y_c - jY_1\tan(\beta_1 d_1) + jY_2\cot(\beta_2 d_2)}{Y_c + jY_1\tan(\beta_1 d_1) - jY_2\cot(\beta_2 d_2)} e^{j\varphi} \\
\rho_{aa} &= \frac{Y_c + jY_1\cot(\beta_1 d_1) + jY_2\cot(\beta_2 d_2)}{Y_c - jY_1\cot(\beta_1 d_1) - jY_2\cot(\beta_2 d_2)} e^{j\varphi}
\end{aligned} \qquad (6.\ 92)$$

The phase shift $\varphi$ has been introduced to take into account the fact that the reference plane does not coincide with the junction of the lines.

The matching and directivity conditions require that

$$Y_1\cot(2\beta_1 d_1) + Y_2\cot(2\beta_2 d_2) = 0 \quad \text{and} \quad Y_1^2 - Y_2^2 = Y_c^2 \qquad (6.\ 93)$$

With help of equation (6.91), we find that $\varphi = \pi/2$.

The system of equations admits a "simple" solution, obtained by setting the two terms of the sum on the left hand side of equation (6.93) equal to zero, yielding :

$$\beta_1 d_1 = \beta_2 d_2 = \pi/4 \quad Y_1 = Y_c/\alpha \quad Y_2 = -Y_c\,\beta/\alpha \tag{6. 94}$$

We find a negative value for $\beta$. (Note: be careful not to mix $\beta_1$, $\beta_2$, which are propagation factors with $\beta$, which is the coupling coefficient!!)

The half length of the two lines are respectively of $\lambda_i/8$, with $i = 1,2$. For microstrip lines, the wavelength varies with the characteristic impedance of the lines, and the coupler is not square. In the case of a hybrid coupler, the power is equally distributed between the two outputs, and we must have $\alpha = -\beta = 1/\sqrt{2}$ yielding

$$Y_1 = \sqrt{2}\,Y_c \quad \text{et} \quad Y_2 = Y_c \tag{6. 95}$$

The transmission of the signal through a microstrip hybrid coupler is shown in figure 6.36. We see clearly that the signal is equally distributed between two ports, the last being isolated.

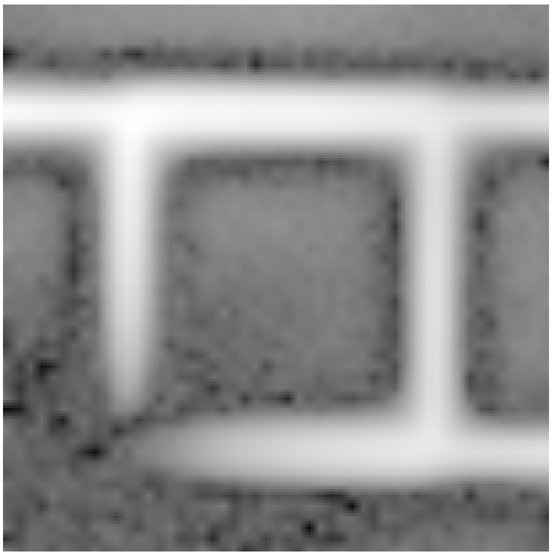


*Fig. 6. 36 : Distribution of the signal on a hybrid coupler*

This figure represents the component of the electric field which is normal to the plane of the circuit, measured just above the circuit.

Equations (6.92) have more solutions, which are not so simple, having line lengths different from $\lambda_i/8$ (one is shorter the other is longer). The characteristic admittances are then given by:

$$Y_1 = \frac{Y_c}{\sqrt{1 - \left[ \cot(2\beta_1 d_1) \big/ \cot(2\beta_2 d_2) \right]^2}}$$
$$Y_2 = \frac{Y_c}{\sqrt{\left[ \cot(2\beta_2 d_2) \big/ \cot(2\beta_1 d_1) \right]^2 - 1}} \quad (6.\ 96)$$

And the terms of the scattering matrix are given by

$$\alpha = \frac{2Y_c \left[ Y_1 \tan(\beta_1 d_1) + Y_2 \tan(\beta_2 d_2) \right]}{Y_c^2 + \left[ Y_1 \tan(\beta_1 d_1) + Y_2 \tan(\beta_2 d_2) \right]^2}$$
$$\beta = \frac{Y_c^2 - \left[ Y_1 \tan(\beta_1 d_1) + Y_2 \tan(\beta_2 d_2) \right]^2}{Y_c^2 + \left[ Y_1 \tan(\beta_1 d_1) + Y_2 \tan(\beta_2 d_2) \right]^2} \quad (6.\ 97)$$

In these developments, we have again neglected the reactive contributions due to the spurious modes at the discontinuities.

### *6.4.3 Particular case: the asymmetric coupler*

We choose

$$\psi = 0 \, , \theta = \pi$$

and we obtain

$$[S] = \begin{bmatrix} 0 & \alpha & 0 & \beta \\ \alpha & 0 & -\beta & 0 \\ 0 & -\beta & 0 & \alpha \\ \beta & 0 & \alpha & 0 \end{bmatrix} \quad (6.\ 98)$$

The flow chart is illustrated in Fig. 6. 37

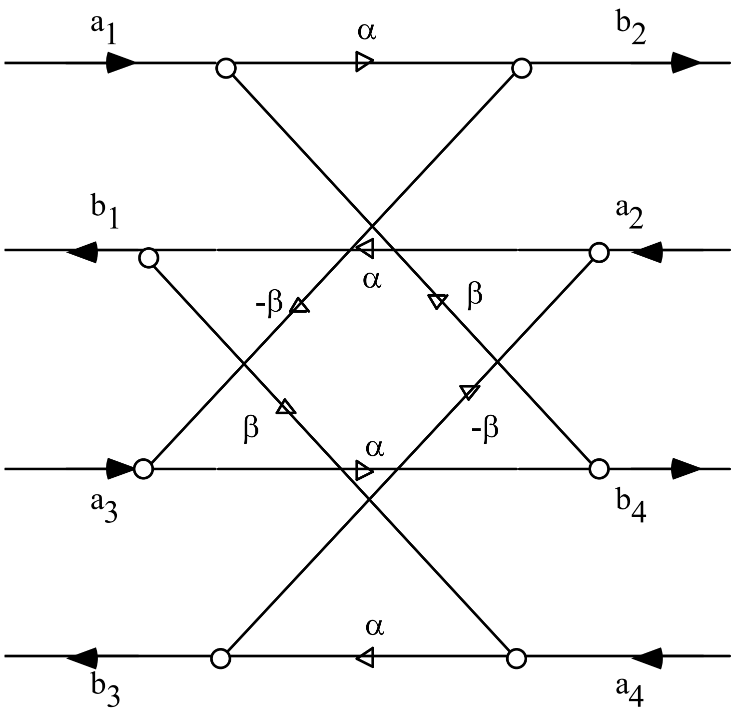


*Fig. 6. 37 : Flow chart of an asymmetric coupler*

Example: The hybrid T, the hybrid circle (magic T, rat-race) $\alpha = \beta = \frac{1}{\sqrt{2}}$

$$[S] = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \qquad (6.\ 99)$$

### *6.4.4 The real directional coupler*

The ports are numbered in a way to obtain

$$\alpha \geq \beta \qquad (6.\ 100)$$

We define the following terms:

- Attenuation level: $LA = -20\log\alpha$
- Coupling level: $LC = -20\log\beta$

In the case of a real (non ideal) coupler, we have moreover

$$s_{ii}, s_{13}, s_{24} \; small\,but \neq 0$$

The coupler is then characterized by its reflection coefficients and by its isolation levels :

- $LI_{13} = -20\log|s_{13}|$
- $LI_{24} = -20\log|s_{24}|$

The quality of a coupler is given by its directivity:

- $LD_{13} = LI_{13} - LC = -20\log\frac{|s_{13}|}{\beta}$
- $LD_{24} = LI_{24} - LC = -20\log\frac{|s_{24}|}{\beta}$

The higher the directivity, the better the coupler

### *6.4.5 Electric symbol of a directional coupler*

the electric symbol of a directional is shown in figure 6.38

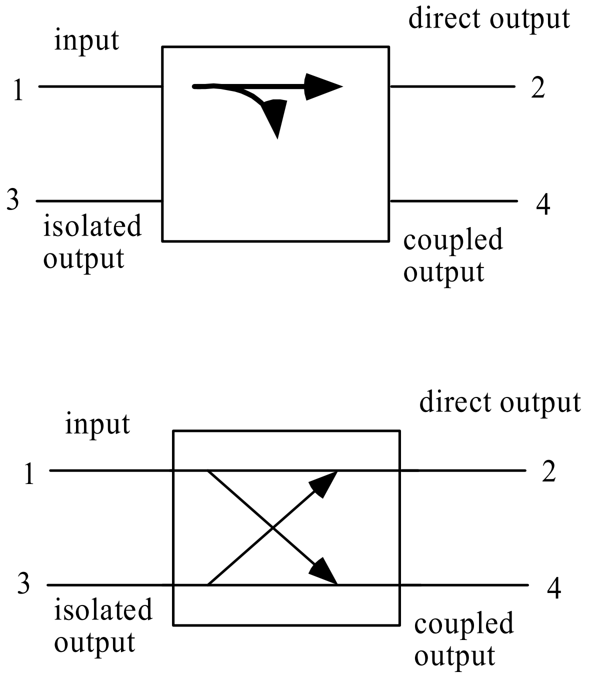



*Fig. 6. 38 : Symbol of the directional coupler*

# 7. Microwave filters

### References

G.L. Matthaei, L. Young and E.M.T. Jones, "Microwave filters, impedance-matching Networks and coupling structures", Mc Graw Hill, New-York, 1964.

J.A.G. Malherbe, "Microwave transmission line filters", Artech House, Dedham, Ma, 1979.

## 7.1 Introduction

Filters are components (two-ports) used to control the frequency response of the signal, by allowing transmission in the pass band of the filter and attenuating the signal in the stop band. At lower frequencies, filters can be passive or active, the former being made of lumped capacitors and inductors, the latter using transistors as well. In the microwave domain, filters are mainly passive. The design principle is very similar to the design principle of passive low frequency filters. The used technology however is fundamentally different, as the required impedances of the different stages of the filters are obtained using transmission line sections rather than lumped elements. We talk about a distributed technology, versus lumped technology. This has two main reasons: the wavelength is small enough to allow us to have transmission lines of a length of few tens of wavelength without becoming prohibitively large, and lumped capacitors and inductors with a good quality factor are difficult to obtain in the microwave range.

We will concentrate in the first section on the so called low pass prototype filter, which is a lumped element filter designed for a normalized frequency and for normalized terminators. This filter is important, as we can easily derive from it lumped low pass, high pass, band pass and band stop filters for any frequency band, which will be done in the following section. We will then see how we go from the lumped element prototype to the distributed element microwave filter, using Richard's transformations and Kuroda's identities.

