Telecommunications Systems Exercises 4: Solutions

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4.1 The Need for Pulse Shaping

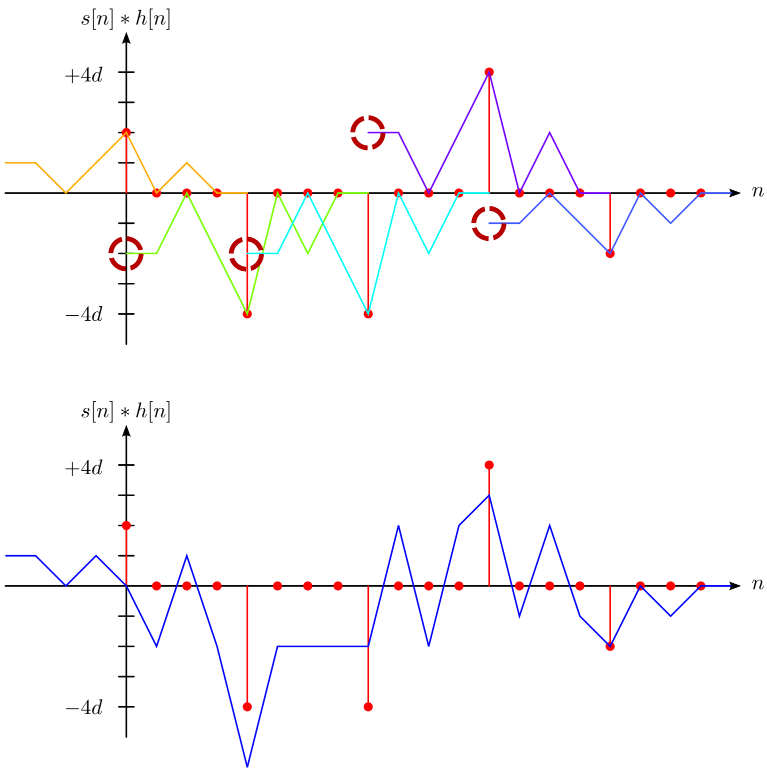

Figure 1: Output of the pulse shaping filter $s[n] * h[n]$ decomposed in pulses above and as one signal below

1. In Figure 1, the graph on the top represents the initial $s[n]$ in red and all the summed $h[n]$ pulses that constitute $s[n] * h[n]$. The latter is drawn in the bottom.

2. The pulse does not satisfy the Nyquist ISI criterion because its most negative sample overlaps with the previous symbol. This can be seen when

the pulses are drawn separately. The samples causing ISI are circled in Figure 1.

3. Because the second Nyquist criterion is not respected, the signal is not ISI-free. This can be seen on the bottom of Figure 1 because on the symbol samples, the filtered signal is not equal to the input signal.

4. We start with:

$$h(t) = \begin{cases} 1, & \text{if } t = 0; \\ 0, & \text{if } t = kT_s,\ k \in \mathbb{Z}^* \end{cases}$$

As hinted, this means that we can sample $h(t)$ every T_s (i.e. multiply by a Dirac comb) and the result will be a single Dirac pulse at 0:

$$h(t) \cdot \sum_{k=-\infty}^{+\infty} \delta\left(t - kT_s\right) \ = \ \delta(t)$$

From here, there are two variants that just differ in the order of the steps:

Variant 1 : We directly take the Fourier transform of the equation above, yielding:

$$H(f) * \sum_{k=-\infty}^{+\infty} e^{-j2\pi fkT_s} = 1$$

The sum term is saddening us as it is difficult to carry such a convolution by hand. But one may observe that it looks like a Fourier serie, and it is. This is the Fourier serie of a Dirac comb in the frequency domain with spacing $\frac{1}{T_s}$ (see Equation 1 of the Dirac comb in time domain). We can rewrite it as such (note that we can change the sign of k at will because we sum to $\pm\infty$):

$$\sum_{k=-\infty}^{+\infty} e^{-j2\pi fkT_s} \ = \ \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} \delta\left(f - \frac{k}{T_s}\right)$$

