The EKV Normalized Functions

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1 Introduction

This document presents the main normalized functions of the EKV model (sEKV) that can be used used for design.

The document is written in Quarto in order to check whether different features like equation numbering and cross-referencing, figures numbering and cross-referencing work correctly. It also checks whether exported documents in HTML and pdf are rendered correctly.

2 Large-signal functions

2.1 Normalized current versus charge

2.1.1 Long-channel

The normalized drain current or inversion coefficient IC is defined as the drain current in saturation normalized to the specific current I_{spec}

$$IC \triangleq \frac{I_D|_{\text{saturation}}}{I_{spec}}, \tag{1}$$

where the specific current I_{spec} is given by

$$I_{spec} = I_{spec\square} \cdot \frac{W}{L} \tag{2}$$

with

$$I_{spec\square} \triangleq 2n \cdot \mu \cdot C_{ox} \cdot U_T^2. \tag{3}$$

The inversion coefficient IC defined in (1) gives the level of inversion of the transistor according to

$$\begin{aligned} IC &< 0.1 && \text{weak inversion (WI)}, && (4)\\ 0.1 \leq IC &< 10 && \text{moderate inversion (MI)}, && (5)\\ 10 &\leq IC && \text{strong inversion (SI)}. && (6) \end{aligned}$$

The inversion coefficient for a long-channel transistor is a function of the normalized source charge q_s according to

$$IC = q_s^2 + q_s = q_s \cdot (q_s + 1), \tag{7}$$

where q_s is the inversion charge Q_i evaluated at the source and normalized to $Q_{spec} \triangleq -2nC_{ox}U_T$

$$q_s \triangleq \frac{Q_i(x=0)}{Q_{spec}}. \tag{8}$$

Expression (7) can be inverted to express the normalized charge as a function of the inversion coefficient according to

$$q_s = \frac{\sqrt{4IC+1}-1}{2}. \tag{9}$$

The normalized current or inversion coefficient IC given by (7) is plotted versus q_s in Figure 1 (curve corresponding to $\lambda_c = 0$).

2.1.2 Short-channel

The normalized drain current in saturation or inversion coefficient in the simplified EKV (sEKV) model is given by

$$IC \triangleq \frac{I_D|_{saturation}}{I_{spec}} = \frac{2(q_s^2 + q_s)}{1 + \sqrt{1 + \lambda_c^2(q_s^2 + q_s)}}, \tag{10}$$

where parameter λ_c is the velocity saturation parameter which scales as

$$\lambda_c = \frac{L_{sat}}{L} \tag{11}$$

where $L_{sat} = 2\mu U_T / v_{sat}$ is the length over which the carriers velocity is saturating to v_{sat}. λ_c is therefore the fraction of the channel over which the carriers are in full velocity saturation. The long-channel case is obtained by setting $\lambda_c = 0$ in (10) which reduces to (7).

The short-channel asymptotes are given by

$$IC \cong \begin{cases} q_s & \text{in weak inversion } (\lambda_c \cdot q_s \ll 1), \\ \frac{2q_s}{\lambda_c} & \text{in strong inversion } (\lambda_c \cdot q_s \gg 1). \end{cases} \tag{12}$$

The normalized current or inversion coefficient IC given by (7) is plotted versus q_s in Figure 1 (curve corresponding to $\lambda_c = 0.5$).

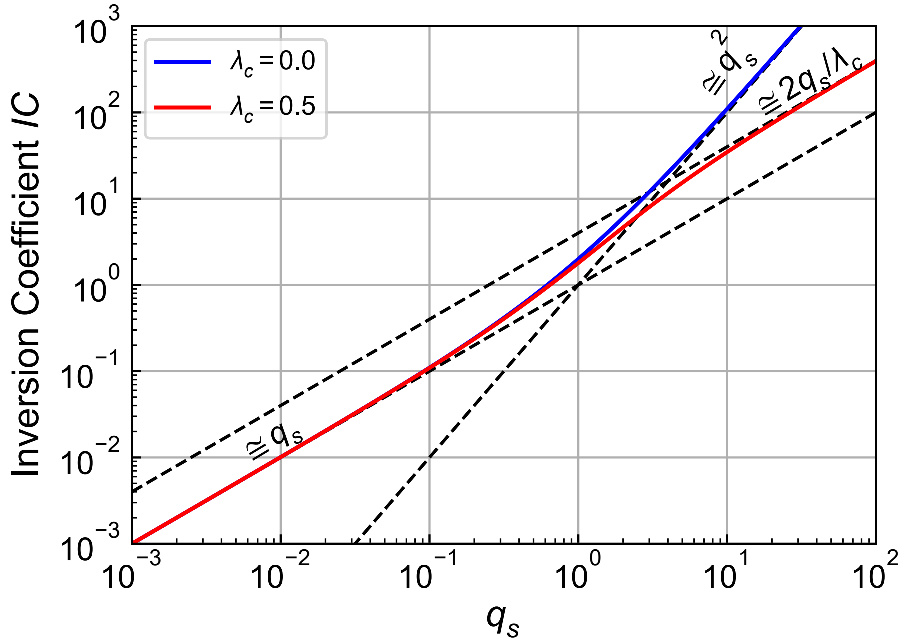

Figure 1: Normalized drain current or inversion coefficient IC versus normalized charge q_s.

2.2 Normalized charge versus current

2.2.1 Long-channel

The long-channel expression of the inversion coefficient (7) can be inverted to express the normalized charge q_s in function of the inversion coefficient IC according to

$$q_s = \frac{\sqrt{4\,IC + 1} - 1}{2}. \tag{13}$$

Equation (7) plotted in Figure 2 (curve corresponding to $\lambda_c = 0$).

2.2.2 Short-channel

Equation (10) can also be inverted to express the normalized charge q_s in function of the inversion coefficient IC according to

$$q_s = \frac{\sqrt{4\,IC + 1 + (\lambda_c\,IC)^2} - 1}{2} \tag{14}$$

Setting $\lambda_c = 0$ we get the long-channel expression (13)

$$q_s = \frac{\sqrt{4\,IC + 1} - 1}{2}. \tag{15}$$

Equation (14) is plotted in Figure 2 (curve corresponding to $\lambda_c = 0.5$).