## 7.2 The low pass prototype filter

An ideal low pass filter would have a transmission coefficient of 1 up to the cut-off frequency and a zero transmission coefficient above. This is clearly not realistic, and in practice a low pass filter is defined by (figure 7.1):

- $\omega_c$, the cut-off frequency
- $A_0$, the maximal tolerated attenuation in the pass band
- $\omega_1$, an angular frequency in the stop band, where we want to specify
- $A_1$, the minimal attenuation at this frequency

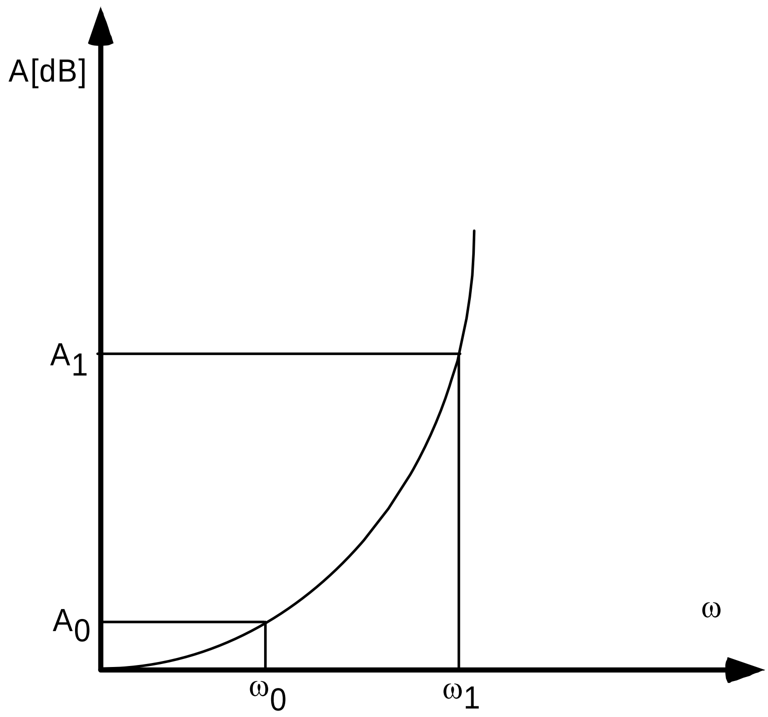


Fig. 7. 1 *: Real low pass filter response*

### *7.2.1 The insertion loss method*

The insertion loss method gives us control over the pass band and stop band amplitude and phase characteristics, and is a systematic way to synthesise a desired response. Let us define the power loss ratio as:

$$P_{LR} = \frac{1}{1-\left|\Gamma(\omega)\right|^2} = \frac{P_{inc}}{P_{load}} \qquad (7.\ 101)$$

and the insertion loss in dB as

$$IL = 10\log P_{LR} \qquad (7.\ 102)$$

The filter is passive network, and the causality requires that $\left|\Gamma(\omega)\right|^2$ is an even function of ω. We can therefore write $\left|\Gamma(\omega)\right|^2$ as a polynomial in $\omega^2$:

$$\left|\Gamma(\omega)\right|^2 = \frac{M\left(\omega^2\right)}{M\left(\omega^2\right)+N\left(\omega^2\right)} \qquad (7.\ 103)$$

where M and N are real polynomials. Substituting in (7.1) yields:

$$P_{LR} = 1 + \frac{M\left(\omega^2\right)}{N\left(\omega^2\right)} \tag{7. 104}$$

which must hold for the filter to be physically realisable. From this point, we can consider several practical filter responses.

### *7.2.1.*1 *Normalization*

The normalized low pass prototype is shown in figure 7.2, and a possible lumped element low pass filter in figure 7.3

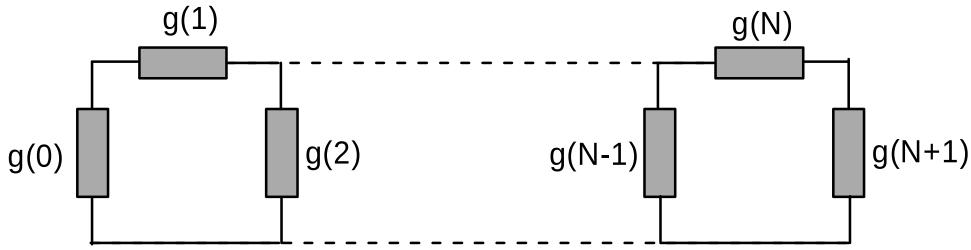



*Fig. 7.2 : low pass prototype filter*

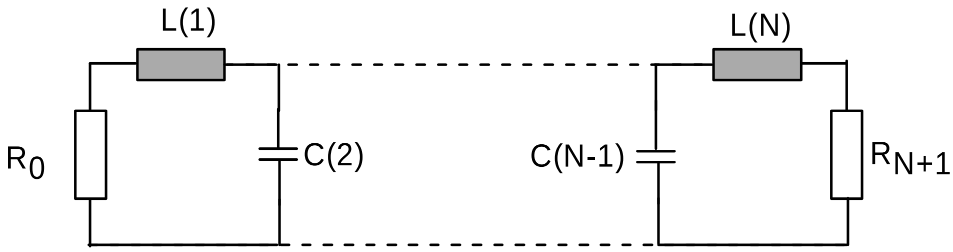



*Fig. 7.3 : Lumped element low pass filter*

The link between the two is given by the following relations:

$$\begin{aligned} C(k) &= \frac{g(k)}{\omega_0 Z_0} \\ L(k) &= \frac{g(k) Z_0}{\omega_0} \\ R(k) &= Z_0 g(k) \end{aligned} \tag{7. 105}$$

where $Z_0$ is the characteristic impedance of the line, which is supposed equal to the source impedance.

In the previous example, the first stage is made of a series inductor. The dual solution, i.e. starting with a parallel capacitor would also have been possible.

### *7.2.12 The Butterworth filter (maximally flat response)*

This filter is optimum in the sense that is provides the flattest possible response in the pass band. This response is defined as:

$$P_{LR} = 1 + k^2 \left( \frac{\omega}{\omega_o} \right)^{2N} \tag{7. 106}$$

where N is the order of the filter, $\omega_o$ the cut-off frequency. The pass band extends from $\omega=0$ to $\omega=\omega_o$ and the power loss ratio is $1 + k^2$ at the band edge. This point is usually chosen at -3dB, yielding k=1.

In practice a Butterworth lumped element filter is synthesized as follows:

- the degree of the filter (the number of stages of the filter) is computed as

$$N = \frac{\ln(10^{\frac{A_1}{10}} - 1)}{2\ln\left(\omega_o / \omega_1\right)} \tag{7. 107}$$

- the normalized elements of the filter (figure 7.2) are obtained as

$$\begin{aligned} &g\left(0\right) = 1 \\ &g\left(N+1\right) = 1 \\ &g(k) = 2\sin\left(\left(2k-1\right)\frac{\pi}{2N}\right) \quad for \quad 1 \le k \le N \end{aligned} \tag{7. 108}$$

- the de-normalized lumped elements are obtained from (7.5)

### 7.2.1.2 The Chebyshev (equal ripple) filter

The equal ripple design is optimal in the sense that it yields the steepest cut-off in the stop band, at the cost of an undulation in the pass band. In this case, the power loss ratio is given by:

$$P_{LR} = 1 + k^2 T_N^2(\omega) \qquad (7.\ 109)$$

where $T_N$ is the Chebyshev polynomial of order N. In this case the filter is synthesized using the following approach:

- the number of stages is obtained using

$$N = \frac{\text{Arcosh}\left(\sqrt{\dfrac{10^{A_1/10}-1}{10^{A_0/10}-1}}\right)}{\text{Arcosh}\left(\dfrac{\omega_1}{\omega_0}\right)} \qquad (7.\ 110)$$

- and the normalized prototype elements by

$$\begin{aligned} &g(0) = 1 \\ &g(1) = \frac{2a_1}{\gamma} \\ &g(N+1) = 1 \quad \text{if N is odd} \\ &\text{g}(\text{N+1}) = \tanh^2\left(\frac{\beta}{4}\right) \text{ if N is even} \\ &\text{g}(\text{k}) = \frac{4a_{k-1}a_k}{b_{k-1}g(k-1)} \end{aligned} \qquad (7.\ 111)$$

with

$$\begin{aligned} &a_k = \sin\left[(2k-1)\pi/2N\right] \\ &b_k = \gamma^2 + \sin^2(k\pi/N) \\ &\beta = \ln\left(\coth\left(\frac{A_0}{17.37}\right)\right) \\ &\gamma = \sinh\left(\frac{\beta}{2N}\right) \end{aligned} \qquad (7.\ 112)$$

A typical Chebyshev low pass response is illustrated in figure 7.4.

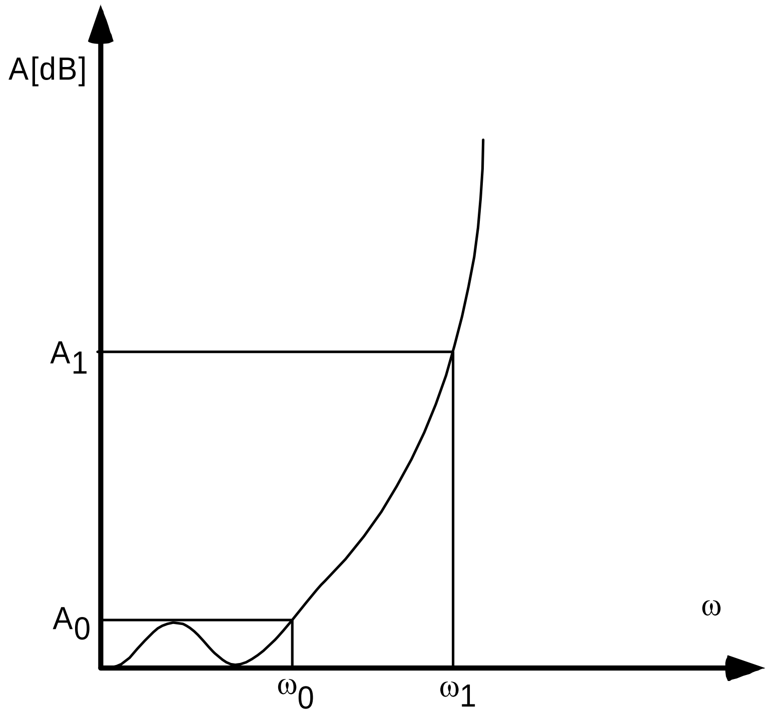



*Fig. 7.4: Chebyshev low pass response*

## 7.3 Impedance and frequency transformations to lumped element high pass, band pass and band stop filters

### *7.3.1 Lumped element high pass filters*

The frequency transform to transform a low pass prototype to a high pass is given by the substitution:

$$\omega \leftarrow -\frac{\omega_o}{\omega} \tag{7. 113}$$

where the negative sign is used to obtain physically realizable elements. The substitution maps $\omega = 0 \;\; to \;\; \omega = \pm\infty$ and $\omega = \pm\infty \;\; to \;\; \omega = 0$. Cut-off occurs when $\omega = \pm\omega_0$. Applying the transform to the impedances of the prototype filter of figure 7.2, yields the high pass filter of figure 7.5 with the following correspondences:

$$C_k = \frac{1}{\omega_o g(k) Z_o}$$
$$L_k = \frac{Z_o}{\omega_o g(k)} \qquad (7.\ 114)$$