Injecting this in our Fourier domain equation, we get:

$$H(f) * \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} \delta\left(f - \frac{k}{T_s}\right) = 1$$

Which can be reformulated by executing the convolution:

$$\frac{1}{T_s} \sum_{k=-\infty}^{+\infty} H\left(f - \frac{k}{T_s}\right) \ = \ 1 \quad \square$$

Variant 2 : The Dirac comb is a periodic function; Thus its Fourier transform is an infinite Fourier serie and it can be expressed as such:

$$h(t) = \sum_{k=-\infty}^{+\infty} \delta\left(t - kT_s\right) \Leftrightarrow h(t) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} e^{j2\pi k \frac{t}{T_s}} \qquad (1)$$

Injected in our startpoint, we have:

$$h(t) \cdot \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} e^{j2\pi k \frac{t}{T_s}} \;=\; \delta(t)$$

Taking the Fourier transform on both sides:

$$H(f) * \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} \delta\left(f - \frac{k}{T_s}\right) = 1$$

Which can be reformulated by executing the convolution:

$$\frac{1}{T_s} \sum_{k=-\infty}^{+\infty} H\left(f - \frac{k}{T_s}\right) \;=\; 1 \quad \square$$

4.2 The Nyquist ISI Criterion

1. We want to find a pulse rate T_s such as the Nyquist ISI criterion is satisfied:

$$\frac{1}{T_s} \sum_{k=-\infty}^{+\infty} H\left(f - \frac{k}{T_s}\right) = 1$$

   From the drawing, we can sketch and figure by trial and error that taking $f_s = \frac{1}{T_s} = 1\ MHz$ will have the tip of the triangle as single non-zero term and yield:

$$\frac{1}{T_s} \sum_{k=-\infty}^{+\infty} H\left(f - \frac{k}{T_s}\right) = 10^6 \cdot 10^{-6} = 1$$

   Which satisfies Nyquist ISI criterion. The development of the next point also provides a mathematical path to find the right pulse rate.

2. To find $p(t)$, we can either directly use the Fourier pair between the triangle function and the $sinc^2$ or walk through the few steps below. Considering the triangle and the rectangle functions:

$$\Lambda(x) = \begin{cases} 1 - 2x, & \text{if } |x| \leq \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$\Pi(x) = \begin{cases} 1, & \text{if } |x| \leq \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

   Then we have :

$$P(f) = 10^{-6} \Lambda\left(\frac{f}{2 \cdot 10^{-6}}\right)$$

A triangle function can be expressed as the convolution between two rectangles with half its width. One have to be careful with the amplitude of the convolution when the rectangles are not of unit width. We have:

$$P(f) = 10^{-12} \cdot \Pi\left(\frac{f}{10^{-6}}\right) * \Pi\left(\frac{f}{10^{-6}}\right)$$

Taking the Fourier transform, we obtain:

$$\begin{aligned} p(t) &= 10^{-12} \cdot 10^6 \cdot sinc(10^6 t) \cdot 10^6 \cdot sinc(10^6 t) \\ &= sinc^2(10^6 t) \end{aligned}$$

It is immediate that $p(t) = 0$ for every $t = n \cdot T_s$ where $T_s = 1\ \mu s$. Hence by spacing symbols at multiples of T_s, we ensure that there is no ISI and the second Nyquist criterion is satisfied. We can see that $T_s = 1\mu s$ is consistent with $f_s = 1MHz$ from the previous point.

3. The pulse satisfies the Nyquist criterion with $T_s = 1\ \mu s$. Hence the Baud rate is

$$f_{Bd} = f_s = 1\ MBd/s$$

Since the system, is binary, the bit rate is equal to the baud rate:

$$f_{bit} = f_{Bd} = 1\ Mbits/s$$

Now, knowing from $P(f)$ that the one-sided bandwidth is $B = 1\ MHz$, we can calculate the roll-off factor:

$$\beta = \frac{2B}{f_s} - 1 \ = \ 1$$

In the end, we have a signal with $B = 1$ MHz, which means according to the Nyquist principle for perfecti signal reconstitution, that the minimum sampling frequency would be 2 MHz. But, with the second Nyquist criterion, we can sample at 1 MHz and we will be able to reconstruct the bits if the signal is sampled exatcly at the right times where there is no ISI.

