Telecommunications Systems Solution 2

Spring semester 2025

1 Stochastic signals: Noise in Wireless

1. The input noise $X(t)$ is modeled as white noise, meaning it has a flat power spectral density:

$$P_X(f) = \frac{N_0}{2}, \quad \text{for all } f$$

The noise passes through an ideal low-pass filter with impulse response $h(t)$ and cutoff frequency B_c, whose frequency response is:

$$H(f) = \begin{cases} 1, & |f| \leq B_c \\ 0, & \text{otherwise} \end{cases}$$

The output noise $Y(t)$ is given by the convolution $Y(t) = X(t) * h(t)$. In the frequency domain, the PSD of $Y(t)$ is:

$$P_Y(f) = P_X(f) \cdot |H(f)|^2 = \begin{cases} \frac{N_0}{2}, & |f| \leq B_c \\ 0, & \text{otherwise} \end{cases}$$

Since the PSD of $Y(t)$ is not flat over all frequencies but limited to the band $[-B_c, B_c]$, the noise is no longer white. Instead, it is called **colored noise**, as the power is not uniformly distributed across the frequency spectrum.

2. The autocorrelation function of the output noise is given as:

$$R_Y(\tau) = A \cdot \frac{\sin(2\pi B_c \tau)}{2\pi B_c \tau}$$

This function is a sinc function, which indicates that the noise samples are correlated over time. The width of the main lobe of the sinc function is inversely proportional to B_c.

When B_c increases, the sinc function becomes narrower. This means the noise values at different times decorrelate more quickly, and the noise behaves more like white noise. When B_c decreases, the sinc function becomes wider. This leads to stronger time-domain correlation between noise samples, and the noise appears more smooth or correlated over time.

2 Digitalize your AM signal

1. When a baseband signal $m(t)$, band-limited to B Hz, is modulated using a sinusoidal carrier at frequency f_0, the modulated signal is:

$$s_{\text{mod}}(t) = m(t)\cos(2\pi f_0 t)$$

   In the frequency domain, this corresponds to a spectrum:

$$S_{\text{mod}}(f) = \frac{1}{2}M(f - f_0) + \frac{1}{2}M(f + f_0)$$

   where $M(f)$ is the Fourier transform of $m(t)$. This means the baseband spectrum is shifted to both $+f_0$ and $-f_0$, resulting in two symmetric sidebands centered around $\pm f_0$, each of bandwidth B.

2. (a) The square wave can be expanded into its Fourier series:

$$\text{sq}(t) = \sum_{k=1,3,5,\ldots}^{\infty} \frac{4}{\pi k}\sin(2\pi k f_0 t)$$

   It contains only odd harmonics of the fundamental frequency f_0, i.e., components at $f_0, 3f_0, 5f_0, \ldots$.

   (b) The modulated signal becomes:

$$s_{\text{mod}}(t) = m(t)\cdot \text{sq}(t) = \sum_{k=1,3,5,\ldots}^{\infty} \frac{4}{\pi k} m(t)\sin(2\pi k f_0 t)$$

   Using the modulation property in frequency domain (multiplication in time $\rightarrow$ convolution in frequency), each harmonic $\sin(2\pi k f_0 t)$ shifts the baseband signal to $\pm k f_0$.

   (c) Therefore, the output spectrum will contain copies of $M(f)$ centered at $\pm f_0, \pm 3f_0, \pm 5f_0, \ldots$, each scaled by $\frac{2}{\pi k}$ (since sin modulation gives imaginary components at $\pm k f_0$).

3. Although square wave mixing introduces spectral components at $3f_0, 5f_0, \ldots$, these components are far from the main carrier f_0 and can be easily separated in frequency. The receiver typically uses low-pass filters to isolate the desired band around f_0, effectively rejecting the out-of-band harmonics.

4. A low-pass filter centered at f_0 with bandwidth slightly wider than the message signal B can be used to suppress the higher-order harmonics. This filter allows only the desired spectral component to pass through while attenuating $3f_0, 5f_0, \ldots$.

5. Square waves (e.g., CLK signal) are easier and more power-efficient to generate in CMOS circuits, making them a practical choice.