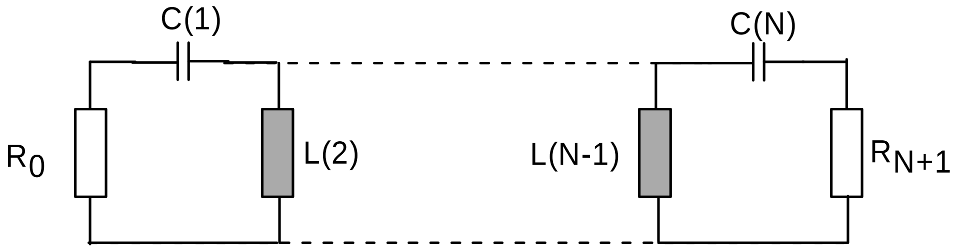



*Fig. 7.5 : Low pass filter prototype*

The strategy in this case is thus to perform the frequency transformation, compute the stage of the filter N and the g(k) coefficients for the equivalent low pass prototype, and do the transformation back in frequency and obtain the $C_k$ and $L_k$ using (7.14).

### *7.3.2 Lumped element band pass filter*

The frequency transformation used in this case is given by

$$\omega \leftarrow \frac{\omega o}{\omega_2 - \omega_1}\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right) = \frac{1}{\Delta}\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right) \qquad (7.\ 115)$$

with

$$\Delta = \frac{\omega_2 - \omega_1}{\omega_o} \qquad (7.\ 116)$$

The centre frequency $\omega_o$ is defined as the geometric mean of $\omega_1$ and $\omega_2$ $\omega_o = \sqrt{\omega_1 \omega_2}$ .

Applying the transform to the impedances of the prototype filter of figure 7.2 yields the band pass filter of figure 7.6.

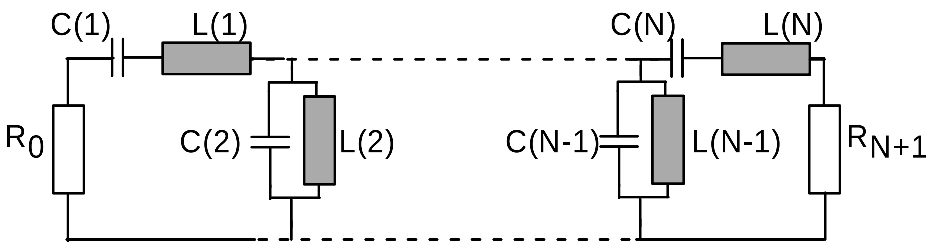


*Fig. 7.6: Band pass filter*

with the following correspondence :

$$\left.\begin{aligned} L(k) &= \frac{g(k)Z_o}{\Delta\omega_o} \\ C(k) &= \frac{\Delta}{\omega_o g(k) Z_o} \end{aligned}\right\} \text{ for the series branches}$$

$$\left.\begin{aligned} L(k) &= \frac{\Delta Z_o}{\omega_o g(k)} \\ C(k) &= \frac{g(k)}{\Delta\omega_o Z_o} \end{aligned}\right\} \text{ for the parallel branches} \qquad (7.\ 117)$$

### *7.3.3 Lumped element band stop filter*

The transform used in this case in the inverse of the one used for the band pass filter, and is given by:

$$\omega \leftarrow \Delta\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right)^{-1} \qquad (7.\ 118)$$

with $\omega_o$ and $\Delta$ defined in §7.3.2. Applying the transform to the impedances of the prototype filter of figure 7.2 yields the band stop filter of figure 7.7.

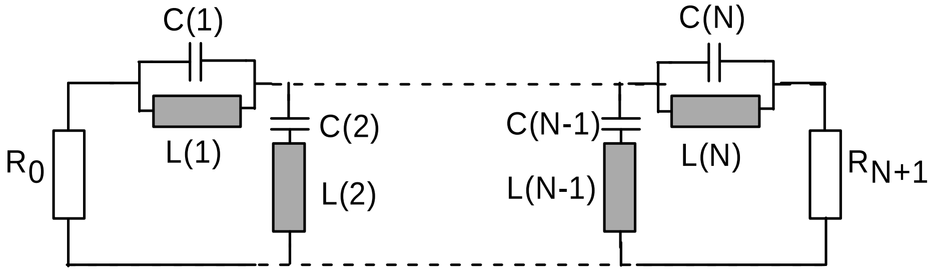



*Fig. 7.7*

with the following correspondence:

$$
\left.\begin{aligned}
L(k) &= \frac{\Delta Z_o g(k)}{\omega_o} \\
C(k) &= \frac{1}{\omega_o g(k) Z_o \Delta}
\end{aligned}\right\} \text{for the series branches}
$$

$$
\left.\begin{aligned}
L(k) &= \frac{Z_o}{\omega_o g(k) \Delta} \\
C(k) &= \frac{\Delta g(k)}{\omega_o Z_o}
\end{aligned}\right\} \text{for the parallel branches} \qquad (7.\ 119)
$$

## 7.4 Low-pass and high-pass filters using transmission line stubs

As mentioned in the introduction, passive lumped element filters as the ones designed above work well at low frequencies. At microwaves frequencies however, the quality of lumped elements is not good enough to make good filters. Moreover, lumped elements may be obtainable only for a limited range of value. Thus at these frequencies, we use distributed elements such as open or short-circuited transmission line stub as reactive elements. In addition, the distance between reactive elements is not negligible at microwave frequencies, but has to be taken into account. We will use Richard's transformation to convert lumped reactance to transmission line stubs, and Kuroda's identities to separate filter elements by transmission line lengths. Because such additional transmission line sections do not affect the filter response, this type of design is called redundant filter synthesis.

### *7.4.1 Richard's transformations.*

The transformation defined by

$$
\Omega = \tan \beta l = \tan \frac{\omega l / \omega_0}{v_p} \qquad (7.\ 120)
$$

maps the ω plane to the Ω plane, which repeats with a period of $\frac{\omega / \omega_o \, l}{v_p} = 2\pi$ . This transformation was introduced by P. Richard to synthesize an LC network using open and short-circuited stubs. Indeed, if we replace the frequency variable ω by Ω, the reactance of an inductor can be written as (figure 7.8):

$$jX_L = j\Omega L = jL\tan\beta l \qquad (7.\ 121)$$

and the susceptance of a capacitor as :

$$jB_C = j\Omega C = jC\tan\beta l \qquad (7.\ 122)$$

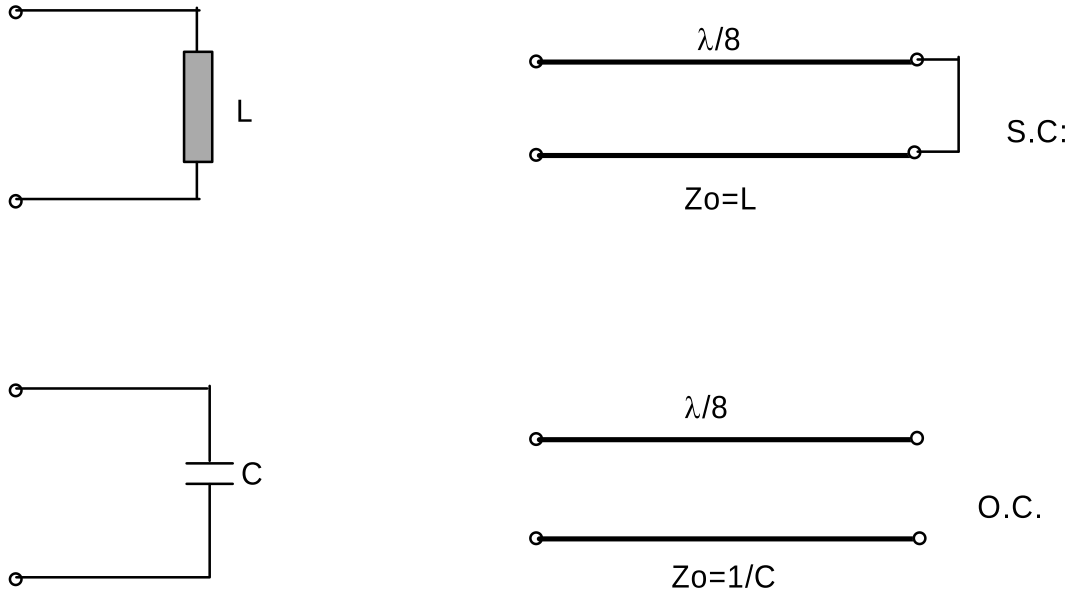



*Fig. 7.8 : Richard's transformations*

These results indicate that an inductor can be replaced by a short-circuited stub of length βl and characteristic impedance L, while a capacitor can be replaced by an open-circuited stub of length βl and characteristic impedance 1/C. A normalized filter impedance is assumed here ($Z_0 = 1$). The cut-off of the prototype filter occurs at a frequency $\omega = \omega_o$ or $\omega/\omega_o = 1$. To obtain the same cut-off for the Richard's transformed filter, we must have :

$$\Omega = \tan\beta l = 1 \qquad (7.\ 123)$$

Which gives a stub length of λ/8, where λ is the wavelength at the cut-off frequency. At a frequency which is the double of the cut-off, corresponding to a stub length of λ/4, an attenuation pole will occur.
We see that Richard's transformation can be used to build a distributed element low pass filter, where the inductors and capacitors of the prototype filter are made of shorted and open-circuited transmission line stubs. All stub lengths will be of λ/8 at the cut-off frequency.

At frequencies away from cut-off, the impedance of the stubs will be different from the impedances of the lumped element prototype, and the filter response will no longer be identical to the response of the lumped element prototype. We should also note that there is a periodicity of $4\omega_o$ in the response of the stub filter.

### 7.4.2 Kuroda's identities

We have seen above that lumped element band pass and band stop filters are made by cascading resonant ant antiresonant branches in series and in parallel. Unfortunately when we come to distributed element filters, we cannot realize at the same time two different types of microwave resonators. We can either couple them in series or in parallel, as is illustrated in figure 7.9 for a case of microstrip lines.