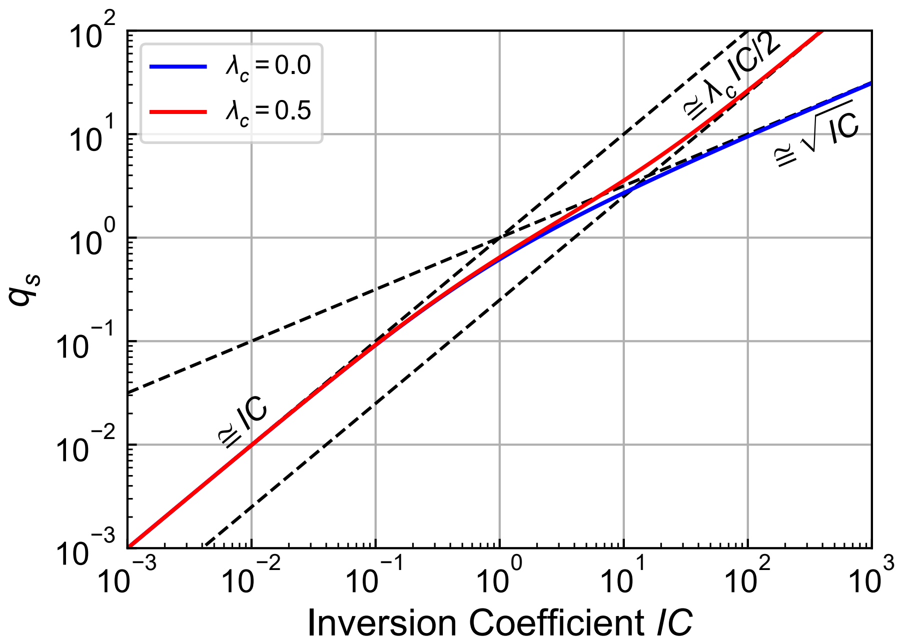

Figure 2: Normalized charge q_s versus inversion coefficient IC.

2.3 Normalized saturation voltage versus charge

The voltage is related to the charge according to

$$v_p - v_s = 2q_s + \ln(q_s) \tag{16}$$

which is plotted in Figure 3.

2.4 Normalized charge versus saturation voltage

The voltage versus charge equation (16) can actually be inverted using the Lambert W-function of order 0. The Lambert function $W(z)$ is defined as the function satisfying

$$W(z) \cdot e^{W(z)} = z. \tag{17}$$

The voltage versus charge equation can be written as

$$2q \cdot e^{2q} = e^v \tag{18}$$

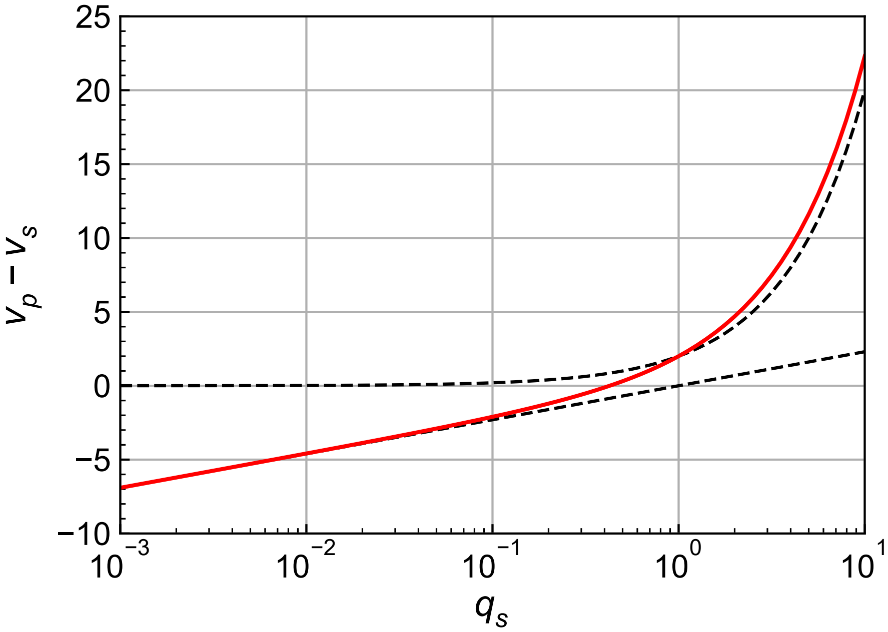

Figure 3: Saturation voltage versus normalized charge q_s.

where $q \triangleq q_s$ and $v \triangleq v_p - v_s$. Equation (18) can now be solved for q using the Lambert W-function by setting $z = 2e^v$ which leads to

$$q(v) = \frac{1}{2} W\left(2e^{v}\right) \tag{19}$$

or

$$q_s = \frac{1}{2} W\left(2e^{v_p - v_s}\right). \tag{20}$$

The Lambert W-function is available in the *scipy* Python package as *lambertw*. It is also available in Mathematica as the ProductLog[z] function.

The charge versus voltage can also be approximated by the EKV function.

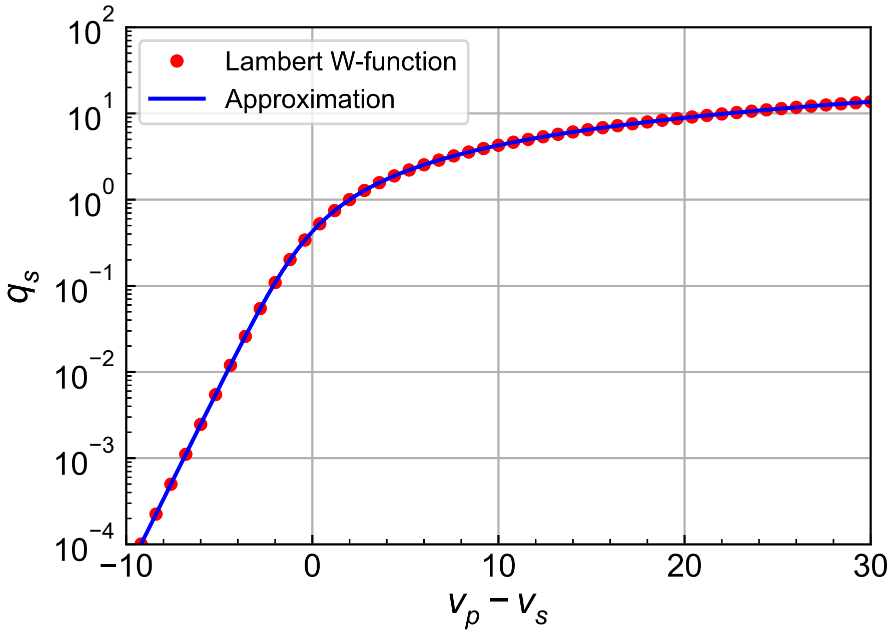

Figure 4: Normalized charge q_s versus $v__p-v__2.

