1 Power and Attenuation in Telecommunications

1.1 Basic Free-Space Path Loss Model

1) The wavelength is given by:

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{2 \times 10^9} = 0.15 \text{ m}$$

where c is the speed of light.

2) Compute P_R for $P_T = 1$ W and $d = 100$ m: Using the FSPL model:

$$P_R = P_T \left(\frac{\lambda}{4\pi d} \right)^2 = 1 \times \left(\frac{0.15}{4\pi \times 100} \right)^2 \approx 1.42 \times 10^{-8} \text{ W}$$

3) The path loss in dB:

$$L(\text{dB}) = 10 \log_{10} \left(\frac{P_T}{P_R} \right) = 10 \log_{10} \left(\frac{1}{1.42 \times 10^{-8}} \right) \approx 78.5 \text{ dB}$$

1.2 Effect of Distance on Power Attenuation

1. **Compute P_R for $d = 200$ m:**
   Using the FSPL model:

   $$P_R = P_T \left(\frac{\lambda}{4\pi d} \right)^2$$

   Substituting $d = 200$ m:

   $$P_R = 1 \times \left(\frac{0.15}{4\pi \times 200} \right)^2 = 3.56 \times 10^{-9} \text{ W}$$

   Compute P_R using empirical model with $\gamma = 3$ and $\gamma = 4$:

   $$P_R = P_T \cdot d^{-\gamma}$$

   For $\gamma = 3$:
   $$P_R = 1 \times 100^{-3} = 10^{-6} \text{ W}$$

   For $\gamma = 4$:
   $$P_R = 1 \times 100^{-4} = 10^{-8} \text{ W}$$

2 Signal Transformation in Time and Frequency

1) Classification of $x(t)$

$$x(t) = \begin{cases} 1, & 0 \leq t < 1 \\ 0, & \text{otherwise} \end{cases}$$

The given signal is: Continuous, Non-Periodic and Deterministic

2) Time Delay by 2 Seconds
A time delay by $T_0 = 2$ seconds results in:

$$x'(t) = x(t-2) = \begin{cases} 1, & 2 \leq t < 3 \\ 0, & \text{otherwise} \end{cases}$$

The Fourier transform property states:

$$x(t - T_0) \Longleftrightarrow X(f)e^{-j2\pi f T_0}$$

Hence,

$$X'(f) = X(f)e^{-j4\pi f}.$$

This introduces a linear phase shift but does not affect the magnitude spectrum.

3) Time Scaling by Factor α
If the signal is scaled in time by a factor α, then the transformed signal is:

$$x''(t) = x(\alpha t)$$

Using the Fourier transform time-scaling property:

$$x(\alpha t) \Longleftrightarrow \frac{1}{|\alpha|} X\left(\frac{f}{\alpha}\right)$$

This implies that:

- The frequency content of the signal is compressed for $\alpha > 1$ and expanded for $\alpha < 1$.
- The bandwidth scales proportionally to $1/|\alpha|$.

3 Power, Energy, and Power Spectral Density

1) Total Energy E_s
The total energy of the signal is given by:

$$E_s = \int_{-\infty}^{\infty} |s(t)|^2 dt.$$

Since $s(t) = 0$ for $t < 0$, we integrate from 0 to ∞:

$$E_s = \int_0^\infty A^2 e^{-2at} dt.$$

Evaluating the integral:

$$E_s = A^2 \int_0^\infty e^{-2at} dt = A^2 \left[\frac{e^{-2at}}{-2a} \right]_0^\infty.$$

$$E_s = \frac{A^2}{2a}, \quad \text{for } a > 0.$$

Thus, the signal has finite energy.

2) Energy and Power for $A = 5$, $a = 2$
Substituting $A = 5$ and $a = 2$:

$$E_s = \frac{5^2}{2 \times 2} = \frac{25}{4} = 6.25.$$

The power of the signal is:

$$P_s = \lim_{T \to \infty} \frac{1}{T} \int_0^T s^2(t) dt.$$

Since $s(t)$ is not periodic, its power is zero.

3) Effect of Gain G on Energy and Power If the receiver introduces a gain G, the new signal is:

$$s'(t) = Gs(t).$$

The new energy is:

$$E'_s = \int_0^\infty |GAe^{-at}|^2 dt = G^2 E_s.$$

The new power is:

$$P'_s = G^2 P_s.$$

4) White Noise and Power Spectral Density
White noise has a flat power spectral density (PSD), meaning:

$$S_n(f) = \text{constant}, \quad \forall f.$$

This indicates equal power distribution across all frequencies.

5) Effect of Noise on Received Signal
The received signal is:

$$y(t) = x(t) + n(t).$$

Since the power of independent signals adds up:

$$P_y = P_x + P_n.$$

6) Noise Mitigation through Filtering
Filtering can reduce noise by eliminating unwanted frequency components. A low-pass filter, for instance, removes high-frequency noise while retaining the signal of interest.

4 Parseval's Theorem and Signal Energy

1) Energy of a Rectangular Pulse in Time Domain

The energy of $x(t)$ is computed as:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt.$$

Since $x(t) = 1$ for $0 \leq t < 1$ and 0 otherwise:

$$E_x = \int_0^1 1^2 dt = \int_0^1 dt = 1.$$

Thus, the total energy of the pulse is:

$$E_x = 1.$$

2) Energy in the Frequency Domain and Parseval's Theorem Verification

Remark: Considering computing this integral is difficult, a precise calculation is not required for this question. It is sufficient to understand that, according to Parseval's Theorem, the energy in the frequency domain is equal to the energy in the time domain.