3 Digitalize your AM signal

1. First, $m(t)$ can be conveniently expressed as a sum of sinuses:

$$m(t) = \frac{1}{2}[\sin(2\pi(f_c + f_1)t) + \sin(2\pi(f_c - f_1)t)]$$

Which leads to a straightforward expression of the Fourier transform:

$$M(f) = \frac{1}{4j}[\delta(f - f_c - f_1) - \delta(f + f_c + f_1) \\ + \delta(f - (f_c - f_1)) - \delta(f + f_c - f_1)]$$

The Fourier transform of the sampled signal then consist of the continuous Fourier transform repeated around multiples of f_s, we can expand it and then only keep the terms within $\pm f_s/2$:

$$\tilde{M}(f) = \frac{1}{4j}\sum_{l=-\infty}^{+\infty}[\delta(f - f_c - f_1 - lf_s) - \delta(f + f_c + f_1 - lf_s)$$

$$+\delta(f - (f_c - f_1) - lf_s) - \delta(f + f_c - f_1 - lf_s)]$$

$$= \frac{1}{4j}[\delta(f - (f_c - f_1)) - \delta(f + f_c - f_1) \\ + \delta(f - f_c - f_1 + f_s) - \delta(f + f_c + f_1 - f_s)]$$

$$= \frac{1}{4j}[\delta(f - 750 \cdot 10^3) - \delta(f + 750 \cdot 10^3) \\ + \delta(f - 850 \cdot 10^3) - \delta(f + 850 \cdot 10^3)]$$

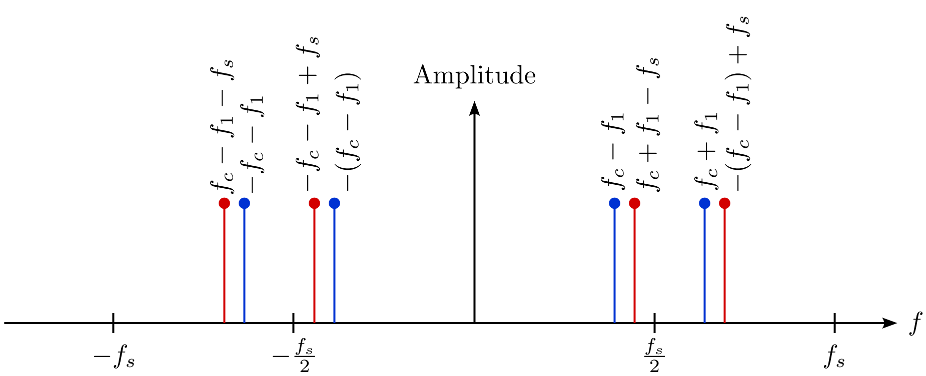


Figure 1: Spectrum $\tilde{M}(f)$, in blue, the original elements of $M(f)$ and in red, the repetitions of $M(f)$ due to sampling.

We can see in Fig. 1 that the elements sinus in $\pm(f_c + f_1)$ is lost beyond $f_s/2$, but its repetition around $\pm f_s$ are re-injected between $\pm f_s/2$, creating a new frequency component at $\pm(f_c + f_1 - f_s)$.

2. By taking the inverse Fourier transform of $\tilde{M(f)}$, we obtain :

$$\mathcal{F}^{-1}\{\tilde{M}(f)\} = \frac{1}{2}\sin(2\pi(f_c - f_1)t) + \frac{1}{2}\sin(2\pi(f_c + f_1 - f_s)t)$$

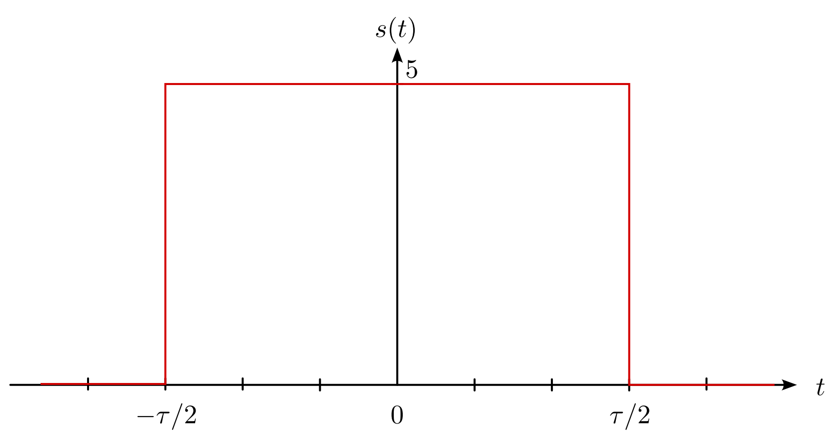
We can indeed see that the sine of $f_c + f_1 = 750$ kHz is lost, but a new sine of $t_c + f_1 - f_s = 850$ kHz appeared. It is the phenomenon of aliasing.