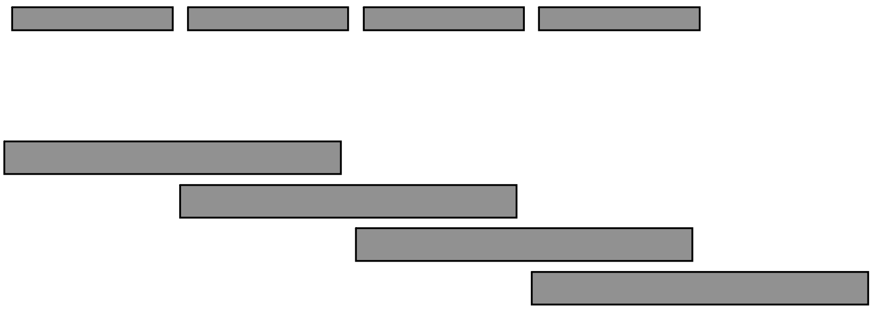


*Fig. 7.9 : Microstrip resonators coupled in series or in parallel*

We will thus use redundant transmission line sections to achieve a more practical distributed element filter implementation, by performing any of the following operations:

- Physically separate transmission line stubs
- Transform series stubs into shunt stubs, or vice versa
- Change impractical impedances into more realizable ones

This is done via the insertion of additional transmission lines which have a length of $\lambda/8$ at $\omega_o$. These lines are called unit elements.

The four Kuroda identities performing these transformations are illustrated in figure 7.10.

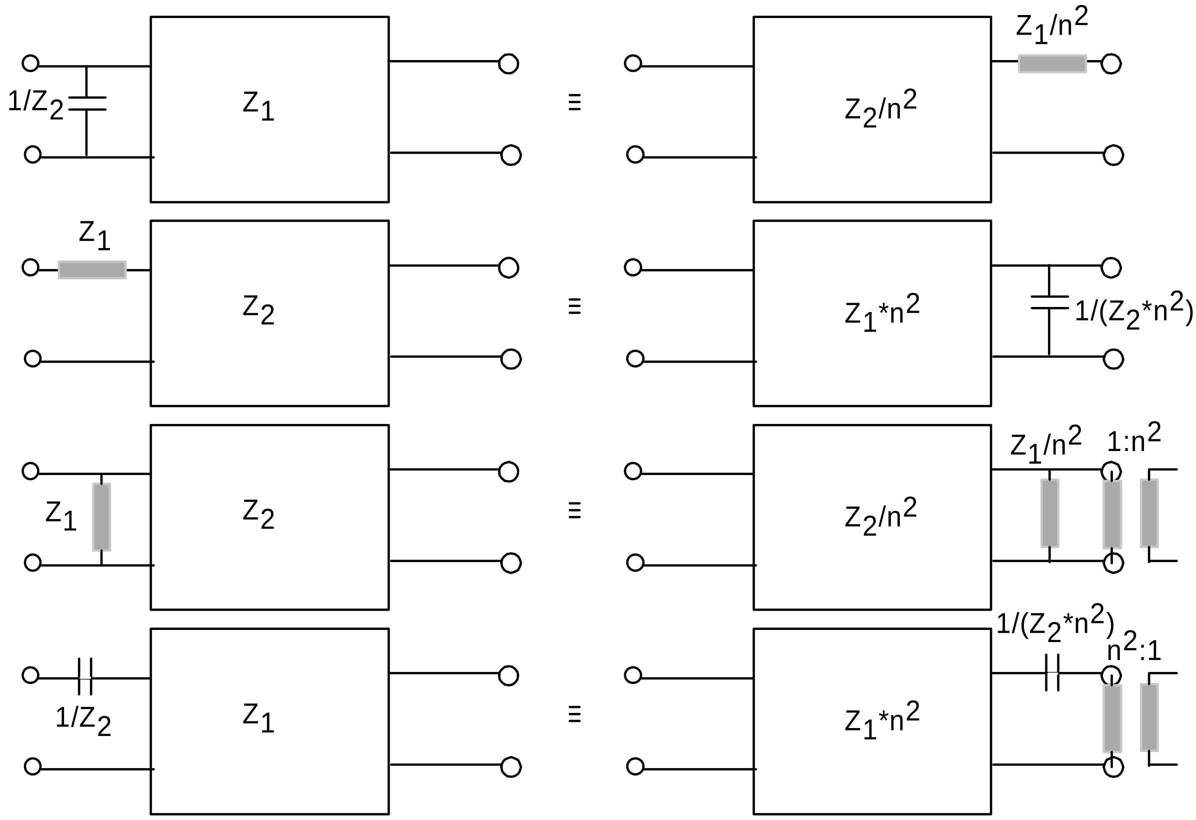



*Fig. 7.10 : The Kuroda identities, where $n^2=1+Z_2/Z_1$*

The proof of these identities is left to the reader.

### *7.4.3 Design of a low pass filter using stubs*

We want to design a low pass filter using stubs. We first compute the low pass prototype filter, and we then compute the corresponding lumped element low pass filter. Let us suppose that we obtain the filters illustrated in figure 7.11

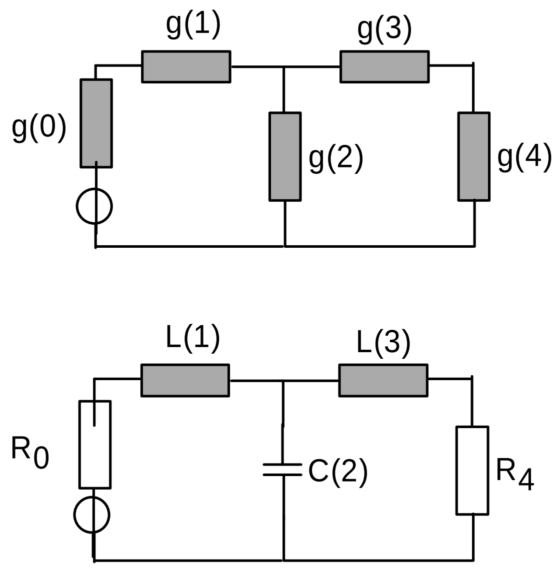



*Fig. 7.11 : Low pass prototype and lumped element version*

We then used Richards transformation to convert series inductors to series short-circuited stubs, and shunt capacitors to shunt open-circuited stubs, and we obtain the circuit depicted in figure 7.12.

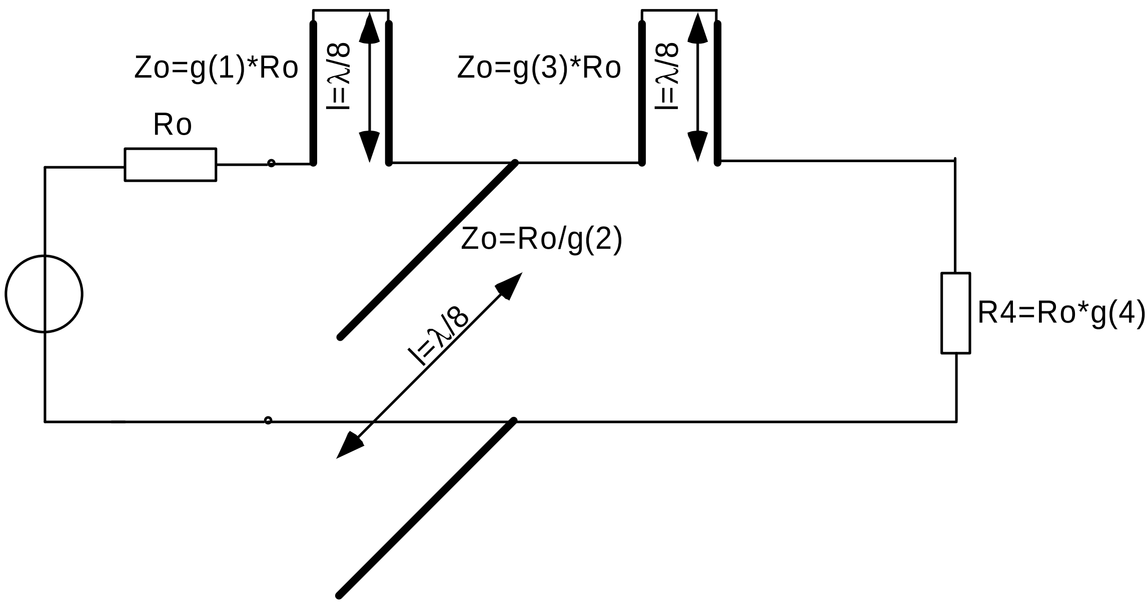



*Fig. 7.12 Low pass filter with series and parallel stubs*

Now let us imagine we want to realize this filter in a microstrip technology. The series stubs would be difficult to realize in this technology, so we will use Kuroda's identities to convert them to shunt stubs. First, we add unit elements at each end of the filter circuit, as is shown in figure 7.13.

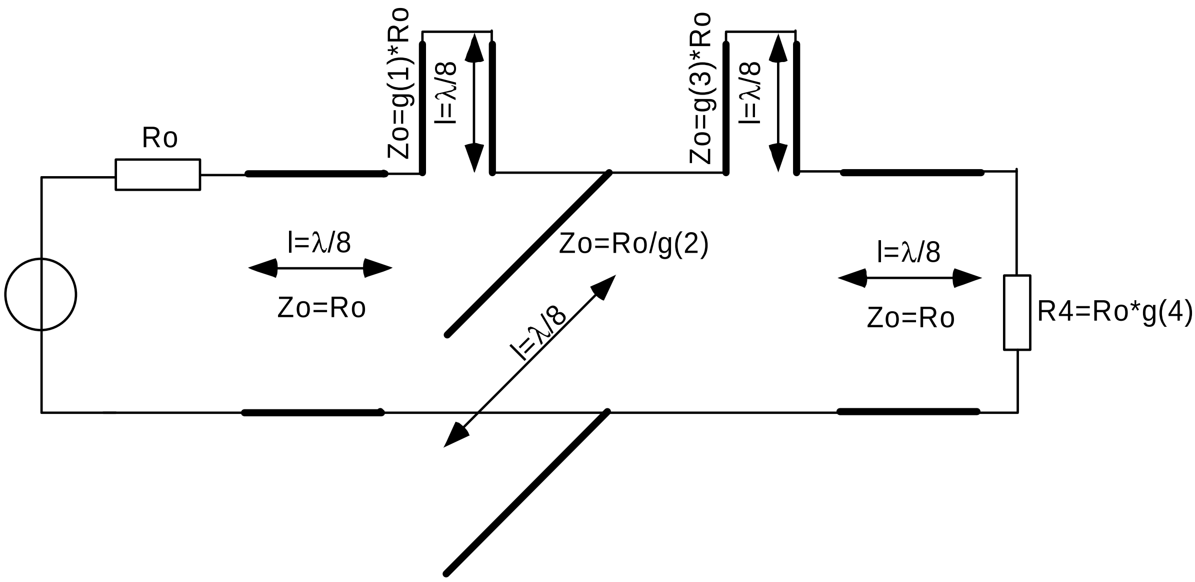



*Fig. 7.13 : Low pass filter with unit elements at the end*

We apply then the second Kuroda identity to both ends of the filter, computing for both cases

$$n_1^2 = 1 + \frac{R_o}{g(1) * R_o}$$
$$n_2^2 = 1 + \frac{R_o}{g(3) * R_o} \qquad (7.\ 124)$$

and we obtain the following filter :