2.5 Inversion coefficient versus saturation voltage

Having expressed the charge q_s versus the saturation voltage $v_p - v_s$, we can now express the inversion coefficient in terms of the saturation voltage using (7) for long-channel or (10) for short-channel with (20). This results in the plots shown in Figure 5.

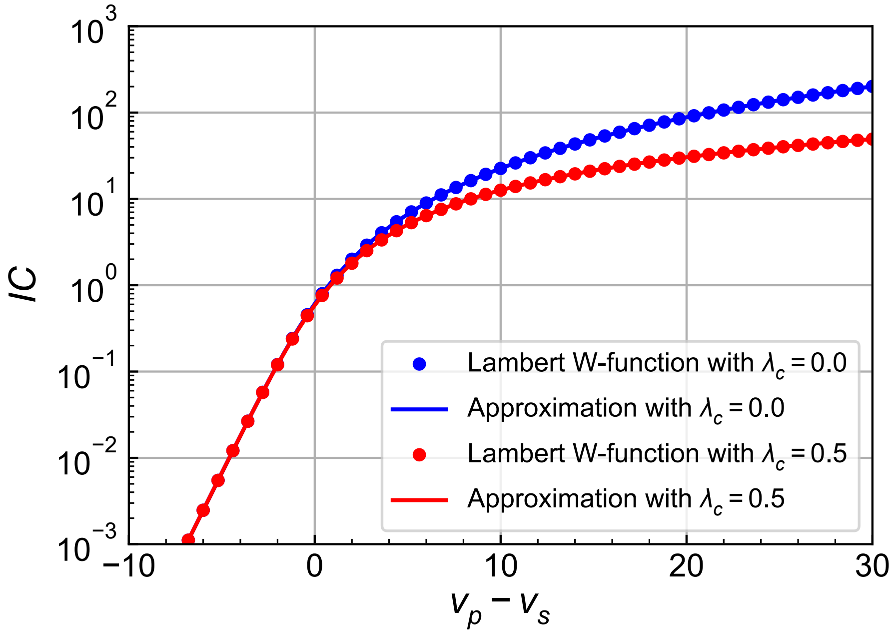

Figure 5: Normalized current or inversion coefficient IC versus $v_p - v_s$.

2.6 Drain-to-source saturation voltage versus inversion coefficient

The saturation voltage $v_p - v_s$ only gives an estimation of the saturation voltage in strong inversion. In weak inversion, it becomes negative instead of saturating to about $4\,U_T$. We can use the following function to estimate the drain-to-source saturation voltage v_{dssat} from weak to strong inversion

$$v_{dssat} = 2\sqrt{IC + v_{dssat,wi}}, \tag{21}$$

where $v_{dssat,wi} = 4$ is the asymptote of the saturation voltage in weak inversion (it should be a few U_T). Equation (21) is plotted in Figure 6.

2.7 Inversion coefficient versus drain-to-source saturation voltage

In the design process, we might need to set the minimum drain-to-source saturation voltage and want to deduce the corresponding inversion coefficient. This is done by inverting (21) resulting in

$$IC = \left(\frac{v_{dssat}}{2}\right)^2 - v_{dssat,wi}, \tag{22}$$

Equation (22) is plotted in Figure 7.

Note that if the inversion coefficient gets to zero, it simply means that the voltage available for biasing the transistor in sturation is not large enough. In other words the limit $4U_T$ is incompressible and if the available voltage is lower it means there is not enough voltage for biasing the transistor in saturation. In reality, the transistor will actually move into the linear region.

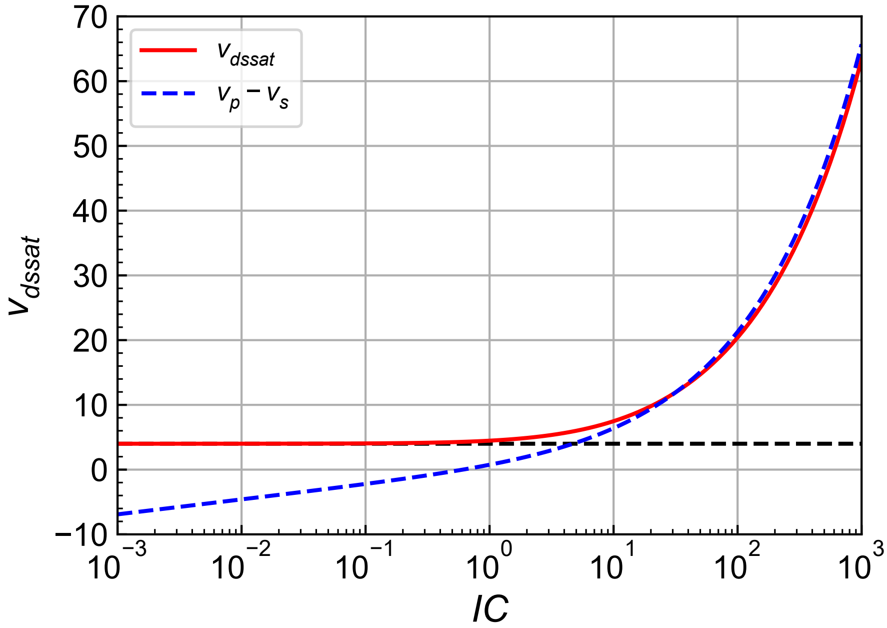

Figure 6: Saturation voltage v_{dssat} versus IC.

