Telecommunications Systems Solution 3

Spring semester 2025

1 AD Convertion and Quantization

1. Let us define SNR' as the SNR at the ADC input: $SNR' = \frac{P'_s}{P'_n}$. Then:

   - For each line, the signal power level and noise power level are reduced by α.
   - For each amplifier, both the incoming signal and noise power levels are increased by G, but there is additional noise level N introduced at input of the amplifier and thus also amplified by G.

   The signal power is therefore maintained thanks to the amplification chain.

   $$P'_s = P_s - \alpha + G - \alpha + G = P_s = 33 \text{ dBm}$$

   But there are two noise sources (each amplifier) and they are also amplified through the chain:

   $$\begin{aligned} P'_{n1} =& N + G - \alpha + G \quad \text{(Noise of the 1st amplifier, attenuated and then re-amplified.)} \\ P'_{n2} =& N + G \quad \text{(Noise of the 2nd amplifier, amplified.)} \end{aligned}$$

   The total noise power at the ADC is therefore:

   $$\begin{aligned} P'_n =& 10 \log \left(10^{P'_{n1}/10} + 10^{P'_{n2}/10}\right) \quad (*) \\ =& 10 \log \left(10^{(N+G)/10} + 10^{(N+G)/10}\right) \\ \approx& 19 \text{ dBm} \end{aligned}$$

   Notice in (*) that we summed both noise powers linear value. When in dB, summing is equivalent to multiplying the physical value, thus summing a power with unit-less gains remains a power, but summing powers together is equivalent to multiplying them together, which does not results in the desired physical quantity of total power. That is why we go back in the liner expression to sum them up, then back in dB to express it. We can now calculate the SNR, by subtracting two powers together, yielding a unit-less ratio:

$$SNR' = P'_s - P'_n = 14 \text{ dB}$$

The requirement is fulfilled.

2. At the input of the ADC, if the SNR is smaller than the SQR means that the power of noise is larger than the power of the error of quantization. In other words, the noise "amplitude" is larger than the quantization error which means that the noise will often cause a wrong value to be sampled and thus cause a lot of errors in the reconstruction;

3. We want to ensure:

$$SNR' \geq SQR + 3$$

Since $SNR' = 14$ dB and $SQR = 2dB$, we can increase the SQR of $9dB$ at much.
And because:

$$SQR = c + Q \cdot 6\text{dB}$$

We can increase Q from 8 to 9 bits which will cause:

$$SQR = 8 \text{ dB} < SNR' - 3$$

4. Finally, Q can also be expressed as the ratio between channel bandwidth and signal bandwidth (i.e., the maximal number of bits we can put in one quantized symbol).

$$Q_{max} = \frac{BW_{ch}}{BW_{signal}}$$

Thus, by increasing Q from 8 to 10, and because the signal is not changed, the fractional increase on channel bandwidth is $\frac{9}{8}$.

2 Audio Quantization

1. For a baseband $B = 15$ kHz, the Nyquist rate is:

$$f_{s,min} = 2 \cdot B = 30 \text{ kHz}$$

2. The number of bits required to encode on 65'536 levels is:

$$Q = \log_2(65'536) = 16 \text{ bits}$$

3. At 30 kS/s and 16 bits per sample, the bit rate is:

$$f_{bit} = f_{s,min} \cdot Q = 480 \text{ kbits/s}$$

4. If the sampling rate is $f_s = 44.1$ kHz then:

$$f_{bit} \;=\; f_s \cdot Q \;=\; 705.6 \text{ kbits/s}$$

And:

$$BW_{ch} \;=\; \frac{f_{bit}}{2} \;=\; 352.8 \text{ kHz}$$

Alternatively:

$$BW_{ch} \;=\; Q \cdot BW_{signal} \;=\; Q \cdot \frac{f_s}{2} \;=\; 352.8 \text{ kHz}$$

3 Frequency Multiplexing of PCM signals

- Let A be the amplitude of the signal. The error must be within 0.2% of the signal amplitude, which leads us to the number of bits to be used to encode a sample:

$$\begin{aligned} e &\leq A \cdot 0.002 \\ \frac{A}{2^Q - 1} &\leq A \cdot 0.002 \\ Q &\geq \log_2(501) \\ Q_{min} &= 9 \text{ bits} \end{aligned}$$

- For a signal bandwidth $BW_s = 1$ kHz, the sampling frequency 20% above the Nyquist rate is given by:

$$\begin{aligned} f_s &= 2 \cdot BW_s \cdot 1.2 \\ &= 2.4 \text{ kHz} \end{aligned}$$

- Knowing the sampling frequency and the bits per sample, the bit rate can be determined. Including the 0.5% overhead for framing and synchronization, we get:

$$\begin{aligned} f_{bit} &= Q \cdot f_s \cdot 1.005 \\ &= 21.708 \text{ kbits/s} \end{aligned}$$