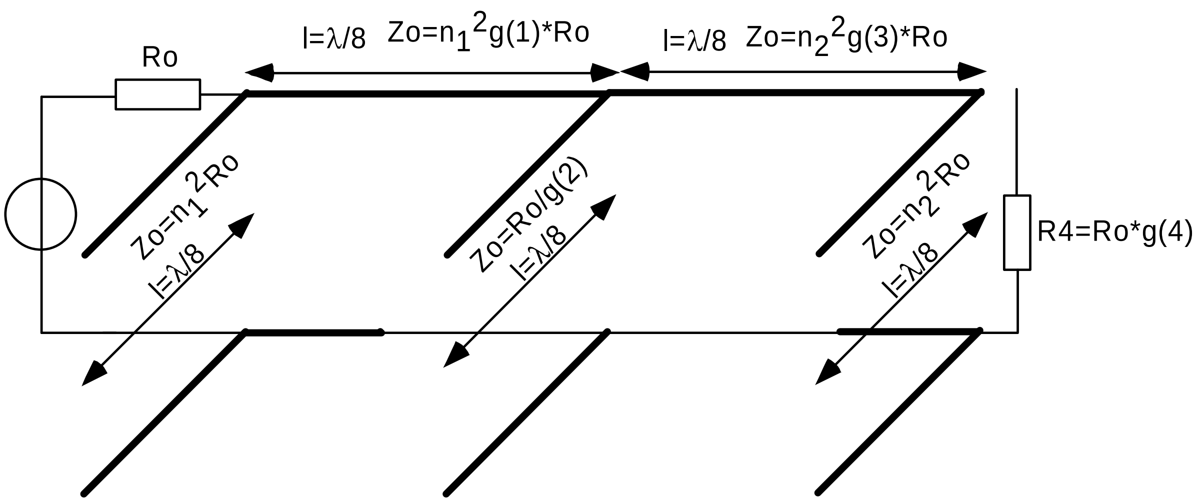



*Fig. 7.14 : Low pass filter after transformation*

Which can be easily realized in microstrip technology, as is shown in figure 7.15

*Fig. 7.15 : Microstrip line layout of a 3 stage low pass filter*

### *7.4.4 Impedance and admittance inverters*

An impedance inverter is an ideal quarter-wave transformer. A load impedance connected at one end is seen as an impedance that has been inverted with respect to the characteristic impedance squared at the input. Impedance inverters can be used to convert a band pass filter network of the type shown in figure 7.16 into a network containing only series tuned circuits. By using admittance inverters, the band pass filter can be converted into a network containing only parallel tuned circuits. Furthermore, by choosing the inverters correctly, all the inductors and capacitors can be chosen to have the same values. Thus impedance and admittance inverters enable us to use identical resonators, either series or parallel tuned, throughout the network.

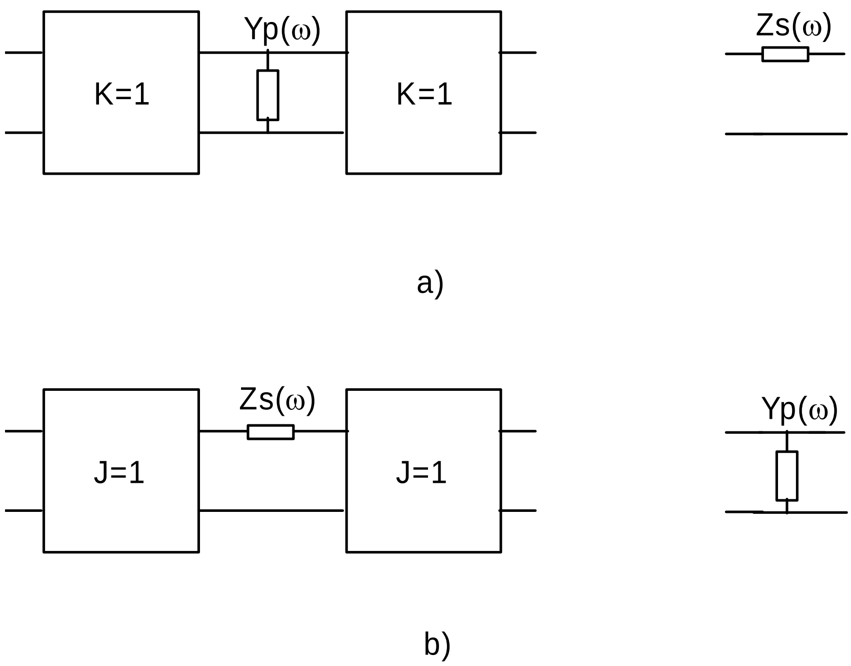



*Fig. 7.16 : (a) impedance inverter used to convert a parallel admittance into an equivalent series impedance; (b) admittance inverter used to convert a series impedance into an equivalent parallel admittance.*

Consider the parallel admittance Yp(ω) with an ideal impedance inverter with characteristic impedance K connected on both sides as shown in figure 7.16. A short circuit at the output will be transformed to an open circuit in parallel with Yp. The input impedance is given by :

$$Z_{in} = \frac{K^2}{Z_p} = K^2 Y_p = Y_p = Z_s \tag{7. 125}$$

Thus the shunt element with two impedance inverters converts the shunt admittance into an equivalent series impedance Zs(ω)=Yp(ω). If Yp is a parallel tuned resonator with

$$Y_p = j\omega C - \frac{j}{\omega L} = j\omega C\left(1 - \frac{\omega_o^2}{\omega^2}\right) \tag{7. 126}$$

it is converted into a series tuned circuit with

$$Z_s = j\omega L\left(1 - \frac{\omega_o^2}{\omega^2}\right) \tag{7. 127}$$

with the inductance L in Henries having the same numerical value as the capacitance C in Farads. If we want to convert an admittance

$$Y_1 = j\omega C_1\left(1 - \frac{\omega_o^2}{\omega^2}\right) \tag{7. 128}$$

into a particular series tuned circuit with arbitrary inductance L , then we must choose K so that

$$K^2 j\omega C_1\left(1 - \frac{\omega_o^2}{\omega^2}\right) = j\omega L\left(1 - \frac{\omega_o^2}{\omega^2}\right) \tag{7. 129}$$

or

$$K = \sqrt{\frac{L}{C_1}} \tag{7. 130}$$

The same considerations can be done for the admittance transformer depicted in figure 7.16 b), and we obtain finally:

$$Y = j\omega C\left(1-\frac{\omega_o^2}{\omega^2}\right) = j\sqrt{\frac{C}{L}}\left(\frac{\omega}{\omega_o}-\frac{\omega_o}{\omega}\right)$$
$$Z = j\omega L\left(1-\frac{\omega_o^2}{\omega^2}\right) = j\sqrt{\frac{L}{C}}\left(\frac{\omega}{\omega_o}-\frac{\omega_o}{\omega}\right) \qquad (7.\ 131)$$

We will illustrate the use of inverters to convert the circuit shown in figure 7.17 into one with two identical parallel tuned resonators or one with two identical series tuned resonators.

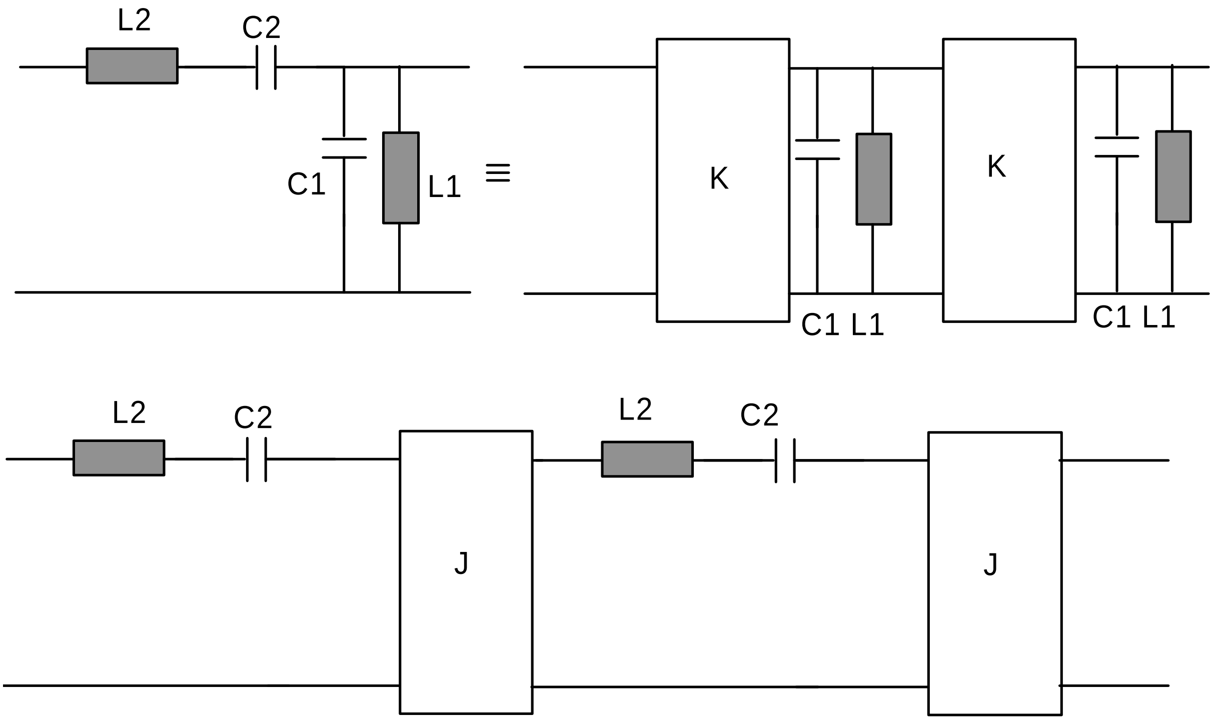



*Fig. 7.17 : Use of inverters to convert resonators*

For the first case, we choose K such as

$$K^2 j\omega C_1\left(1-\frac{\omega_o^2}{\omega^2}\right) = j\omega L_2\left(1-\frac{\omega_o^2}{\omega^2}\right)\ ,\ \Rightarrow K = \sqrt{\frac{L_2}{C_1}} \qquad (7.\ 132)$$

For the second case, we choose J such as :

$$J^2 j\omega L_2\left(1-\frac{\omega_o^2}{\omega^2}\right) = j\omega C_1\left(1-\frac{\omega_o^2}{\omega^2}\right)\ ,\ \Rightarrow J = \sqrt{\frac{C_1}{L_2}} \qquad (7.\ 133)$$

Impedance and admittance transformers are ideal quarter-wave transformers. There is no basic difference in their inverting properties. The only distinction that we make is to use the symbol K to denote the characteristic impedance of the impedance inverter and we use J to denote the characteristic admittance of an admittance inverter.