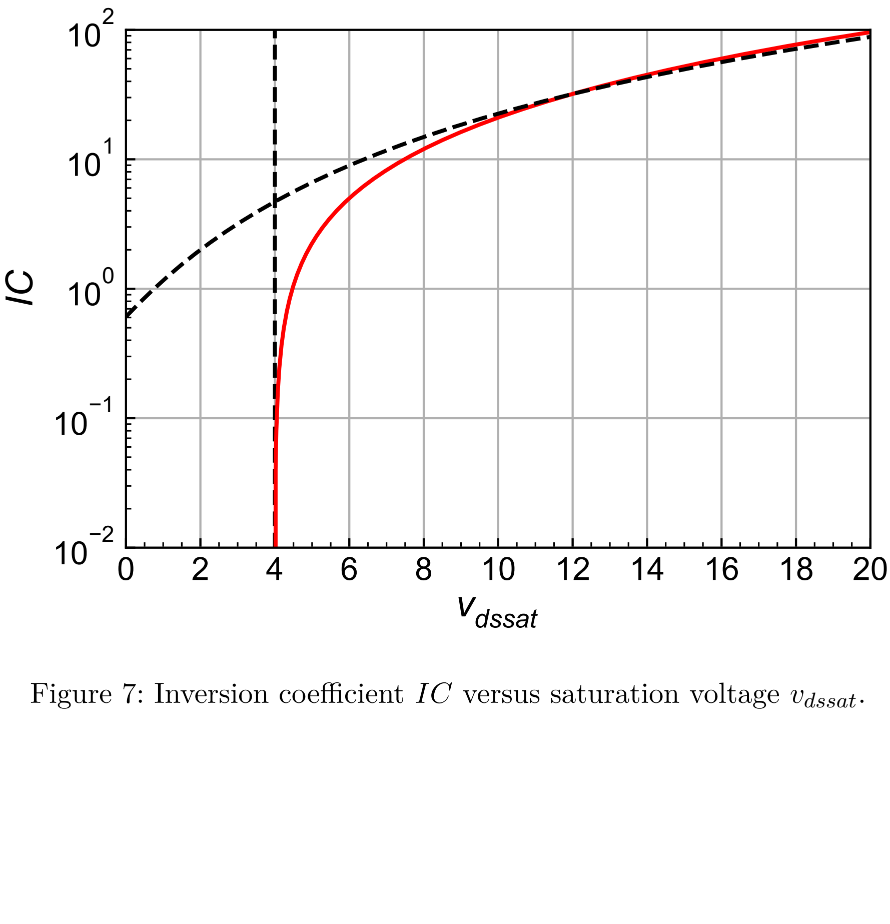

Figure 7: Inversion coefficient IC versus saturation voltage v_{dssat}.

2.8 Slope factor versus inversion coefficient

The slope factor n in weak inversion is actually depending on the pinch-off voltage according to

$$n = 1 + \frac{\Gamma_b}{2\sqrt{\Psi_0 + V_P}} = 1 + \frac{\gamma_b}{2\sqrt{\psi_0 + v_p}} \tag{23}$$

where Γ_b is the substrate factor given by

$$\Gamma_b = \gamma_b \cdot \sqrt{U_T} = \frac{\sqrt{2qN_b\epsilon_{Si}}}{C_{ox}} \tag{24}$$

and $\Psi_0 \cong 2\Phi_F +$ a few U_T. The slope factor normally depends on the pinch-off voltage which depends on the gate voltage. For $v_s = 0$, we can express v_p as a function of IC and plot the slope factor versus IC. From Figure 8 we see that the slope factor does not change dramtically and we can approximate it by its value in moderate inversion for $v_p = 0$ or $IC \cong 1$

$$n_0 = 1 + \frac{\Gamma_b}{2\sqrt{\Psi_0}} = 1 + \frac{\gamma_b}{2\sqrt{\psi_0}}. \tag{25}$$

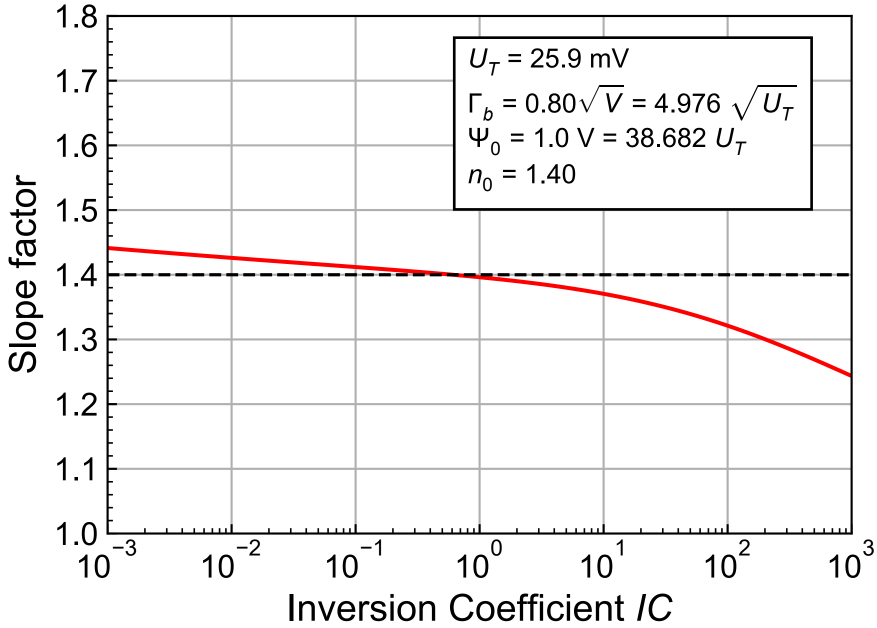

Figure 8: Slope factor n versus inversion coefficient IC.

3 Small-signal functions

3.1 Transconductance versus inversion coefficient

3.1.1 Long-channel

It can be shown that for a long-channel transistor the source transconductance G_{ms} is actually proportional to the inversion charge taken at the source $Q_i(x = 0)$. The normalized source transconductance can therefore be expressed in terms of the inversion coefficient according to

$$g_{ms} \triangleq \frac{G_{ms}}{G_{spec}} = q_s(IC) = \frac{\sqrt{4IC + 1} - 1}{2}, \tag{26}$$

where the specific conductance G_{spec} is defined as

$$G_{spec} = \frac{I_{spec}}{U_T} = 2n \cdot \mu \cdot C_{ox} \cdot U_T. \tag{27}$$

3.1.2 Short-channel

If velocity saturation is accounted for, the normalized source transconductance becomes

$$g_{ms} = \frac{\sqrt{4\,IC + 1 + (\lambda_c\,IC)^2} - 1}{2 + \lambda_c^2\,IC}. \tag{28}$$

The normalized source transconductance in weak and in strong inversion reduce to

$$g_{ms} = \begin{cases} IC & \text{in weak inversion } (IC \ll 1), \\ \frac{1}{\lambda_c} & \text{in strong inversion } (IC \gg 1). \end{cases} \tag{29}$$

As shown in Figure 9, we see that the normalized source transconductance saturates to $1/\lambda_c$ in SI.