4.3 PAM Constellation

First, we define the constellation as a set of equally spaced points:

$$x_n = \left\{ \pm d \frac{2n+1}{2} \right\}, \quad n \in \{0, 1, 2, 3\}$$

The expression above places each symbol at distance d from its neighbors in a bipolar way symmetrically around 0. Now, let us define d to ensure unit power:

$$P = \frac{1}{N} \sum_n |x_n|^2 \quad = \frac{d^2}{8} \cdot \left(2 \cdot \frac{1}{4} + 2 \cdot \frac{9}{4} + 2 \cdot \frac{25}{4} + 2 \cdot \frac{49}{4}\right) = d^2 \cdot \frac{21}{4} \quad = 1$$

Which implies:

$$d = \sqrt{\frac{4}{21}} \approx 0.44$$

If a symbol is falsely sampled as its neighbor due to noise, the decoded bits will be wrong. One can minimize the number of erroneous bits by ensuring that only one bit changes from one symbol to the other. The Gray code is a solution to do that as proposed in Figure 1.

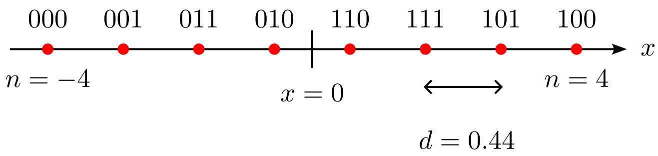

Figure 2: The constructed 8-PAM constellation

4.4 Error Rate for 3-PAM Constellation

We compare the error rate performance of the three 3-PAM constellation options based on minimum distance and average energy.

Option 1: $\mathcal{O}'_3 = \{-1, 0, 1\}$

- Spacing between adjacent points: 1
- Average energy:

$$E_{\text{avg}} = \frac{1}{3}((-1)^2 + 0^2 + 1^2) = \frac{2}{3}$$

- Minimum distance: 1

Option 2: $\mathcal{O}'_3 = \{-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\}$

- Spacing between adjacent points: 1
- Average energy:

$$E_{\text{avg}} = \frac{1}{3}\left(\left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2\right) = \frac{1}{3}\left(\frac{1}{4} + \frac{1}{4} + \frac{9}{4}\right) = \frac{11}{12}$$

- Minimum distance: 1

Option 3: $\mathcal{O}'_3 = \{-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}\}$

- Spacing between adjacent points: 1
- Average energy:

$$E_{\text{avg}} = \frac{1}{3}\left(\left(-\frac{3}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2\right) = \frac{1}{3}\left(\frac{9}{4} + \frac{1}{4} + \frac{1}{4}\right) = \frac{11}{12}$$

- Minimum distance: 1

Conclusion: All three options have the same minimum distance (1), but Option 1 has the lowest average energy ($\frac{2}{3}$). Hence, it has the best error rate performance under equal noise conditions.

$$\text{Best Option: } \mathcal{O}_3' = \{-1, 0, 1\}$$

4.5 Error Rate for M-PAM Constellations

We consider an M-PAM (Pulse Amplitude Modulation) constellation with alphabet given by:

$$\mathcal{O}_M = \left\{\pm\frac{d}{2}, \pm\frac{3d}{2}, \ldots, \pm\frac{(M-1)d}{2}\right\}$$

This consists of M equally spaced symbols centered around 0, where d is the minimum distance between adjacent constellation points.
The average power of this constellation is:

$$\bar{P}_{\mathcal{O}_M} = \frac{M^2-1}{3}\left(\frac{d}{2}\right)^2$$

1. Symbol Error Rate as a Function of M and SNR

$$\begin{aligned}\epsilon &= \frac{1}{M}Pr\left(n_k < -\frac{1}{2}d\right) + \frac{M-2}{M}Pr\left(|n_k| > \frac{1}{2}d\right) + \frac{1}{2}Pr\left(n_k > \frac{1}{2}d\right)\\ &= 2\frac{M-1}{M}Pr\left(n_k > \frac{1}{2}d\right) = 2\frac{M-1}{M}Q\left(\frac{d}{2\sigma}\right)\end{aligned}$$