3. Aliasing is caused by attempting to sample signals with a too low sampling frequency and thus violating the Nyquist theorem. It can be avoided by:

   - augmenting f_s beyond $2 \cdot (f_1 + f_c)$;
   - using a low-pass filter which cut-offs every component of the signal above $\frac{f_s}{2}$.

4 You can't have the cake and eat it too

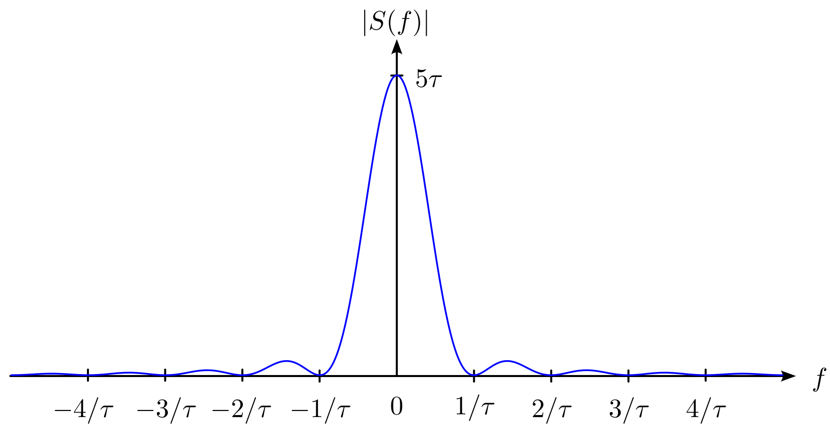
1. The sent signal is:



2. The Fourier transform of the rectangle $s(t)$ is

$$S(f) = 5\tau\,\text{sinc}(f\tau)$$

Ans its graph is:



3. Because the sinc is non-zero until infinity, it would require an infinite sampling frequency to sample it, which is impossible.

5 Time to sing

The spectrum is drawn below, it evolves in the following way:

- The ideal low-pass filter removes everything that is beyond $\pm f_0$.
- The ADC turns the continuous spectrum in samples, and repeats it around f_s.
- The up-sampler interleaves zeros in the time domain as such :

$$x_{(2)}[n] = \begin{cases} x[r], & n = 2r, \quad r \in \mathbb{Z}^* \\ 0, & \end{cases}$$

Consider the discrete time Fourier transform of $x[n]$ defined as $X(e^{j\omega})$. The discrete Fourier transform of $x[n]$ over time period T is therefore :

$$X[k] = X(e^{j\frac{2\pi f_k}{f_s}}) = X(e^{j\frac{2\pi k}{M}}) = X(e^{j\frac{2\pi k}{Tf_s}})$$

Where M is the number of samples of the signal acquired in period T. We can see that