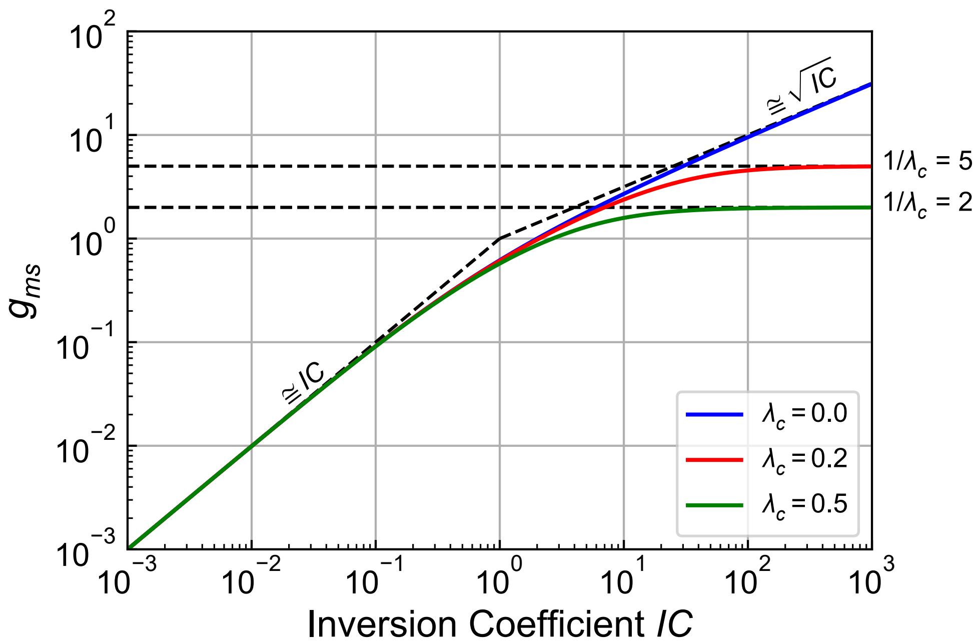

Figure 9: Normalized source transconductance g_{ms} versus inversion coefficient IC.

3.2 Inversion coefficient versus transconductance

3.2.1 Long-channel

For a long-channel transistor, it is easy to invert the normalized source transconductance to express the inversion coefficient IC required to achieve a given normalized source transconductance

$$IC = g_{ms} \cdot (g_{ms} + 1). \tag{30}$$

3.2.2 Short-channel

The formula becomes much more complicated when including velocity saturation. For $\lambda_c > 0$ we get

$$IC = \frac{2 - \lambda_c^2\,g_{ms}\,(1 + 2\,g_{ms}) - \sqrt{4 - (\lambda_c\,g_{ms})^2\,(4 - \lambda_c^2)}}{\lambda_c^2\,((\lambda_c\,g_{ms})^2 - 1)}, \tag{31}$$

which for $\lambda_c \to 0$ reduces to (30).

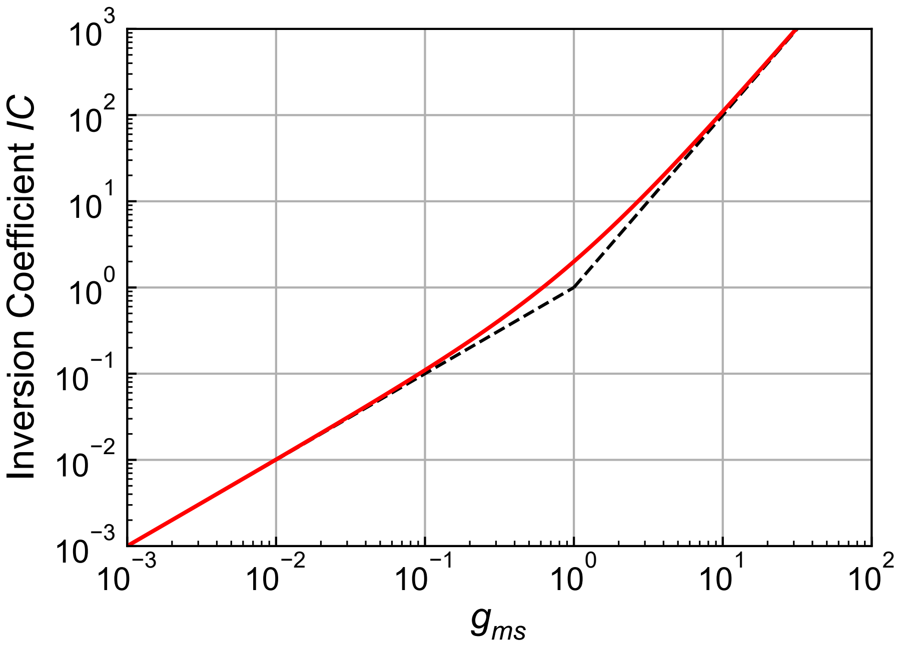

Figure 10: Inversion coefficient IC versus normalized source transconductance g_{ms}.