- Finally, this bit rate corresponds to a band:

$$BW_{ch} = \frac{f_{bit}}{2} = 10.854 \text{ kHz}$$

  And since $N = 5$ of these signals have to be transmitted, we can multiplex them in the shortest band possible (see spectrum below) which gives us:

$$\begin{aligned} BW_{ch,total} &= BW_{ch} \cdot (2N - 1) \\ &= 97,686 \text{ kHz} \end{aligned}$$

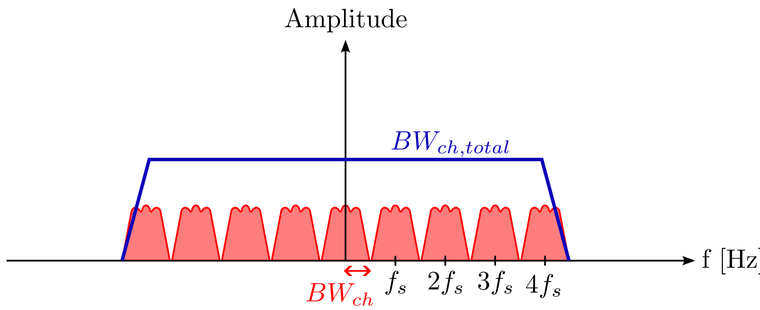

Figure 1: Spectrum of the 5 signals

4 Hints Towards Pulse Shaping

1. The error allows us to compute the number of bits

$$e = \frac{A}{2^Q - 1}$$
$$Q = \log_2\left(\frac{A}{e}\right) + 1$$
$$Q = 4 \text{ bits}$$

2. The minimum sampling frequency is given by the Nyquist rate:

$$f_s = 2 \cdot B = 10 \text{ kHz}$$

Knowing the bits per sample, we can calculate the bit rate:

$$f_{bit} = Q \cdot f_s = 40 \text{ kbits/s}$$

3. Signed values mean that 4 bits range from -8 to 7. Converting this with the amplitude of the signal, we get the quantization steps:

$$\nu = \frac{2A}{2^Q - 1} = 0.106$$

We want the closest step from $s(t_0) = 0.425$:

$$N = \frac{s(t_0)}{\nu} \approx 4.009$$

Therefore, the step value that minimizes the quantization error is 4, which is 0010 in binary (LSB on the left).

4. One pulse is one bit. T can be calculated from the bit rate:

$$T = \frac{1}{f_{bit}} = 25\ \mu s$$

The binary code of the sample is 0010, hence the signal for this pulse train is:

$$\begin{aligned} m(t) &= \sum_{k=0}^{3} b_k \cdot \text{rect}\left(\frac{t-kT}{T}\right), \text{ where } b_0, b_1, b_3 = 0 \text{ and } b_2 = 1 \\ &= \text{rect}\left(\frac{t-2T}{T}\right) = \text{rect}\left(\frac{t}{T} - 2\right) \end{aligned}$$

Which is sketched below:

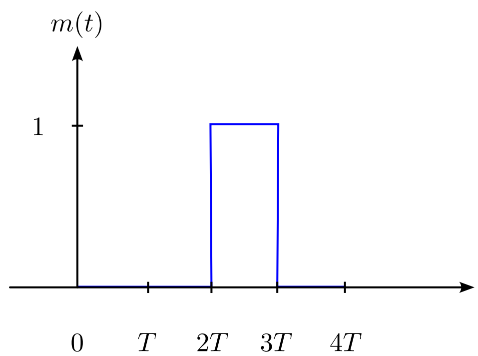

Figure 2: Rectangular pulses for $s(t_0)$ expressed as 0010 in binary

5. The signal can be expressed as the convolution between a rectangle and a time delay. Using the Fourier pairs, the spectrum can easily be analytically expressed:

$$\begin{aligned} m(t) &= \text{rect}\left(\frac{t}{T} - \frac{2}{T}\right) \\ &= \text{rect}\left(\frac{t}{T}\right) * \delta\left(t - 2T\right) \\ &\Leftrightarrow \\ M(2\pi f) &= T \cdot sinc(fT) \cdot e^{-j2\pi f 2T} \end{aligned}$$

The amplitude is sketched below:

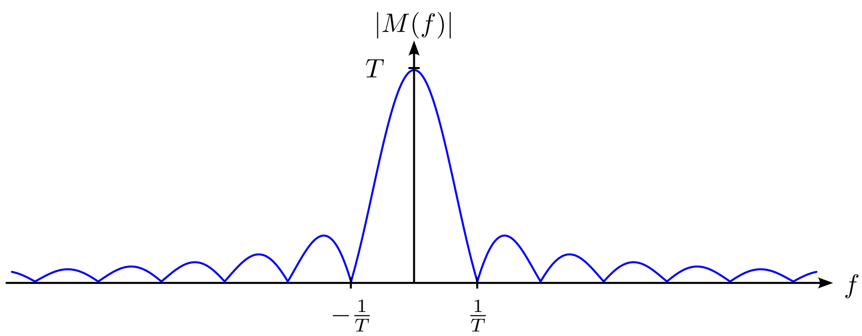

Figure 3: Spectrum of the perfect rectangular pulses

6. The spectrum is a sinc and is therefore not band limited. This is a problem because it occupies all of the spectrum which is undesirable in order to multiplex signal and share frequencies.