$$\text{SNR}_M = \bar{P}_{\mathcal{O}_M}/\sigma^2$$

write d as a function of $\bar{P}_{\mathcal{O}_M}$, the ϵ can be expressed as

$$\epsilon = 2\frac{M-1}{M}Q\left(\sqrt{\frac{3\bar{P}_{\mathcal{O}_M}}{(M^2-1)\sigma^2}}\right) = 2\frac{M-1}{M}Q\left(\sqrt{\frac{3}{(M^2-1)}\text{SNR}_M}\right)$$

2. SNR Penalty: 4-PAM to 8-PAM We estimate the required SNR increase (in dB) to maintain the same symbol error rate when moving from 4-PAM to 8-PAM. the SNR penalty comparison is based only on the Q-function argument, which dominates error performance in high SNR regimes. The $\frac{M-1}{M}$ term is a slowly-varying scaling factor that cancels out or contributes minimally compared to the exponential decay of the Q-function.
From the SER expression, we compare the arguments of the Q-function:

$$\sqrt{\frac{3\cdot\text{SNR}_4}{4^2-1}} = \sqrt{\frac{3\cdot\text{SNR}_8}{8^2-1}} \Rightarrow \frac{\text{SNR}_8}{\text{SNR}_4} = \frac{8^2-1}{4^2-1} = \frac{63}{15} = 4.2$$

$$\text{SNR penalty (dB)} = 10\log_{10}(4.2) \approx \boxed{6.2 \text{ dB}}$$

3. SNR Penalty: 8-PAM to 16-PAM

$$\frac{\mathrm{SNR}_{16}}{\mathrm{SNR}_8} = \frac{16^2 - 1}{8^2 - 1} = \frac{255}{63} \approx 4.05$$

$$\text{SNR penalty (dB)} = 10\log_{10}(4.05) \approx \boxed{6.1\text{ dB}}$$

4. Asymptotic SNR Penalty per Bit (Large M) Let $M = 2^R$, where $R = \log_2 M$ is the number of bits per symbol.
From the SER approximation:

$$P_s \approx 2Q\left(\sqrt{\frac{2\cdot\mathrm{SNR}}{M^2}}\right), \quad \text{as } M \to \infty$$

To keep a constant SER as M increases (i.e., fixed Q-function argument), with 1-bit increase

$$\frac{2\mathrm{SNR}_M}{M^2} = \frac{2\mathrm{SNR}_{2M}}{(2M)^2}$$

The SNR penalty per additional bit is:

$$\text{SNR gain per bit} = 10\log_{10}(2^2) = \boxed{6\text{ dB/bit}}$$

4.6 Link between Power, Symbol/Bit Duration, Rate, and Error Rate

(1) Based on Figure: $E_b/N_0 = 8.7\,\mathrm{dB} = 10^{8.7/10} \approx 7.41$

(2) Bit Duration and Bit Rate From the relationship:

$$\frac{E_b}{N_0} = \frac{P_r \cdot T_b}{N_0} \quad \Rightarrow \quad T_b = \frac{E_b/N_0 \cdot N_0}{P_r}$$

Substitute the values:

$$T_b = \frac{7.41 \cdot 3.98 \times 10^{-11}}{3.16 \times 10^{-3}} \approx \frac{2.95 \times 10^{-10}}{3.16 \times 10^{-3}} \approx 9.34 \times 10^{-8}\,\mathrm{s}$$

Therefore:

$$\boxed{T_b \approx 93.4\,\mathrm{ns}}, \quad \boxed{R_b = \frac{1}{T_b} \approx 10.7\,\mathrm{Mbps}}$$

(3) Required Signal Bandwidth (Ideal SINC Filter) For an ideal sinc pulse shaping filter, the required bandwidth is:

$$B = \frac{1}{2T_b} = \frac{1}{2 \cdot 9.34 \times 10^{-8}} \approx 5.35\,\mathrm{MHz}$$

$$\boxed{B \approx 5.35\,\mathrm{MHz}}$$