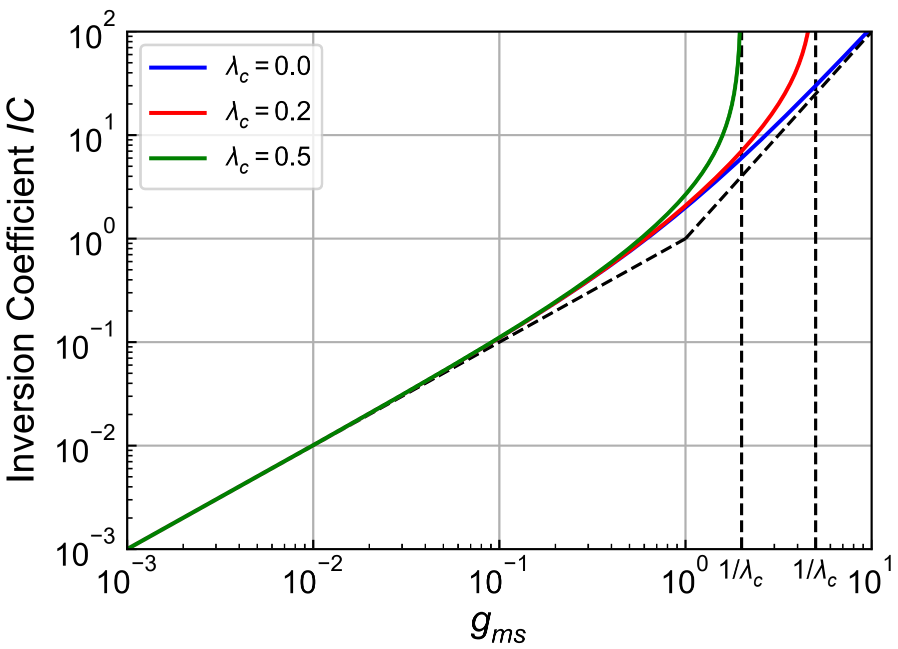

Figure 11: Inversion coefficient IC versus normalized source transconductance g_{ms}.

3.3 Normalized G_m/I_D versus inversion coefficient

3.3.1 Long-channel

The normalized G_m/I_D for the long-channel transistor can then be expressed in terms of the inversion coefficient as

$$\frac{G_m \cdot n\,U_T}{I_D} = \frac{G_{ms} \cdot U_T}{I_D} = \frac{g_{ms}(IC)}{IC} = \frac{\sqrt{4IC+1}-1}{2IC} \tag{32}$$

The normalized G_m/I_D function for a long-channel device is plotted versus IC in Figure 12 (curve with $\lambda_c = 0$).

3.3.2 Short-channel

The normalized G_m/I_D for short-channel devices is given by

$$\frac{G_m \cdot n\,U_T}{I_D} = \frac{G_{ms} \cdot U_T}{I_D} = \frac{g_{ms}}{IC} = \frac{\sqrt{4\,IC+1+(\lambda_c\,IC)^2}-1}{IC\,(2+\lambda_c^2\,IC)}. \tag{33}$$

The normalized source transconductance in strong inversion reduces to

$$\frac{g_{ms}}{IC} = \begin{cases} \frac{1}{\sqrt{IC}} & \text{for long-channel } (\lambda_c = 0), \\ \frac{1}{\lambda_c \cdot IC} & \text{for short-channel } (\lambda_c > 0). \end{cases} \tag{34}$$

The normalized G_m/I_D function for short-channel devices is plotted versus IC in Figure 12.

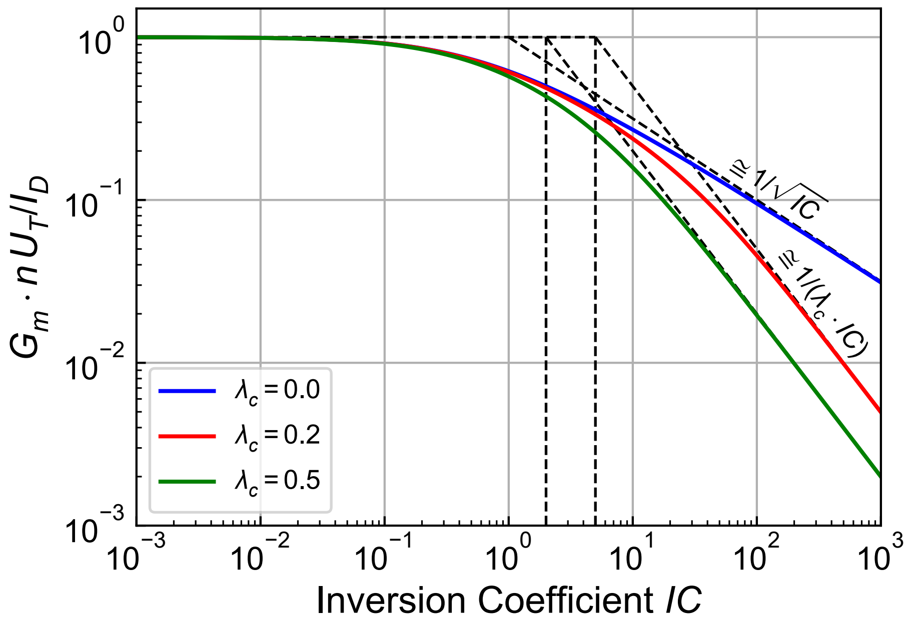

Figure 12: Inversion coefficient IC versus normalized source transconductance g_{ms}.

3.4 Inversion coefficient versus normalized G_m/I_D

For long-channel transistors, the normalized G_m/I_D can be inverted to express the inversion coefficient required to achive a given normalized G_m/I_D. This results in

$$IC = \frac{1 - gmsid}{gmsid^2}, \tag{35}$$

where $gmsid = G_m n U_T / I_D$. Equation (35) is plotted in Figure 13.

There is unfortunately no simple expression that accounts for velocity saturation.

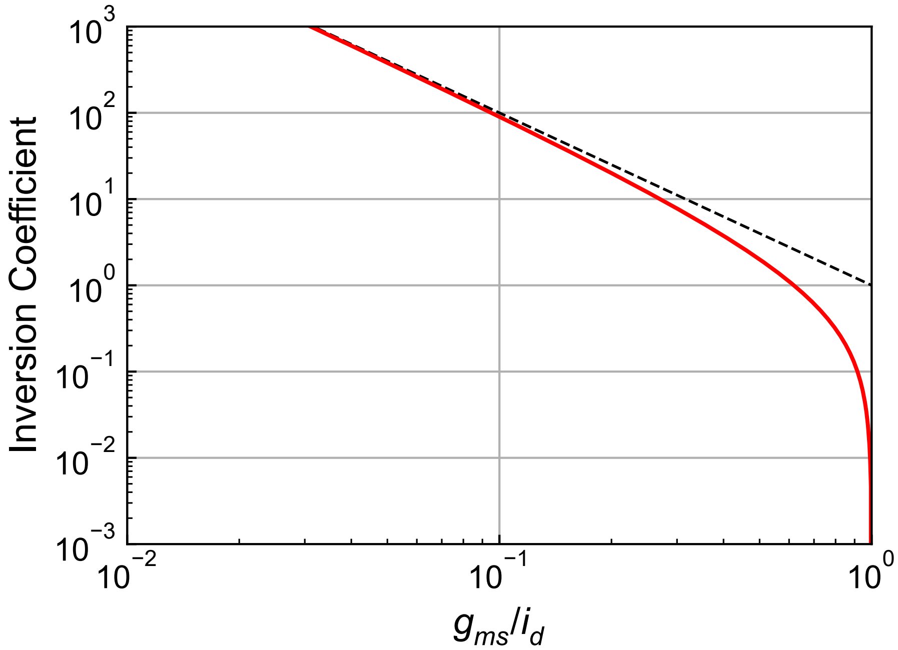

Figure 13: Inversion coefficient IC versus normalized g_{ms}/i_d.