3) Energy of Modulated Signal in Frequency Domain

A modulated signal is given by:

$$s(t) = x(t)e^{j2\pi f_0 t}.$$

The Fourier transform property states that modulation shifts the frequency spectrum:

$$S(f) = X(f - f_0).$$

Using Parseval's theorem:

$$E_s = \int_{-\infty}^{\infty} |S(f)|^2 df = \int_{-\infty}^{\infty} |X(f - f_0)|^2 df.$$

Thus, the energy remains unchanged by modulation.

4) Practical Relevance of Parseval's Theorem

Parseval's theorem is crucial in signal processing and communications:

- It ensures energy conservation between time and frequency domains.
- It helps in analyzing signal power distribution in frequency-selective channels.
- It is used to design filters that minimize energy loss in communication systems.

5 Orthogonality in Modulation and Its Effects

1) Orthogonality of $\cos(2\pi f_0 t)$ and $\sin(2\pi f_0 t)$

Evaluating the integral over one period $T = \frac{1}{f_0}$:

$$\int_0^{1/f_0} \cos(2\pi f_0 t) \sin(2\pi f_0 t) dt.$$

Using the identity:

$$2 \cos A \sin A = \sin(2A),$$

we rewrite the integral as:

$$\frac{1}{2} \int_0^{1/f_0} \sin(4\pi f_0 t) dt.$$

Since $\sin(4\pi f_0 t)$ completes full cycles over the integral limits, the result is zero, proving orthogonality.

2) Quadrature Amplitude Modulation (QAM) and Orthogonality

QAM is based on the combination of two orthogonal signals, typically:

$$s(t) = I(t) \cos(2\pi f_0 t) + Q(t) \sin(2\pi f_0 t),$$

where $I(t)$ and $Q(t)$ are independent data streams.

Since $\cos(2\pi f_0 t)$ and $\sin(2\pi f_0 t)$ are orthogonal, they can be separated at the receiver without interference. This property allows efficient bandwidth usage and increased data transmission rates.

3) Effect of Noise $n(t)$ in Demodulation

When noise $n(t)$ is present in the received signal:

$$y(t) = s(t) + n(t),$$

applying matched filtering with the orthogonal basis functions helps in extracting $I(t)$ and $Q(t)$ independently, reducing interference effects. Since noise is typically spread across frequencies, the orthogonal components remain largely unaffected, improving signal demodulation and reducing error rates.

6 Orthogonality in CDMA-like Waveforms

6.1 Compute the Inner Product

1) The inner product of two signals is defined as:

$$\langle \phi_1, \phi_2 \rangle = \int_0^T \phi_1(t)\phi_2(t)\,dt.$$

Breaking the integral into appropriate intervals:

$$\langle \phi_1, \phi_2 \rangle = \int_0^{T/4} (1)(1)dt + \int_{T/4}^{T/2} (1)(-1)dt + \int_{T/2}^{3T/4} (-1)(-1)dt + \int_{3T/4}^{T} (-1)(1)dt.$$

Evaluating each term:

$$\int_0^{T/4} dt = T/4,$$
$$\int_{T/4}^{T/2} -dt = -T/4,$$
$$\int_{T/2}^{3T/4} dt = T/4,$$
$$\int_{3T/4}^{T} -dt = -T/4.$$

Summing the values:

$$\langle \phi_1, \phi_2 \rangle = \frac{T}{4} - \frac{T}{4} + \frac{T}{4} - \frac{T}{4} = 0.$$

2) Since the inner product is zero, the signals are fully orthogonal.

6.2 Signal Separation in a CDMA-like System

A received composite signal is given by:

$$r(t) = a_1\phi_1(t) + a_2\phi_2(t).$$

1) Extracting a_1

Taking the inner product with $\phi_1(t)$:

$$\langle r, \phi_1 \rangle = \int_0^T r(t)\phi_1(t)dt.$$

Substituting $r(t)$:

$$\langle r, \phi_1 \rangle = \int_0^T (a_1\phi_1(t) + a_2\phi_2(t))\phi_1(t)dt.$$

Using linearity and orthogonality:

$$\langle r, \phi_1 \rangle = a_1\langle \phi_1, \phi_1 \rangle + a_2\langle \phi_2, \phi_1 \rangle.$$

Since $\langle \phi_2, \phi_1 \rangle = 0$:

$$\langle r, \phi_1 \rangle = a_1\langle \phi_1, \phi_1 \rangle.$$

Thus,

$$a_1 = \frac{\langle r, \phi_1 \rangle}{\langle \phi_1, \phi_1 \rangle}.$$

2) Extracting a_2

Similarly, taking the inner product with $\phi_2(t)$:

$$a_2 = \frac{\langle r, \phi_2 \rangle}{\langle \phi_2, \phi_2 \rangle}.$$

3) Effect of Non-Orthogonality

If the signals were not orthogonal, the cross-term $\langle \phi_1, \phi_2 \rangle$ would be nonzero, leading to interference. This would make it impossible to perfectly separate a_1 and a_2, degrading performance in a CDMA system.