7. Again, with binary code 0010, the pulse train is:

$$
\begin{aligned}
m(t) &= \sum_{k=0}^{3} b_k \cdot sinc\left(\frac{t-kT}{T}\right), \text{ where } b_0, b_1, b_3 = 0 \text{ and } b_2 = 1 \\
&= sinc\left(\frac{t-2T}{T}\right)
\end{aligned}
$$

Which is sketched below:

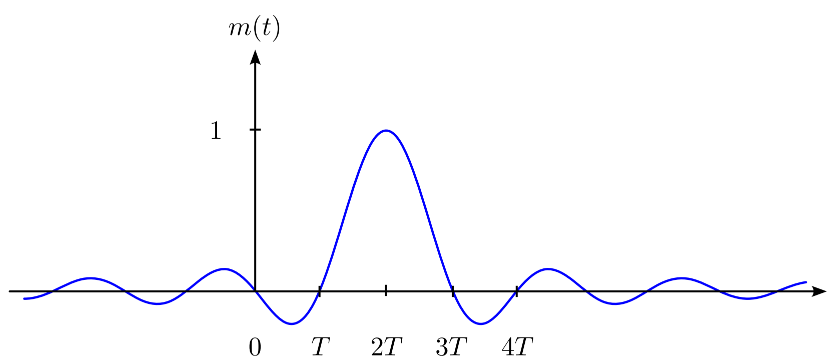

Figure 4: Sinc pulses for $s(t_0)$ expressed as 0010 in binary

8. The signal can be expressed as the convolution between a sinc and a time delay. Using the Fourier pairs, the spectrum can easily be analytically

expressed:

$$
\begin{aligned}
m(t) &= sinc\left(\frac{t-2T}{T}\right) \\
&= sinc\left(\frac{t}{T}\right) * \delta\left(t-2T\right) \\
&\Leftrightarrow \\
M(f) &= T \cdot \text{rect}\left(|f| \cdot T\right) \cdot e^{-j4\pi fT}
\end{aligned}
$$

And is sketched below:

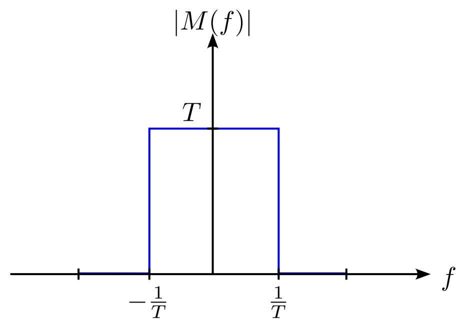

Figure 5: Spectrum of the sinc pulses

The bandwidth is:

$$B = \frac{1}{T} = f_{bit} = 40\ kHz$$

9. The pulse train is made of sinc and is therefore not limited in time. This is a problem because we do not have an infinity of time to transmit a signal. It is also problematic for time multiplexing.

10.
$$G(2\pi f) = G(2\pi f) \cdot \text{rect}\left(\frac{|f|}{B'}\right)$$

Ensures that the spectrum is band limited to at least B'. It is equivalent to its time domain expression:

$$g(t) = g(t) * B' sinc(B't)$$

If $g(t)$ is time limited, it means that:

$$\exists\ t_m \mid\ |t| > t_m \Rightarrow g(t) = 0$$

Now let there be $-t_m < t_0 < t_m$ such that $g(t_0) \neq 0$.

We can expand $g(t)$ as a convolution:

$$g(t) = B' \int_{-\infty}^{\infty} g(t-\tau) \cdot sinc(B'\tau) d\tau$$

Let there now be $t_1 > t_m$, then:

$$g(t_1) = B' \int_{-\infty}^{\infty} g(t_1-\tau) \cdot sinc(B'\tau) d\tau$$

Then, there will be τ such that:

$$t_1 - \tau = t_0 \text{ and } B'\tau \notin \mathbb{Z}$$

Which means that:

$$g(t_1 - \tau) \cdot sinc(B'\tau) \neq 0 \;\Rightarrow\; g(t_1) \neq 0$$

Which contradicts the initial assumption. If:

$$sinc(B'\tau) = 0$$

Then let there be $t_2 = t_1 + \epsilon$ where ϵ is as small as required to ensure $sinc(B'(\tau + \epsilon)) \neq 0$, then:

$$g(t_2 - \tau - \epsilon) \cdot sinc(B'(\tau + \epsilon)) \neq 0 \;\Rightarrow\; g(t_2) \neq 0$$

And since $t_2 > t_1 > t_m$, it contradicts the existence of any t_m beyond which the signal is 0.

Hence, the signal cannot be both spectrum and time limited. $\square$