4 Noise functions

4.1 Thermal noise

4.1.1 Long-channel

The thermal noise parameter expresses the ratio of the thermal noise conductance and the channel conductance for $V_{DS} = 0$

$$\delta_n \triangleq \frac{g_n}{g_{ms}} = \frac{g_n}{q_s}, \tag{36}$$

where g_n is the normalized thermal noise conductance

$$g_n \triangleq \frac{G_n}{G_{spec}} = \frac{1}{6} \cdot \frac{4q_s^2 + 3q_s + 4q_s q_d + 3q_d + 4q_d^2}{q_s + q_d + 1}. \tag{37}$$

Replacing (37) in (36) results in

$$\delta_n = \frac{1}{6} \cdot \frac{4q_s^2 + 3q_s + 4q_s q_d + 3q_d + 4q_d^2}{q_s \left(q_s + q_d + 1\right)}. \tag{38}$$

In saturation, δ_n is equal to

$$\delta_n = \frac{2}{3} \cdot \frac{q_s + \frac{3}{4}}{q_s + 1} = \begin{cases} \frac{1}{2} & \text{WI and saturation } (q_s \ll 1) \\ \frac{2}{3} & \text{SI and saturation } (q_s \gg 1). \end{cases} \tag{39}$$

The thermal noise parameter δ_n is plotted versus the inversion coefficient IC in Figure 14.

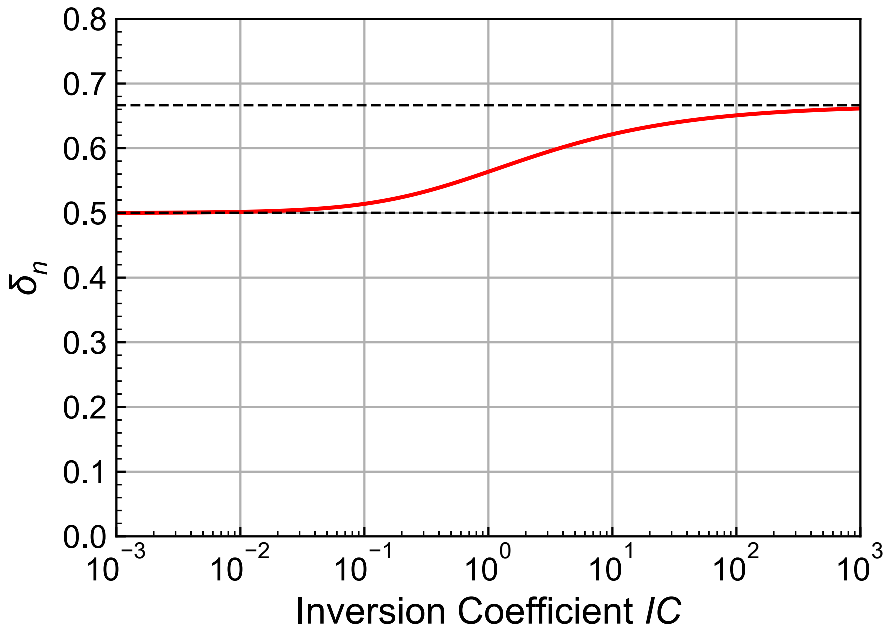

Figure 14: Thermal noise parameter δ_n versus inversion coefficient IC.

Note that the thermal noise parameter δ_n compares the thermal noise conductance evaluated at a given operating point that is not necessarily the same as the one used to define the output conductance G_{dso} (i.e. $v_{DS} = 0$). It is therefore not very useful for circuit design and is used more for modeling purposes.

For circuit design, it is more useful to define another figure-of-merit (FoM) γ_n, named the thermal noise excess factor related to the drain and defined as

$$\gamma_n \triangleq \frac{G_n}{G_m} = \frac{g_n}{g_m} = \frac{n \cdot q_I}{q_s - q_d}. \tag{40}$$

The γ_n FoM represents how much noise is generated at the drain of a transistor for a given gate transconductance. Contrary to the δ_n thermal noise parameter, the noise conductance G_n and the gate transconductance G_m used in the definition (40) are evaluated at the same operating point.

The δ_n thermal noise parameter and the γ_n thermal noise excess factor are obviously related by

$$\gamma_n = \frac{G_n}{G_{dso}} \cdot \frac{G_{dso}}{G_m} = \delta_n \cdot \frac{G_{dso}}{G_m} = \delta_n \cdot n \cdot \frac{G_{dso}}{G_{ms} - G_{md}} = \delta_n \cdot n \cdot \frac{q_s}{q_s - q_d}. \tag{41}$$

In saturation, $G_{md} = 0$ and $q_d = 0$, resulting in

$$\gamma_n = \delta_n \cdot n = \begin{cases} \frac{n}{2} & \text{WI and saturation} \\ \frac{2}{3} \cdot n & \text{SI and saturation,} \end{cases} \tag{42}$$

since $G_{dso} = G_{ms}$. For $n = 1.5$, the thermal noise factor in strong inversion and saturation is approximately equal to unity and the thermal noise conductance is about equal to the gate transconductance $G_n \cong G_m$.

The PSD of the drain current thermal noise fluctuations can then be written in terms of the noise excess factors and the transconductances as

$$S_{\Delta I_D^2} = 4kT \cdot \delta_n \cdot G_{ms} = 4kT \cdot \gamma_n \cdot G_m. \tag{43}$$

The thermal noise excess factor for a long-channel transistor is plotted versus the inversion coefficient in Figure 15. We see that its variation is small and that the thermal noise excess can be conveniently approximated by $\gamma_n \cong 1$.