$$f_k = \frac{k}{T}$$

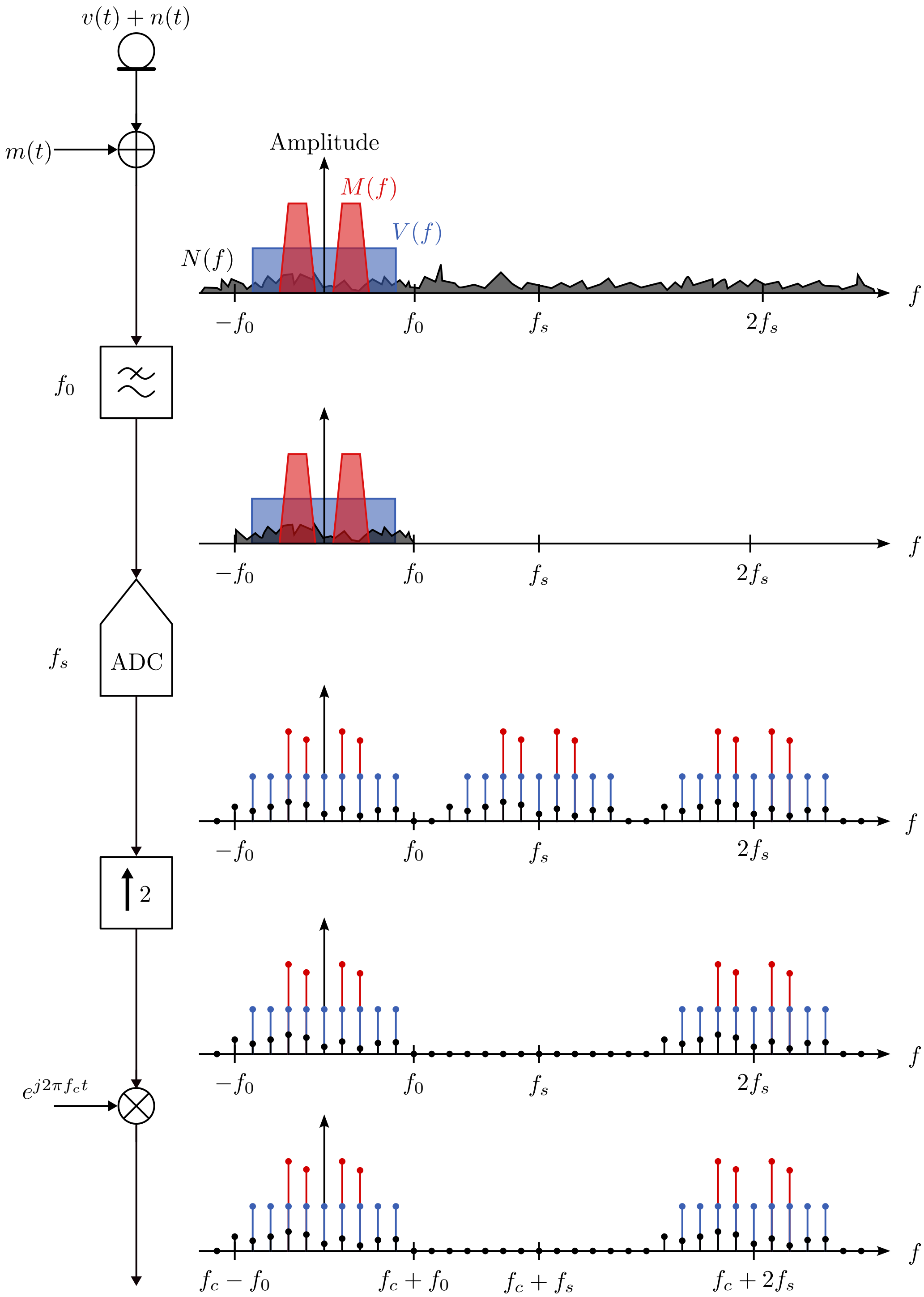
which means that the frequency resolution is of $\frac{1}{T}$ [Hz]. When up-sampling the signal into $x_{(2)}[n]$, thew pulsation ω is now defined relative to $2f_s$ instead of f_s and the total number of samples becomes $2M$ because there are twice as much samples in the same period T. Using the Fourier transform identity for up-sapling, we have:

$$X_{(2)}(e^{j\omega}) = X((e^{j\omega})^2)$$
$$X_{(2)}(e^{j\frac{2\pi f_k}{2f_s}}) = X(e^{j\frac{2\pi f_k}{f_s}})$$
$$X_{(2)}(e^{j\frac{2\pi k}{2M}}) = X(e^{j\frac{2\pi k}{M}})$$
$$X_{(2)}(e^{j\frac{2\pi k}{2f_s T}}) = X(e^{j\frac{2\pi k}{f_s T}})$$
$$X_{(2)}[k] = X[k], k < M$$

Carefully note that the last line only holds for $k < M$ because when k reaches M, X has made a period and starts over but $X_{(2)}$ has not. In other words, we have covered the whole ω for X but only half of it for $X_{(2)}$. So what of the terms in $k = M, ..., 2M - 1$? Well, they are 0, because they are "created" by the up-sampling process but no additional information can be created out of nowhere. In the end, the up-sampling process maintained the same frequency resolution, and simply increased the bandwidth available without new information. We therefore see on the graph that the repeated spectra simply got further apart from each other and the "free space" was filled with zeros. Such an operation can be useful when some further operations will be applied to the signal and might create higher frequency components, in this case, up-sampling allows one to avoid the creation of aliasing.

You might then say "okay, if interleaving zeroes adds bandwidth, then padding a bunch of zeros at the end of my signal to increase the acquisition time will result in a finer frequency resolution $\frac{1}{T}$!". Nice try ! But unfortunately, your zero-padded signal still contains the same information, the zeros didn't add any. You will ultimately end up with a finer discrete Fourier transform, true, but the additional points will correspond to an interpolation of the discrete Fourier transform with the minimum amount of samples to contain all the useful signal.

- Finally, the modulator shifts the signal on the pass-band, which we represent by changing the labels with f_c.



$v(t)+n(t)$
$m(t)$
Amplitude
$M(f)$
$V(f)$
$N(f)$
f
$-f_0$
f_0
f_s
$2f_s$
f_0
f_s
ADC
2
$e^{j2\pi f_c t}$
$f_c - f_0$
$f_c + f_0$
$f_c + f_s$
$f_c + 2f_s$