Impedance and admittance level changing can be accomplished by using different input and output inverters as shown in figure 7.18. For example, as in figure 7.18 a) the parallel admittance appears as a series element $K_1^2 Y_p$ at the left side of the port, and as a series element $K_2^2 Y_p$ at the right side of the port. In figure 7.18 c), the impedance level $\sqrt{L_1/C_1}$ of the resonator is changed to $\sqrt{L_o/C_o}$ by changing the impedance of the inverters form K' to K, where K is chosen as:

$$\frac{K'^2}{\sqrt{\frac{L_1}{C_1}}} = \frac{K^2}{\sqrt{\frac{L_o}{C_o}}} = \frac{K'^2}{\omega_o L_1} = \frac{K^2}{\omega_o L_o} \quad \Rightarrow \quad K = K' \sqrt{\frac{L_o}{L_1}} \tag{7. 134}$$

From the terminals, the new circuit is equivalent to the old one. A similar transformation is shown in figure 7.18d).

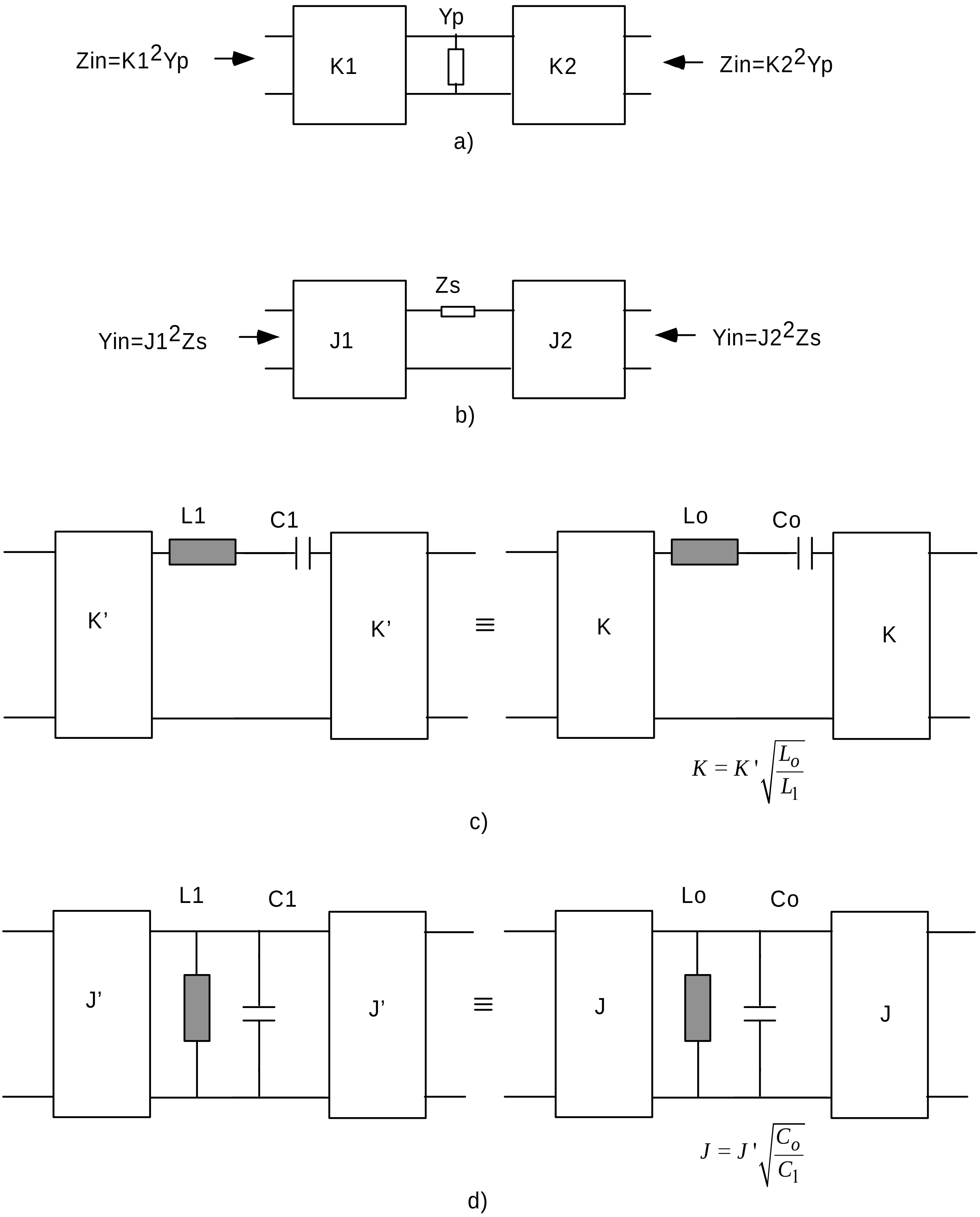



*Fig. 7.18*

## *7.4.5 Design of band pass filter using quarter –wave resonators*

It can be shown that a short-circuited stub which has a length of a quarter wavelength looks like a parallel resonant circuit (figure 7.19)

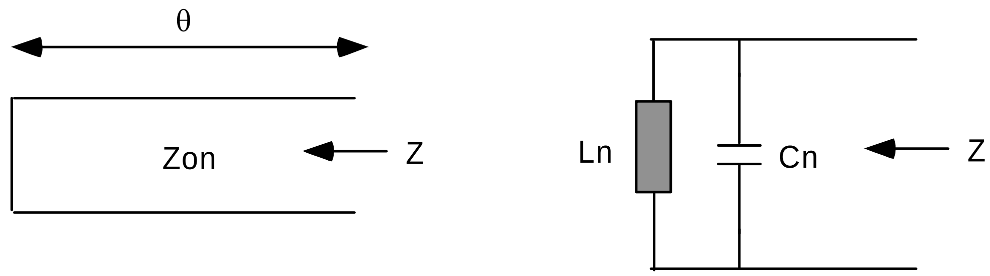



*Fig. 7.19: Quarter-wave resonator*

Thus, quarter-wave short circuited transmission line stubs can be used as the shunt parallel LC resonators for band pass filters. Quarter-wavelength connecting lines between the stubs will act as admittance inverters, converting shunt stubs to series resonators. Such a filter is depicted in figure 7.21, where both the stub length and line length is $\theta=\lambda/4$ for the centre frequency of the pass band $\omega_o$. The characteristic impedance of the connecting lines is Zo, the impedance of the filter.

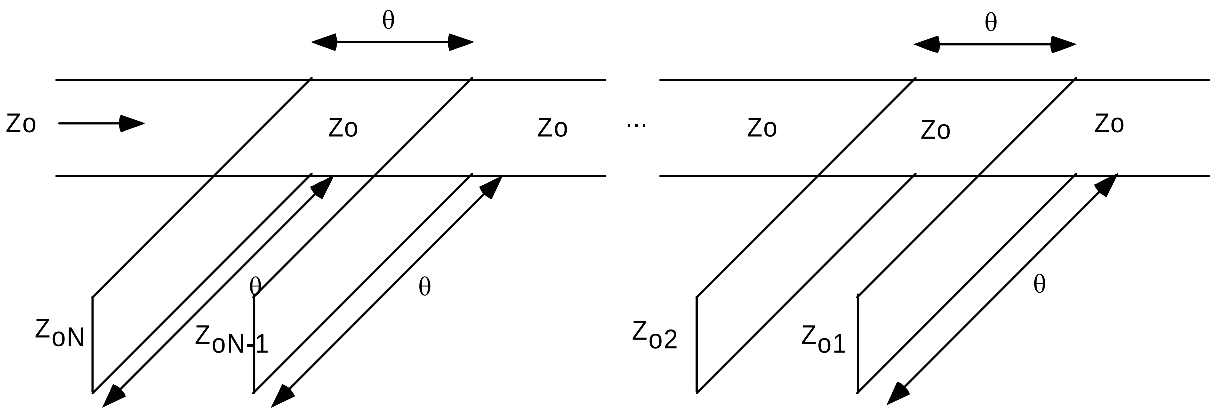



*Fig. 7.20 : Band pass filter using shunt short-circuited quarter-wave resonators*

For a narrow band filter, the response of such a filter using N stubs is essentially the same than that of a lumped element filter of order N.

Design example:

A certain band pass characteristic leads to the lumped element filter of figure 7.21, and we want to find its equivalent in the distributed form of figure 7.20

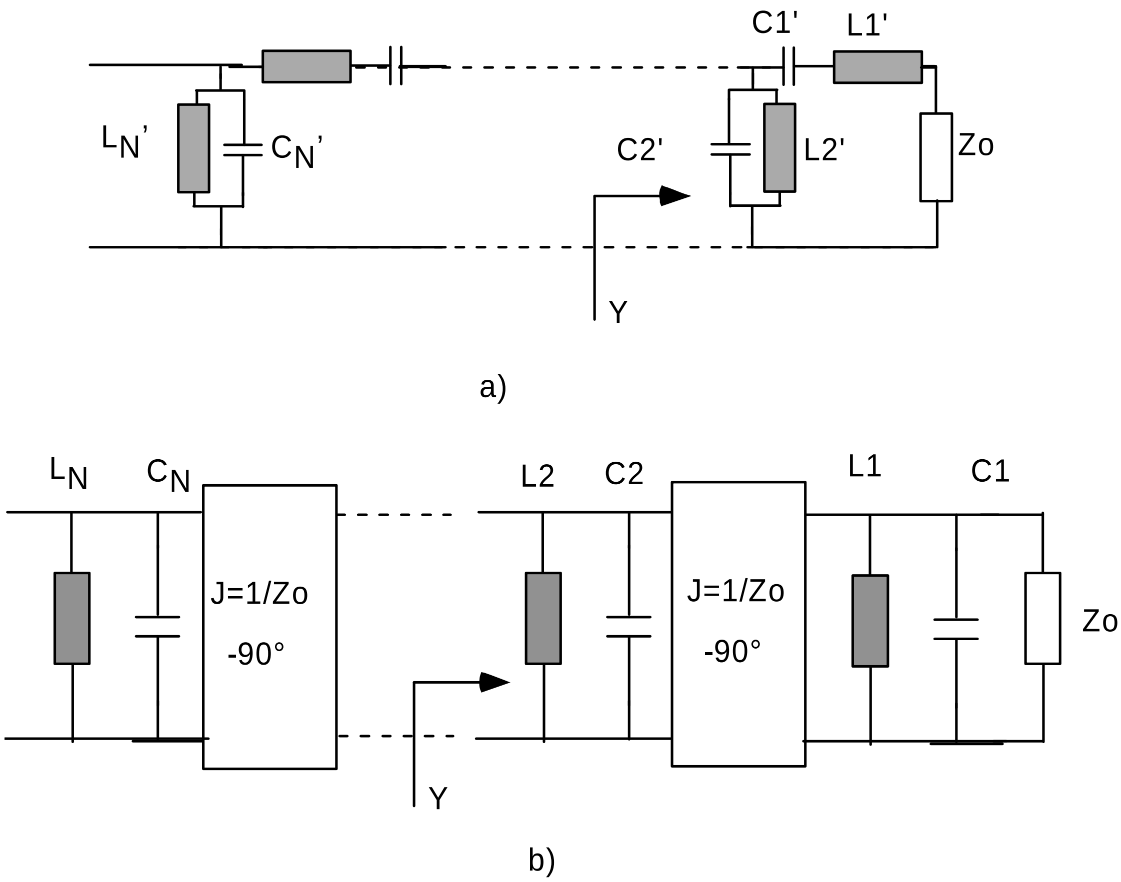



*Fig. 7.21*

The equivalent circuit of a short-circuited transmission line stub can be approximated as a parallel LC resonator when its length is near 90°. The input admittance of a short-circuited transmission line of characteristic impedance $Z_{on}$ is:

$$Y = \frac{-j}{Z_{on}} \cot\theta \qquad (7.\ 135)$$

where $\theta=\pi/2$ for $\omega=\omega_o$. If we let $\omega=\omega_o+\Delta\omega$, where $\Delta\omega<<\omega_o$, then $\theta = \frac{\pi}{2}\left(1+\frac{\Delta\omega}{\omega_o}\right)$, which allows the admittance to be approximated as

$$Y = \frac{-j}{Z_{on}} \cot\left(\frac{\pi}{2}+\frac{\pi\Delta\omega}{2\omega_o}\right) = \frac{-j}{Z_{on}} \tan\frac{\pi\Delta\omega}{2\omega_o} \simeq \frac{j\pi\Delta\omega}{2Z_{on}\omega_o} \qquad (7.\ 136)$$

for frequencies in the vicinity of the centre frequency. The admittance near resonance of the parallel LC network of figure 7.19 can be approximated as:

$$
\begin{aligned}
Y &= j\omega C_n + \frac{1}{j\omega L_n} = j\sqrt{\frac{C_n}{L_n}}\left(\omega\sqrt{C_n L_n} - \frac{1}{\omega\sqrt{C_n L_n}}\right) \\
&= j\sqrt{\frac{C_n}{L_n}}\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right) \simeq 2jC_n\Delta\omega
\end{aligned} \qquad (7.\ 137)
$$

where $C_n L_n = 1/\omega_o^2$ . Equating (7.36) and (7.37) gives the characteristic impedance of the transmission line stub in terms of the resonator parameters as:

$$
Z_{on} = \frac{\pi\omega_o L_n}{4} = \frac{\pi}{4\omega_o C_n} \qquad (7.\ 138)
$$

Next, we consider the quarter-wave sections of line between the stubs as ideal admittance inverters, with J=1/$Z_o$. Then, the band pass filter of figure 7.20 can be represented by the equivalent circuit of figure 7.21 b), which in turn is equivalent to the circuit of 7.21a). Thus, with reference to the terminated (by $Z_o$) circuit of figure 7.21 b), the admittance Y seen towards the load is given by:

$$
\begin{aligned}
Y &= j\omega C_2 + \frac{1}{j\omega L_2} + \frac{1}{Z_o^2}\left[j\omega C_1 + \frac{1}{j\omega L_1} + \frac{1}{Z_o}\right]^{-1} \\
&= j\sqrt{\frac{C_2}{L_2}}\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right) + \frac{1}{Z_o^2}\left[j\sqrt{\frac{C_1}{L_1}}\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right) + \frac{1}{Z_o}\right]
\end{aligned} \qquad (7.\ 139)
$$

where we have used the fact that $L_1 C_1 = L_2 C_2 = 1/\omega^2$ . The admittance at the corresponding point of the circuit given in 7.21 a) can be found as:

$$
\begin{aligned}
Y &= j\omega C'_2 + \frac{1}{j\omega L'_2} + \frac{1}{Z_o^2}\left[j\omega L'_1 + \frac{1}{j\omega C'_1} + \frac{1}{Z_o}\right]^{-1} \\
&= j\sqrt{\frac{C'_2}{L'_2}}\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right) + \frac{1}{Z_o^2}\left[j\sqrt{\frac{L'_1}{C'_1}}\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right) + \frac{1}{Z_o}\right]
\end{aligned} \qquad (7.\ 140)
$$

The two results are equivalent for all frequencies if:

$$\sqrt{\frac{C_2}{L_2}} = \sqrt{\frac{C'_2}{L'_2}} \qquad and \qquad Z_o^2\sqrt{\frac{C_1}{L_1}} = \sqrt{\frac{L'_1}{C'_1}} \tag{7. 141}$$

We use the fact that $L'_1 C'_1 = L'_2 C'_2 = 1/\omega^2$ and solve these equations for $L_1$ and $L_2$, obtaining:

$$L_1 = \frac{Z_o^2}{\omega_o^2 L'_1} \quad ; \quad L_2 = L'_2 \tag{7. 142}$$

We then use (7.38) and (7.19) to obtain the impedance of the two stubs:

$$\begin{aligned} Z_{o1} &= \frac{\pi\omega_o L_1}{4} = \frac{\pi Z_o^2}{4\omega_o L'_1} = \frac{\pi Z_o \Delta}{4g(1)} \\ Z_{o2} &= \frac{\pi\omega_o L_2}{4} = \frac{\pi Z_o^2}{4\omega_o L'_2} = \frac{\pi Z_o \Delta}{4g(2)} \end{aligned} \tag{7. 143}$$

By extension, we can show that

$$Z_{on} = \frac{\pi\omega_o L_n}{4} = \frac{\pi Z_o^2}{4\omega_o L'_n} = \frac{\pi Z_o \Delta}{4g(n)} \tag{7. 144}$$

These results apply only to filters having input and output impedances of Zo, and so cannot be used for Chebyshev filters with an even number of stages.

### *7.4.6 Design of band pass filter using capacitive coupled quarter –wave resonators*

The filters described in 7.4.5 have often very low line impedances, which render them hardly suitable for use in microstrip technology. a related type of bandpass filter is proposed in figure 7.22. In this topology, short circuited shunt resonators are capacitive coupled using series capacitors. An N order filter will use N stubs, which are slightly shorter than λ/4 at the filter centre frequency.

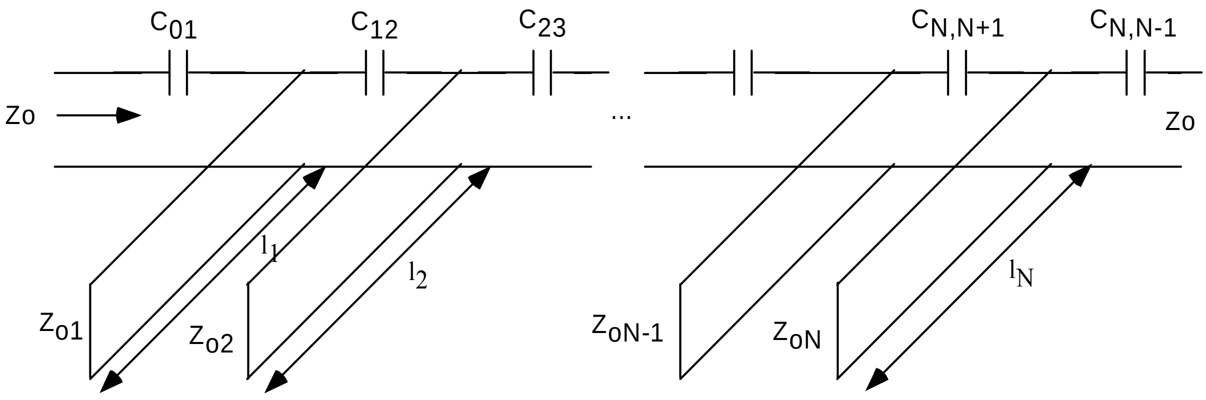



*Fig. 7.22: a bandpass filter using capacitive coupled quarter wave resonators*

Usually, the short-circuited stub resonators are made using sections of ceramic loaded coaxial lines, resulting in a compact design. To design this type of filter, we begin with the general bandpass circuit of figure 7.23 a. In this design, shunt LC resonators alternate with admittance inverters, which convert shunt resonators to series resonators. The extra inverters at the ends serve to scale the impedance level of the filter to a realistic level. Using an analysis similar to the one of § 7.4.5, the admittance inverters constants can be derived as (the complete derivation of the equations can be found in: G.L. Matthaei, L. Young and E.M.T. Jones, "Microwave Filters, Matching Networks and Coupling Structures", Artech House, 1980.) :

$$Z_0 J_{01} = \sqrt{\frac{\pi\Delta}{4g_1}} \tag{7.145}$$

$$Z_0 J_{n,n+1} = \frac{\pi\Delta}{4\sqrt{g_n g_{n+1}}} \tag{7.146}$$

$$Z_0 J_{N,N+1} = \sqrt{\frac{\pi\Delta}{4 g_N g_{N+1}}} \tag{7.147}$$

Similarly, the coupling capacitor values can be found as :

$$C_{01} = \frac{J_{01}}{\omega_o \sqrt{1-\left(Z_0 J_{01}\right)^2}} \tag{7.148}$$

$$C_{n,n+1} = \frac{J_{n,n+1}}{\omega_o} \tag{7.149}$$

$$C_{N,N+1} = \frac{J_{N,N+1}}{\omega_o \sqrt{1-\left(Z_0 J_{N,N+1}\right)^2}} \tag{7.150}$$

The admittance inverters are in a second step replaced by their equivalent π network (figure 7.23b). The capacitors of the equivalent network are negative, and they are combined in parallel with the capacitor of the LC resonator to yield a positive capacitor. The resulting circuit is depicted in figure 7.23c. The effective resonator capacitor values are given by:

$$C'_n = C_n + \Delta C_n = C_n - C_{n-1,n} - C_{n,n+1} \tag{7. 151}$$

Finally, the shunt LC resonators are replaced with short-circuited transmission stubs, as shown in figure 7.22. The resonant frequency of the stubs are no longer $\omega_o$, because the value of the capacitors was corrected into to take into account the impedance inverters. This implies that the length of the resonators is less than the quarter of a wavelength at $\omega_o$, the filter centre frequency. The transformation of the stub's length to take into account for the change of capacitance is shown on figure 7.23d. A short circuited length of line with a shunt capacitor at its input has an admittance of:

$$Y = Y_L + j\omega_o C \ \ , Y_L = \frac{-j}{Z_o}\cot(\beta l) \tag{7. 152}$$

We can replace the capacitor with a short length of line Δl, and obtain the following input admittance:

$$Y = \frac{1}{Z_o}\frac{Y_L + j\frac{1}{Z_o}\tan\beta\Delta l}{\frac{1}{Z_o} + jY_L\tan\beta\Delta l} \cong Y_L + j\frac{\beta\Delta l}{Z_o} \tag{7. 153}$$

the approximation being valid for $\beta\Delta l \ll 1$ . From (7.52) and (7.53), we can get the final length of the stub:

$$l_n = \frac{\lambda}{4} + \left(\frac{Z_o\omega_o\Delta C_n}{\beta}\right) = \frac{\lambda}{4} + \left(\frac{Z_o\omega_o\Delta C_n}{2\pi}\right)\lambda \tag{7. 154}$$

where $\Delta C_n$ is defined in (7.61).