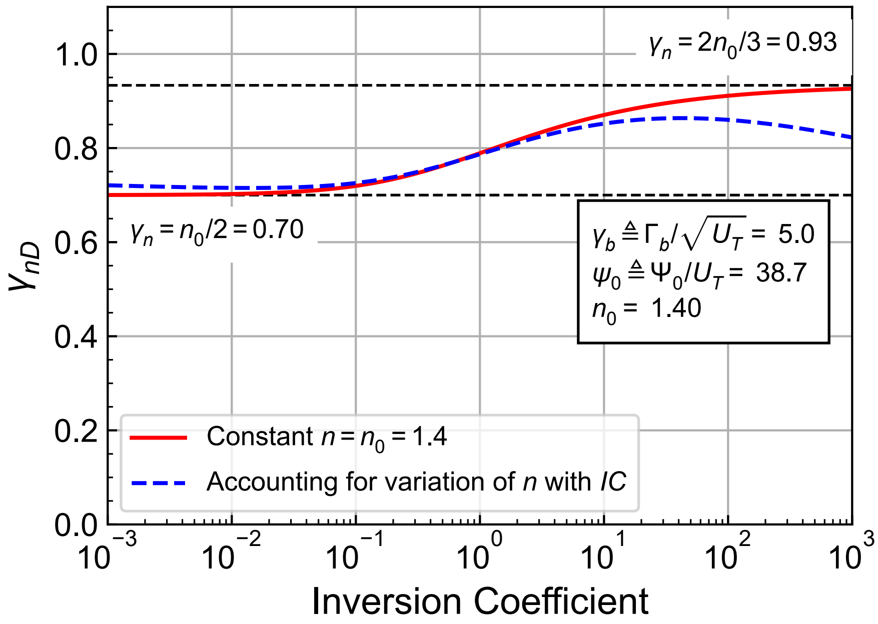

Figure 15: Thermal noise excess parameter γ_n versus inversion coefficient IC.

4.1.2 Short-channel

For short-channel transistors, the thermal noise excess factor increases in strong inversion proportionnally to the inversion coefficient according to

$$\gamma_n = \gamma_{n,wi} + \alpha_n \cdot IC \tag{44}$$

where

$$\gamma_{n,wi} = \frac{n}{2}, \tag{45}$$

$$\alpha_n = \frac{n}{2} \cdot \lambda_c^2. \tag{46}$$

The thermal noise excess factor γ_n is plotted versus the inversion coefficient IC in Figure 16 with a log x-axis and Figure 17 with a linear x-axis.

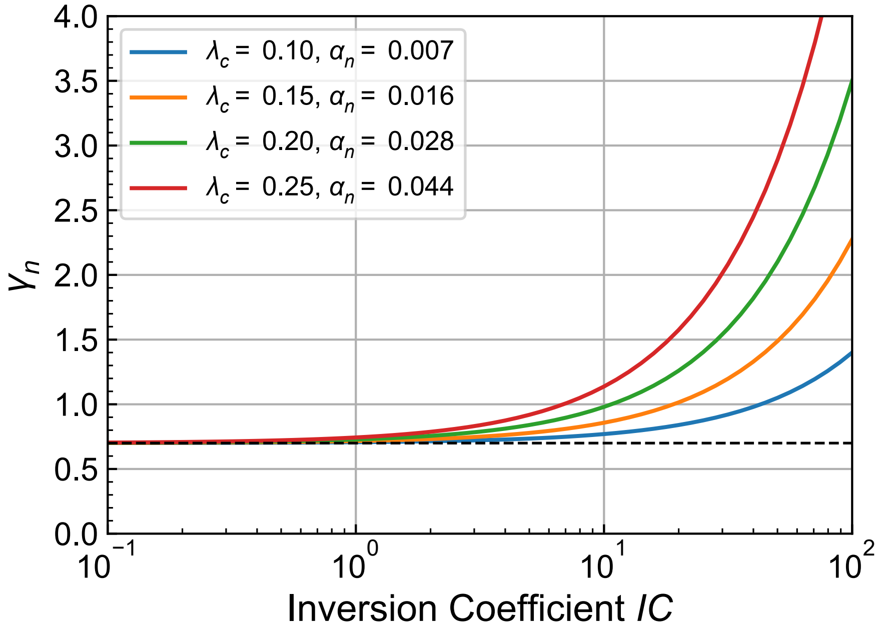

Figure 16: Thermal noise excess parameter γ_n versus inversion coefficient IC including short-channel effects.

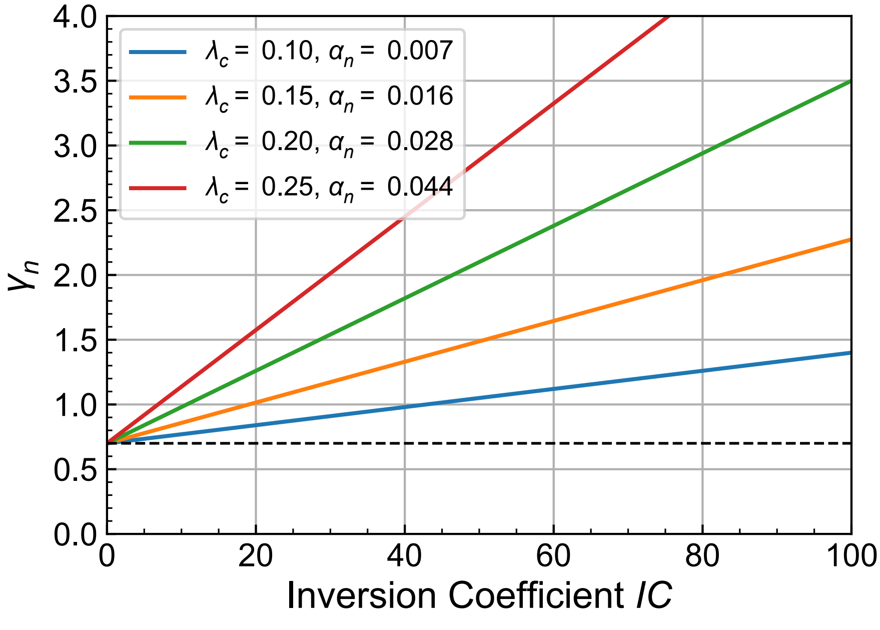

Figure 17: Thermal noise excess parameter γ_n versus inversion coefficient IC including short-channel effects.