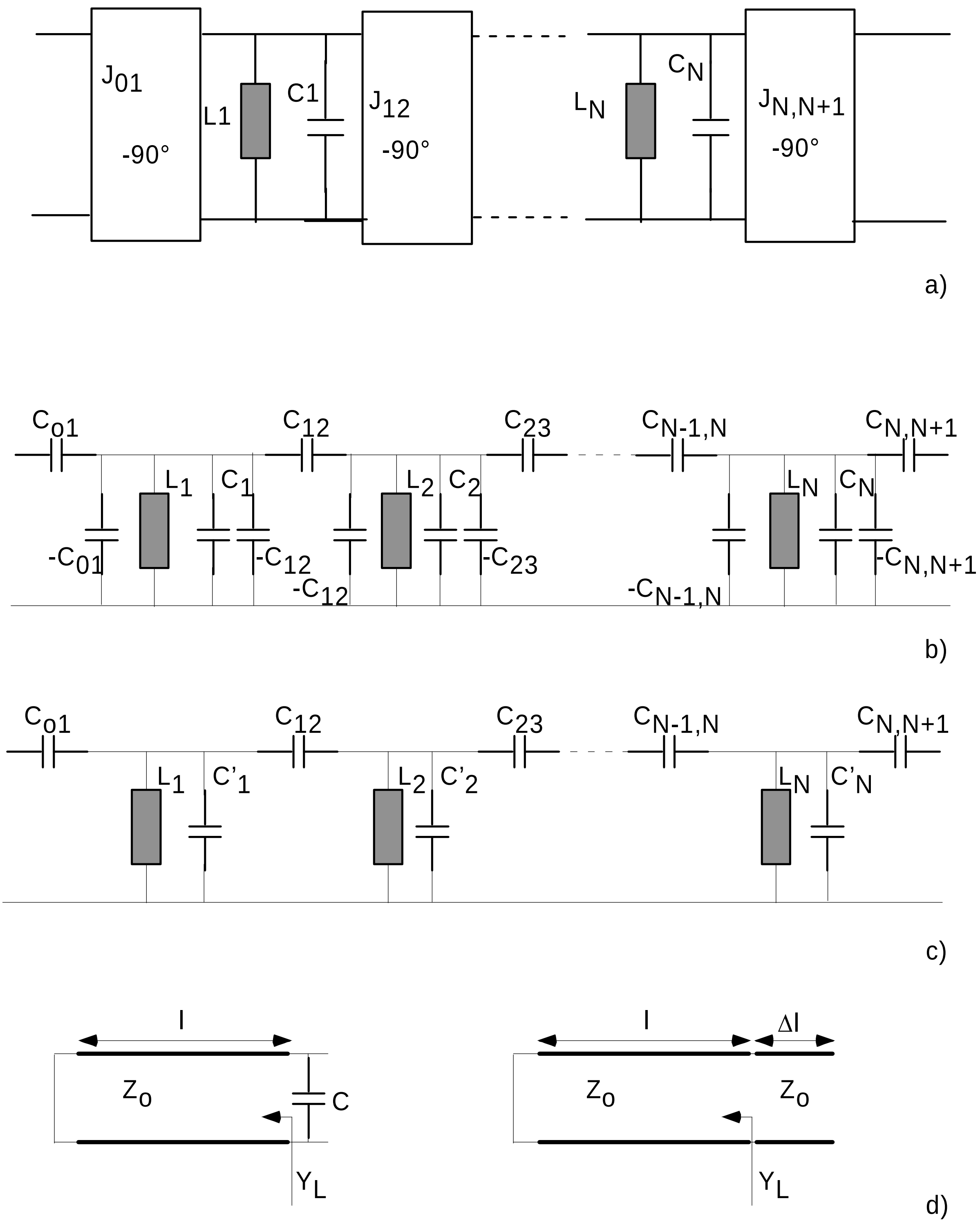



*Fig. 7.23: Bandpass filters using capacitive coupled quarter wave resonators.*

### *7.4.7 Parallel-coupled transmission line resonator filters*

The band pass filter described above are unfortunately not well suited for a realization using microwave printed circuit technologies (microstrip lines, striplines or co-planar waveguides), due either to the low characteristic impedances of the involved transmission line sections, to the presence of short circuited stub and lumped capacitors which are always cumbersome to realize or integrate in printed circuit technology. This is why for these technologies, yet another bandpass filter topology is preferred: the Parallel coupled transmission line resonator filter. The topology of such a structure id depicted in figure 7.24.

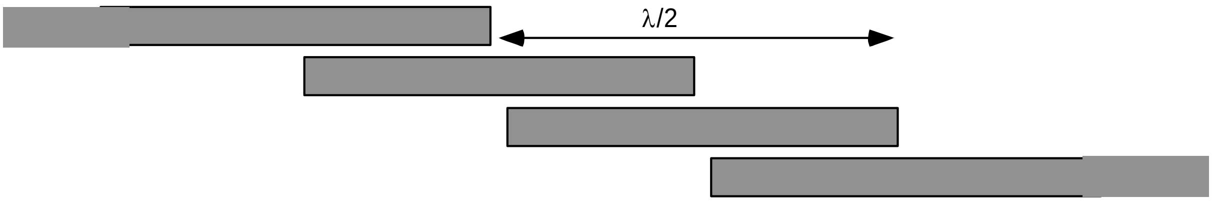



*Fig. 7.24: parallel coupled resonator filter*

It consists N (N is the degree of the filter) sections of resonators of length approximately equal to λ/2, which are cascaded through parallel coupling. An approximate equivalent circuit of the half wave resonators is given in figure 7.25.

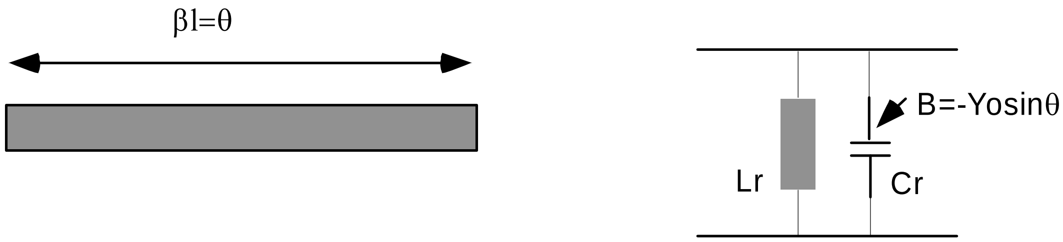



*Fig. 7.25: equivalent circuit of half wave resonator*

Moreover, it can be shown (the details of the analysis are beyond the scope of this course, but can be found in : S.B. Cohn, "Parallel  Coupled Transmission Line Resonator Filters", IRE Transactions on Microwave Theory and Techniques, MTT-6, nr 2, April 1958), that an appropriate equivalent circuit for a pair of open circuited parallel coupled lines having a length of λ/4 is given in figure 7.26, with the following conditions, which are valid for a narrow bandwidth :

$$\frac{Z_{oe}}{Z_o} = 1 + \frac{Z_o}{K} + \frac{Z_o^2}{K^2}$$
$$\frac{Z_{oo}}{Z_o} = 1 - \frac{Z_o}{K} + \frac{Z_o^2}{K^2} \qquad (7.\ 155)$$

where Zoe is the characteristic impedance of the even mode of the coupled lines while Zoo is the characteristic impedance of the odd mode of the coupled lines.

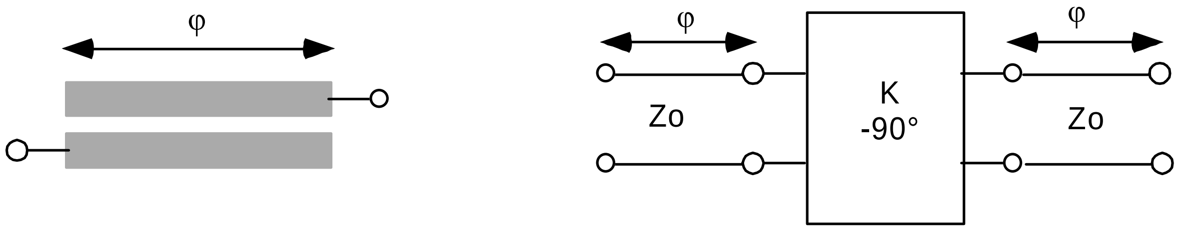



*Fig. 7.26 : Equivalent circuit of quarter wave coupled lines*

Thus, an approximate equivalent circuit of the filter shown in figure 7.24 is depicted in figure 7.27. The lumped element equivalent is shown in figure 7.27, which is valid for a narrow bandwidth close to the resonance.

From this lumped element equivalent circuit, we finally get the rules linking the prototype filter to the printed circuit :

$$\frac{Z_o}{K_{i-1,i}} = \pi\left(\frac{f_2 - f_1}{f_2 + f_1}\right)\sqrt{\frac{1}{g_{i-1,i}}}$$

$$Z_{o\,e\,i} = Z_o\left\{1+1+\frac{Z_o}{K_{i-1,i}}+\frac{Z_o^2}{K_{i-1,i}{}^2}\right\} \qquad (7.\ 156)$$

$$Z_{ooi} = Z_o\left\{1-\frac{Z_o}{K_{i-1,i}}+\frac{Z_o^2}{K_{i-1,i}{}^2}\right\}$$

where $f_2$ is the upper edge of the passband and $f_1$ the lower edge of the passband.

The even and odd impedances of the coupled lines are linked to the width of the strips and the gab between lines by transcendental approximate formulas, which have to be solved numerically (see M. KIRSCHNING AND R. H. JANSEN, Accurate Wide-Range Design Equations for the Frequency-Dependent Characteristic of Parallel Coupled Microstrip Lines IEEE TRANSACTIONS ON MICROWAVE THEORY AND TECHNIQUES, VOL. MTT-32, NO 1, JANUARY1984).

The length of the lines is obtained by computing the effective permittivity of the coupled lines, and knowing that each coupled line section has a length of a quarter wavelength.